Benchmarking Algorithms for the Jones Polynomial

Evan Ashoori, Peter Bai, Hansen Shieh

Mentor: Ryan Maguire

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Background: Knots and Knot Diagrams

Definition (Knot)

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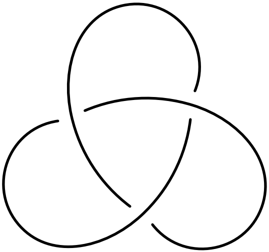

Figure: Knot diagram of the trefoil knot

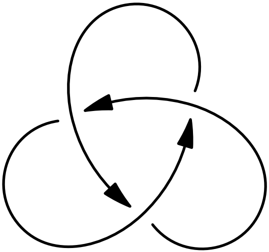

Figure: Oriented knot diagram of the trefoil knot

Definition (Equivalence)

We call two knot diagrams *equivalent* if one deform one into the other without breaking any part of the knot.

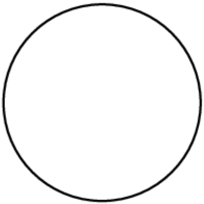

Figure: The unknot

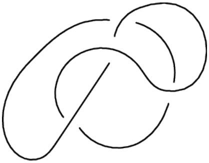

Figure: The unknot, but twisted

Background: Timeline of Knot Tabulation

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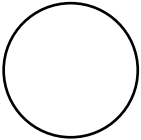
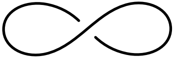
$f\Big($  $\Big) = f\Big($  $\Big)$

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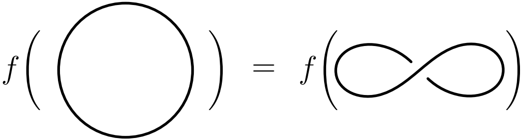

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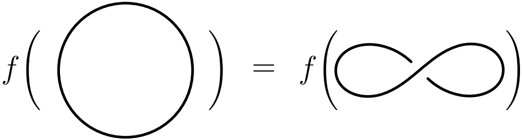

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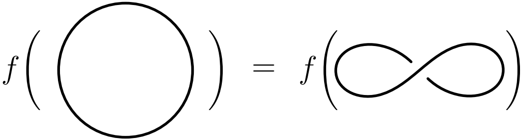

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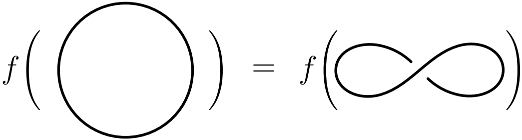

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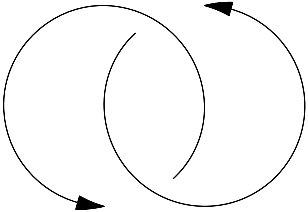

Figure: Projection of the Hopf Link

Kauffman Bracket

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The Kauffman Bracket polynomial is *not* an invariant, *but* it's only ever off by a power of $(-A)^3$, and we can correct this with something called the *writhe*.

Writhe

Definition (Sign of Crossing)

The *sign* of a crossing is defined as shown below:

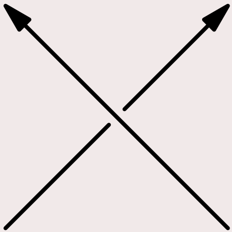
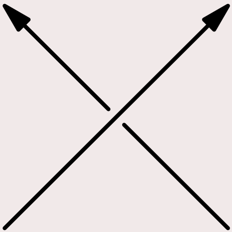


Figure: A negative (-1) crossing



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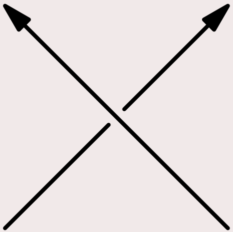

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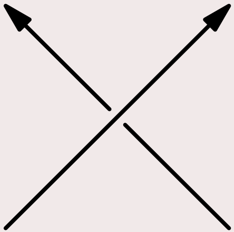

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Definition (Writhe)

The *writhe* of a link L, or $w(L)$, is the sum of the signs of all of the crossings of L.

Jones Polynomial: Definition

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The *normalized bracket polynomial* of an oriented link L is given by $X(L) = (-A^3)^{-w(L)}\langle L \rangle$. The *Jones Polynomial* of L, denoted by $V(L)$, is obtained by setting $A = t^{-1/4}$.

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- One of the most widely recognized knot invariants
- With another invariant, used to tabulate knots up to 20 crossings

Jones Polynomial: Goals

- Naive implementation for a knot with n crossings is $O(n2^n)$
- Computation can take hours or days as knots grow in size
- Faster/sub-exponential algorithms will aid in tabulating 21-crossing prime knots.

Tangles: Definitions

Definition (Tangle Diagram)

A *tangle diagram* is a section of a knot/link diagram embedded in a disk.

