Fast Spectral Approximation of Multi-Valued Functions

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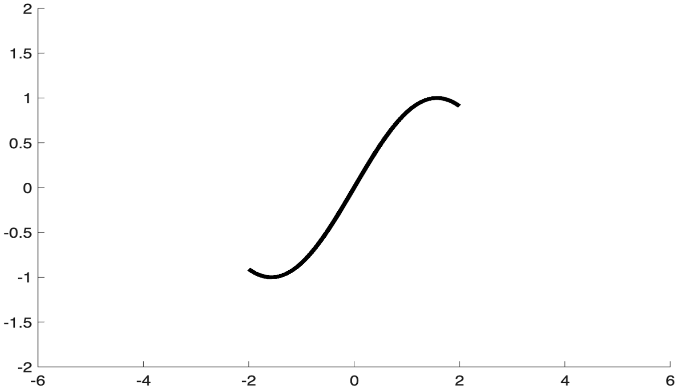
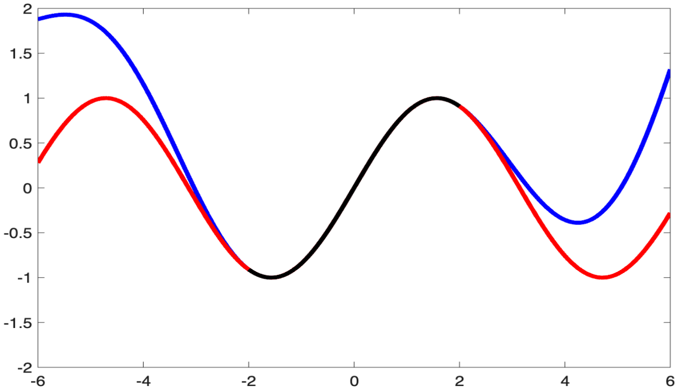
Real Differentiable Functions



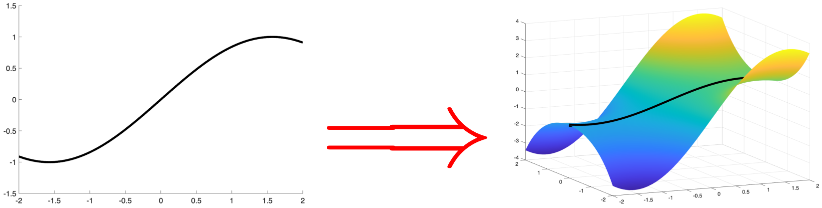
Real Differentiable Functions



Analytic and Harmonic Functions

Complex differentiability is a much more rigid requirement than real differentiability:

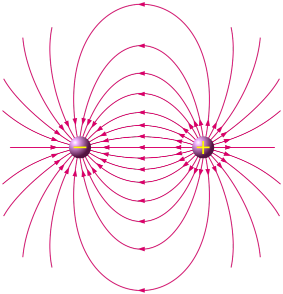
$$f'(z) = \lim_{w \to 0} \frac{f(z+w) - f(z)}{w}, \qquad \textit{from any direction!}$$

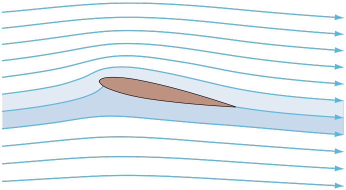


Complex differentiable functions are instrumental in forming 'harmonic' functions, which satisfy Laplace's equation,

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

Harmonic functions are ubiquitous in math and physics:





(source: Wikimedia)

The AAA Algorithm

Goal: Approximate analytic functions, with **unknown** pole locations, with **rational** functions.

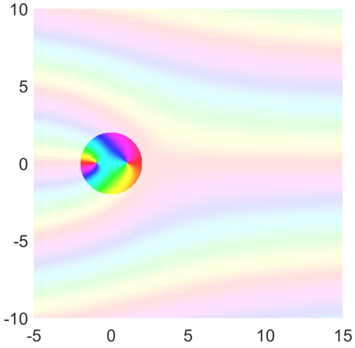
Definition (Barycentric Form)

Rational functions can be expressed as

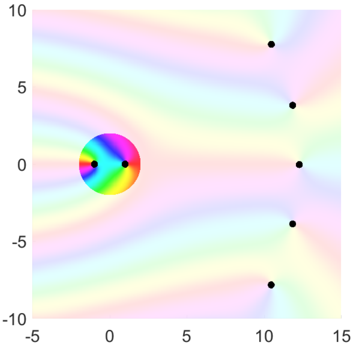
$$r(z) = \frac{\sum \frac{w_j f_j}{z - z_j}}{\sum \frac{f_j}{z - z_j}},$$

for weights w_j, support points z_j, and data f_j.

The AAA Algorithm



(a) Phase portrait of $f(z) = \frac{e^z}{z^2-1}$

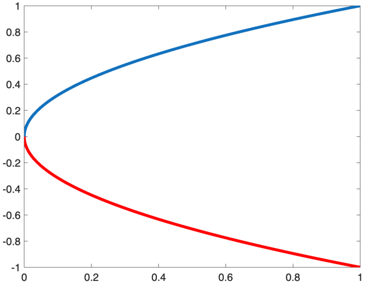


(b) Phase portait of AAA approximant

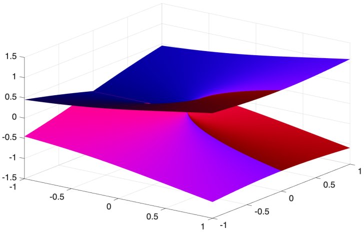
Multi-Valued Function

Recall analytic functions extend uniquely to the entire complex plane from a single curve.

