Data-Driven Latent State Estimation and Dynamical System Identification

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Los Altos High School

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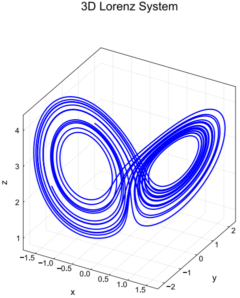
Dynamical Systems

Dynamical System: A mathematical model that describes how a system changes over time (dynamics).

Two Components:

1. **State:** A set of variables that describes the system at a given time
2. **Rule:** A function that describes how the state variables evolve over time



Dynamical Systems

Example: Lotka-Volterra (predator-prey) equations.

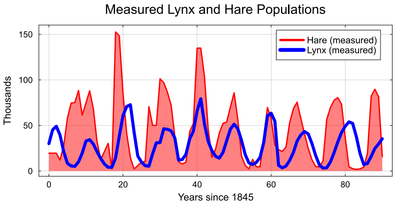
1. **State:** $x =$ prey population, $y =$ predator population
2. **Rule:**

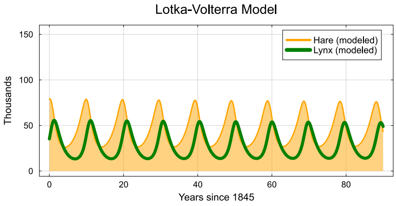
$$\dot{x} = \alpha x - \beta xy$$
$$\dot{y} = -\gamma y + \delta xy$$

Independently discovered by Lotka and Volterra using **physical intuition**.

Dynamical Systems

Example: Snowshoe hare predation by lynxes (Hudson Bay time series).

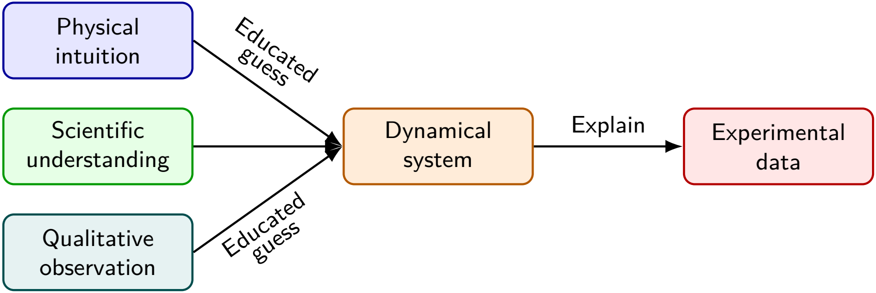




Odum, E. P. (1953). *Fundamentals of Ecology*. W.B. Saunders.

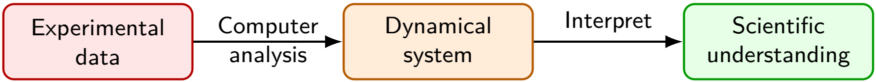
System Identification

Classical Approach:



System Identification Problem

System Identification Approach:



System Identification Problem:

Given measurements $\mathbf{x}(t_i) \in \mathbb{R}^n$ collected at k discrete time points $t_1, t_2, \ldots, t_k$, determine the function $\mathbf{f}\colon \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)).$$

SINDy Algorithm

Powerful approach: **Sparse Identification of Nonlinear Dynamics (SINDy)**

- **Library-based**
- Solves for sparse coefficient matrix $\mathbf{\Xi} \in \mathbb{R}^{p \times n}$ in

$$\dot{\mathbf{X}} \approx \mathbf{\Theta}(\mathbf{X})\mathbf{\Xi},$$

where $\mathbf{\Theta}(\mathbf{X}) \in \mathbb{R}^{k \times p}$ is a library matrix.

$$\text{Ex: } \mathbf{\Theta}(\mathbf{X}) := \begin{bmatrix} 1 & \mathbf{X} & \mathbf{X}^2 & \cdots & \sin(\mathbf{X}) & \cos(\mathbf{X}) & \cdots & \exp(\mathbf{X}) \end{bmatrix}$$

Example: Lotka-Volterra Equations

$$\dot{\mathbf{X}} = \mathbf{\Theta}(\mathbf{X})\mathbf{\Xi}$$

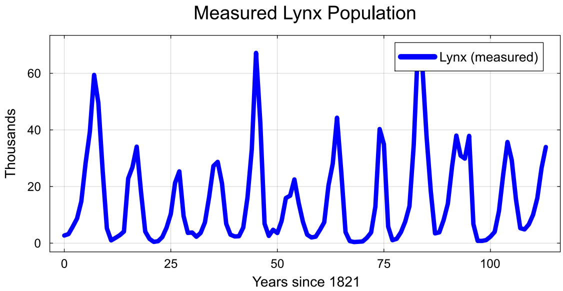
$$\implies \begin{bmatrix} \dot{x}(t_1) & \dot{y}(t_1) \\ \dot{x}(t_2) & \dot{y}(t_2) \\ \vdots & \vdots \\ \dot{x}(t_k) & \dot{y}(t_k) \end{bmatrix} = \begin{bmatrix} x(t_1) & y(t_1) & x(t_1)y(t_1) \\ x(t_2) & y(t_2) & x(t_2)y(t_2) \\ \vdots & \vdots & \vdots \\ x(t_k) & y(t_k) & x(t_k)y(t_k) \end{bmatrix} \begin{bmatrix} \alpha & 0 \\ 0 & -\gamma \\ -\beta & \delta \end{bmatrix}$$