Example

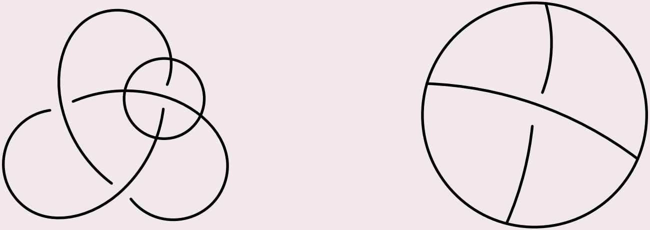

Figure: A tangle with a crossing from the trefoil

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Let T be a tangle. The *type* of T consists of the disc, along with the points on the boundary of T. A tangle is a *base tangle* if it contains no crossings or loops.

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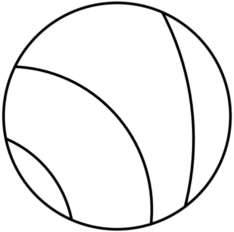

Figure: A base tangle with 6 boundary points

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- For tangle on previous slide, $\langle$  $\rangle = A\langle$  $\rangle + A^{-1}\langle$  $\rangle$
- If a tangle has many crossings, we can first find the KBP of smaller tangles within, and glue the base tangles back in

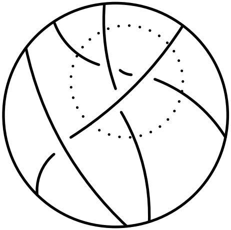

Figure: A smaller tangle with 3 crossings contained in a larger one

Tangles: Cutting

Let L be a link with n crossings. A *cutting* of L is a sequence of $n+1$ tangles T_0, T_1, ..., T_n such that:

- T_0 is empty
- T_n is L embedded in a disk
- For $0 \leq i < n$, the graph $T_{i+1} - T_i$ consists only of a crossing and the edges connecting the boundary of T_i to T_{i+1}

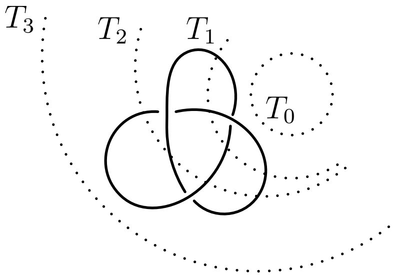

Figure: A cutting of the trefoil knot

Tangles: Algorithm

Algorithm (Ellenberg et al., 2013)

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  - Obtain the KBP of T_{i+1} in terms of its base tangles
- Compute the Jones polynomial of L using the *KBP* of T_n and the writhe of L

Tangles: Performance

Theorem (Djidjev and Vrťo, 2003)

Given a cutting of a knot with n crossings, the number of boundary vertices in any tangle of the cutting is bounded above by $c\sqrt{n}$, where $c = 6\sqrt{2} + 5\sqrt{3}$.

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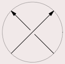
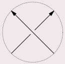
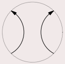
Theorem (Ellenberg, Shieh, et al., 2025)

The time complexity for the tangle algorithm for a knot with n crossings is $O(n^{5/4}2^{c\sqrt{n}})$.

Definition (Jones)

The Jones polynomial is the unique function satisfying

- $t^{-1}V$  $(t) - tV$  $(t) = (\sqrt{t} - \frac{1}{\sqrt{t}})V$  (t)

A Different Recursion for Jones

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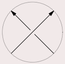
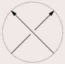
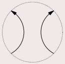
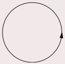
- $t^{-1}V$  $(t) - tV$  $(t) = (\sqrt{t} - \frac{1}{\sqrt{t}})V$  (t)
- V  $(t) = 1$
- Notice that there are two terms with crossings in the above skein relation, so we can't induct on crossing number as with the original skein relation.

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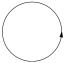
- V  $_{L}(t) = -(\sqrt{t} + \frac{1}{\sqrt{t}})V_L(t).$
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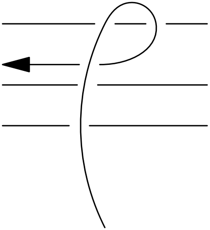

Figure: Part of an Unknot

Unknots

- We can decorate each overcrossing we encounter as we progress along each link component (and not skein decorated crossings) so that eventually we get all unknots. We can induct on the number of undecorated crossings instead of the number of crossings.

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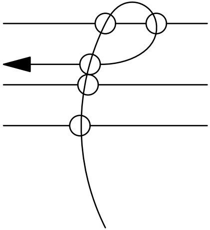

Figure: Decorated Part of Unknot

Skein Template Algorithm

Algorithm (Gouesbet, Meunier-Guttin-Cluzel, and Letellier, 1999)

Input: A link L, a starting crossing, and a starting direction.
Output: The Jones polynomial of L
Repeat the following procedure until I have only unlinks:

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- Proceed to the next crossing in each case.

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Since each recursion splits the current link into two links with one fewer undecorated crossing and each step of the recursion takes polynomial time to compute, the worst case time complexity of the algorithm is $O^{\sim}(2^n)$.