Unfortunately, many functions extend to 'multi-valued' functions:



(a) Graph of $f(z) = \sqrt{z}$



(b) Surface of $f(z) = \text{real}\,\sqrt{z}$

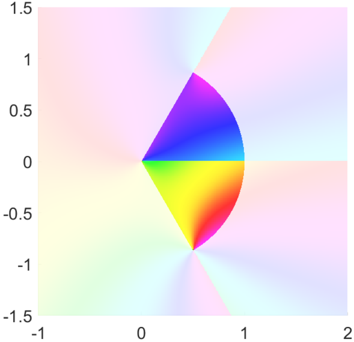
Log-Lightning

Goal: Approximate multi-valued functions, with **known** branch structure, with **rational-log** functions.

Definition (Rational-log function)

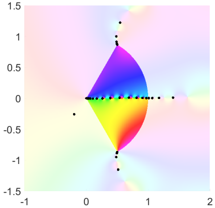
$$r(z) = p\left(\frac{1}{\log(z) - s_0}\right),$$

for a polynomial p and (semi-arbitrary) support point s_0.

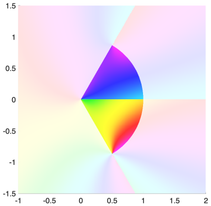


$$f(z) = z\,(1 - z/\omega)^{\frac{1}{2}}\,(1 - z/\bar{\omega})^{\frac{3}{2}}\,\log\left(-\tfrac{1}{2}z\right),$$

$$\omega = \exp\left(i\pi/3\right)$$



(a) AAA



(b) Log-Lightning

Joint Algorithm

Overview of joint algorithm:

- Use AAA to cluster poles around a branch point.
- Use a best fit line to approximate the branch cut location.
- Iteratively shrink the window size to improve accuracy.
- Use edge detection to locate multiple branch points at once.
- Finally, approximate the function using log-lightning.

Branch Point Location

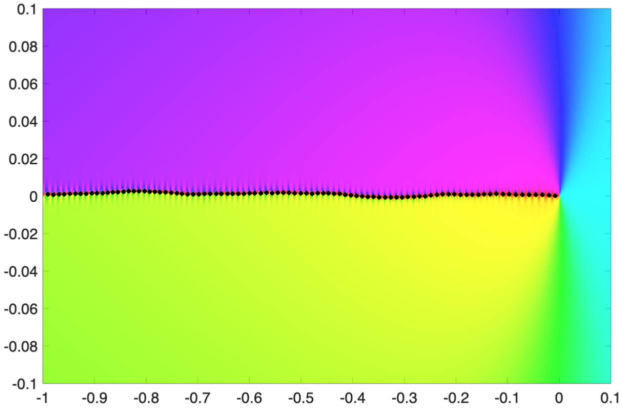


Figure: Poles generated by AAA when approximating $f(z) = z\log(z)$

Multiple Branch Points

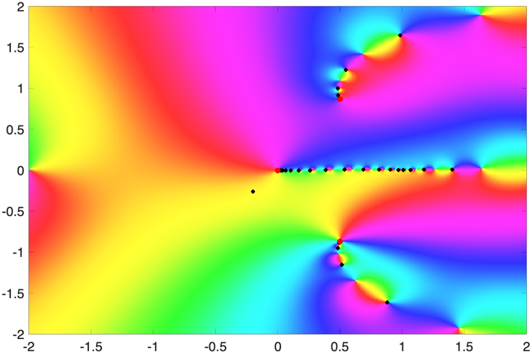


Figure: Poles generated by AAA with
$f(z) = z\left(1 - z/\omega\right)^{\frac{1}{2}}\left(1 - z/\bar{\omega}\right)^{\frac{3}{2}} \log\left(-\frac{1}{2}z\right)$, where $\omega = \exp\left(\frac{\pi i}{3}\right)$

Edge Detection

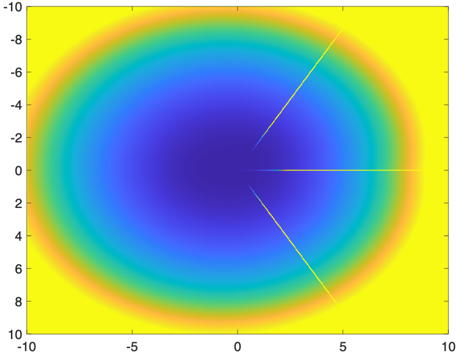
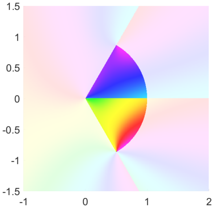
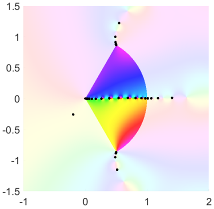


Figure: Edge detection algorithm for $f(z) = z\left(1 - z/\omega\right)^{\frac{1}{2}}\left(1 - z/\bar{\omega}\right)^{\frac{3}{2}}\log\left(-\frac{1}{2}z\right)$, where $\omega = \exp\left(\frac{\pi i}{3}\right)$

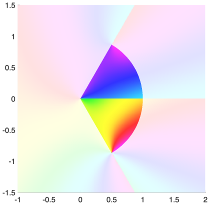
(a) Original function



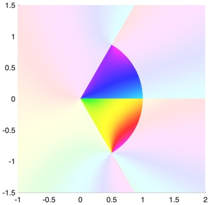
(b) AAA



(c) Log-lightning



(d) Joint



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