$$\implies \begin{bmatrix} 0.02 & 15.06 \\ 0.01 & 4.00 \\ \vdots & \vdots \\ -65.9 & 5.7 \end{bmatrix} = \begin{bmatrix} 19.58 & 30.09 & 588.38 \\ 19.60 & 45.15 & 884.94 \\ \vdots & \vdots & \vdots \\ 15.76 & 35.40 & 557.90 \end{bmatrix} \begin{bmatrix} \alpha & 0 \\ 0 & -\gamma \\ -\beta & \delta \end{bmatrix}$$

$\implies$ regression for $\alpha, \beta, \gamma, \delta$.

Limitations

What if we can only access the **lynx population?** (Mackenzie River time series)



Brockwell, P. J., & Davis, R. A. (1991). *Time Series: Theory and Methods*. Springer, New York.

Limitations

$$\dot{\mathbf{X}} = \mathbf{\Theta}(\mathbf{X})\mathbf{\Xi}$$

$$\implies \begin{bmatrix} \dot{x}(t_1) & \dot{y}(t_1) \\ \dot{x}(t_2) & \dot{y}(t_2) \\ \vdots & \vdots \\ \dot{x}(t_k) & \dot{y}(t_k) \end{bmatrix} = \begin{bmatrix} x(t_1) & y(t_1) & x(t_1)y(t_1) \\ x(t_2) & y(t_2) & x(t_2)y(t_2) \\ \vdots & \vdots & \vdots \\ x(t_k) & y(t_k) & x(t_k)y(t_k) \end{bmatrix} \begin{bmatrix} \alpha & 0 \\ 0 & -\gamma \\ -\beta & \delta \end{bmatrix}$$

$$\implies \begin{bmatrix} 0.52 & \mathbf{?} \\ 2.64 & \mathbf{?} \\ \vdots & \vdots \\ 7.39 & \mathbf{?} \end{bmatrix} = \begin{bmatrix} 2.69 & \mathbf{?} & \mathbf{?} \\ 3.21 & \mathbf{?} & \mathbf{?} \\ \vdots & \vdots & \vdots \\ 33.96 & \mathbf{?} & \mathbf{?} \end{bmatrix} \begin{bmatrix} \alpha & 0 \\ 0 & -\gamma \\ -\beta & \delta \end{bmatrix}$$

$$\implies \mathbf{???}$$

Partial Observation

Partially-Observed System Identification Problem:

Given measurements of the observed trajectories $\mathbf{x}_{\text{obs}}(t) \in \mathbb{R}^n$ at k discrete time points $t_1, \ldots, t_k$, determine functions $\mathbf{f}_x, \mathbf{f}_y : \mathbb{R}^{n+m} \to \mathbb{R}^n \times \mathbb{R}^m$ and latent trajectories $\mathbf{y}(t) \in \mathbb{R}^m$ such that

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}_x(\mathbf{x}(t), \mathbf{y}(t)) \\ \dot{\mathbf{y}}(t) = \mathbf{f}_y(\mathbf{x}(t), \mathbf{y}(t)) \end{cases}$$

Our proposed algorithm: **Latent-SINDy (L-SINDy)**

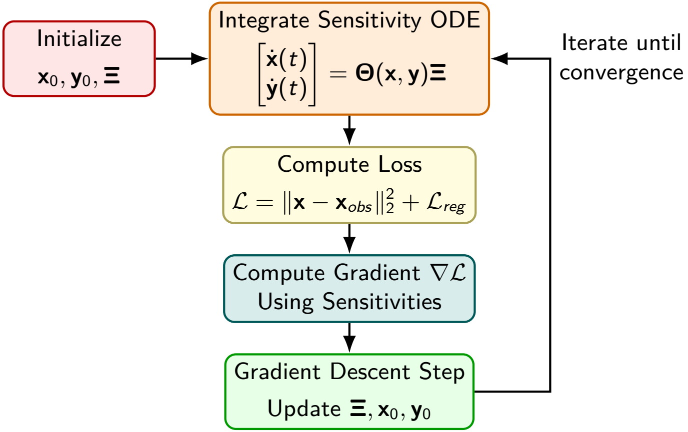
L-SINDy

Goal: Identify sparse coefficient matrix $\boldsymbol{\Xi} \in \mathbb{R}^{p \times (n+m)}$ such that

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\mathbf{y}}(t) \end{bmatrix} = \boldsymbol{\Theta}(\mathbf{x}(t), \mathbf{y}(t))\boldsymbol{\Xi}.$$

Latent-SINDy Algorithm:

Gradient Descent



Prediction Error Method

Prediction Error Method (PEM):

Loss based on simulation $\longrightarrow$ Loss based on short-term prediction

$$\begin{bmatrix}\dot{\mathbf{x}}(t)\\ \dot{\mathbf{y}}(t)\end{bmatrix} = \boldsymbol{\Theta}(\mathbf{x}(t), \mathbf{y}(t))\boldsymbol{\Xi} \quad \longrightarrow \quad \begin{bmatrix}\dot{\mathbf{x}}(t)\\ \dot{\mathbf{y}}(t)\end{bmatrix} = \boldsymbol{\Theta}(\mathbf{x}(t), \mathbf{y}(t))\boldsymbol{\Xi} - \boldsymbol{K}\cdot(\mathbf{x}(t) - \mathbf{x}_{\text{obs}}(t))$$

$\boldsymbol{K} \in \mathbb{R}^{(n+m)\times n}$ is the (linear) observer.