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Load Balanced Algorithm

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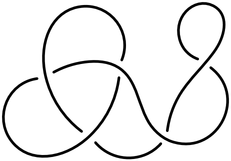

Figure: Trefoil with some twists

Load Balanced Algorithm

Definition (Kauffman Bracket Skein Relation)

$$\langle \quad \rangle = A\langle \quad \rangle + A^{-1}\langle \quad \rangle$$

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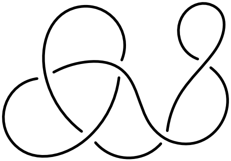

Figure: Trefoil with some twists

- Selecting crossings that yield immediate simplifications after applying a Skein relation

Load Balanced Algorithm

We can look for simple patterns with crossings that can be smoothed to remove *more* than one crossing at once:

Example

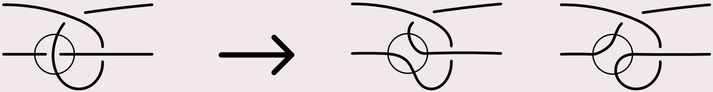

Figure: Smoothing a crossing

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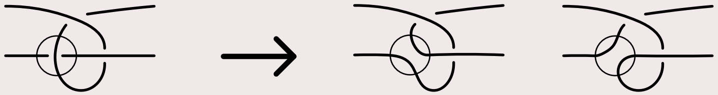

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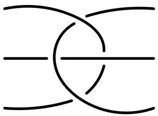

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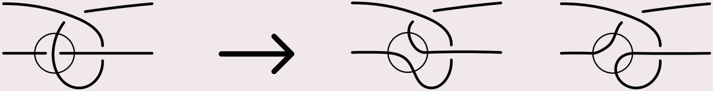

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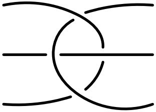

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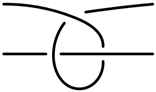

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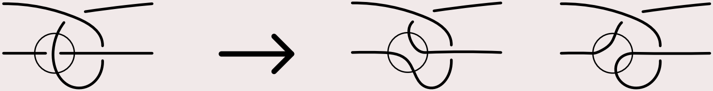

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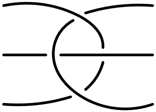

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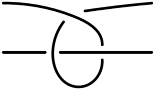

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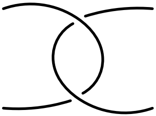

Figure: Bigon

Load Balanced Algorithm

Definition (Complexity Type)

Suppose that smoothing a crossing in a knot with n crossings and performing subsequent simplifications results in two knot diagrams with $n - a$ and $n - b$ crossings. Then, this smoothing has complexity type (a, b).

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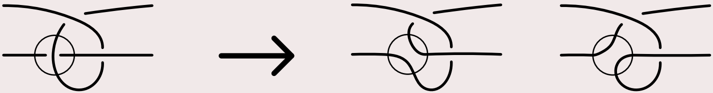

Figure: Smoothing a crossing in a gamma

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Estimate the time complexity of the algorithm as $\Theta(\alpha^n)$ for a knot diagram with n crossings and a hard-coded constant α

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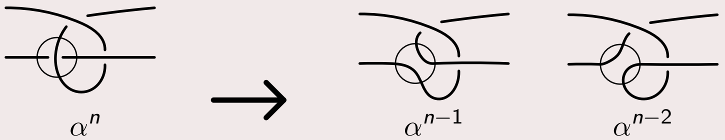

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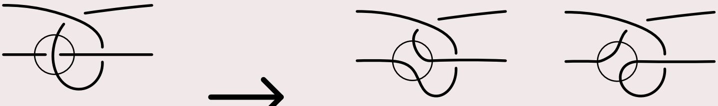

α^n α^{n-1} α^{n-2}

Example (Comparing Complexity Types)

For $\alpha = 1.75$, we estimate a smoothing with complexity type $(2, 2)$ to be better than a smoothing with complexity type $(1, 4)$ because

$$\alpha^{-2} + \alpha^{-2} \approx 0.65 < 0.68 \approx \alpha^{-1} + \alpha^{-4}.$$

Load Balanced Algorithm

Algorithm (Ewing and Millet, 1991)

Input: A link L

Output: The KBP of L

- If L is the unknot, return 1.
- If L has a component with no crossings, get rid of this component and multiply the result by $-A^2 - A^{-2}$.

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- Evaluate the Skein relation and recurse

Benchmarking

Preliminary Results

Algorithm	Time (seconds)
Naive	33.819
Circle Counting	13.394
Skein Template Algorithm	1.611
Symbolic Calculus	0.121
Tangle Algorithm	0.052

Figure: Benchmarks on a random sample of 24-crossing knots

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- Implement and benchmark two additional algorithms
- Tabulate the Jones polynomial for all prime knots up to 20 crossings

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