Guarantees stable trajectories and **smooths the loss function**.

Sparsification

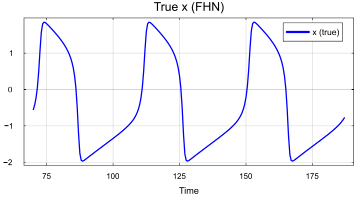
For each parameter p in the system, compute significance:

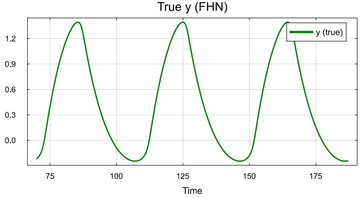
$$s_p = \sqrt{\frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}\left[\frac{\partial x_i(t_j)}{\partial p}\cdot\frac{p}{\sigma_{x_i}}\right]^2}.$$

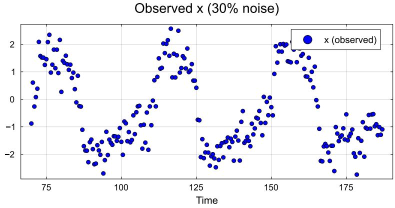
Sequentially delete the least significant parameter $\rightarrow$ recover a sparse solution.

FitzHugh-Nagumo Model

True System:
$$\dot{x} = 0.5 + x - y - 0.33x^3$$
$$\dot{y} = 0.056 + 0.08x - 0.064y$$







FitzHugh-Nagumo Model

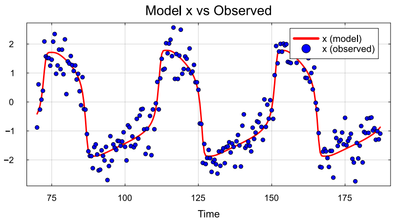
True System:

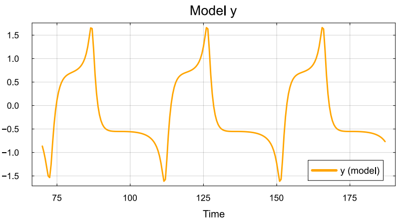
$$\dot{x} = 0.5 + x - y - 0.33x^3$$
$$\dot{y} = 0.056 + 0.08x - 0.064y$$

$$\downarrow$$

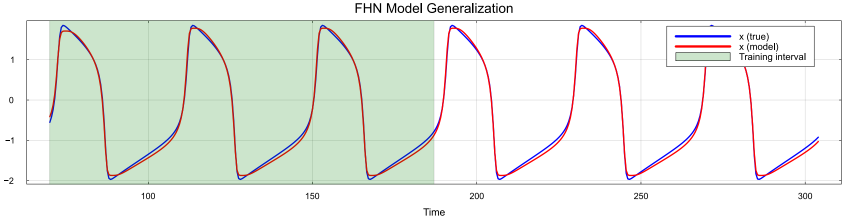
L-SINDy System:

$$\dot{x} = -0.2907y^3$$
$$\dot{y} = 0.3012y + 0.01839x^2 - 0.3137x^2y + 0.0792x^3$$





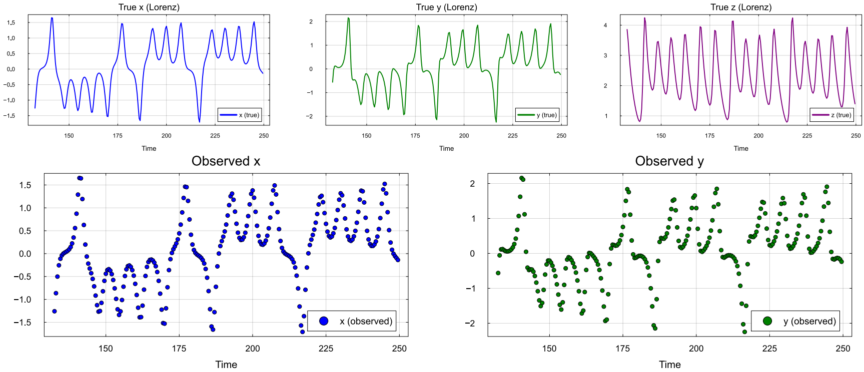
FitzHugh-Nagumo Model



Lorenz System

True System:

$$\dot{x} = y - x$$
$$\dot{y} = 2.8x - 0.1xz - 0.1y$$
$$\dot{z} = 0.1xy - 0.267z$$



Lorenz System

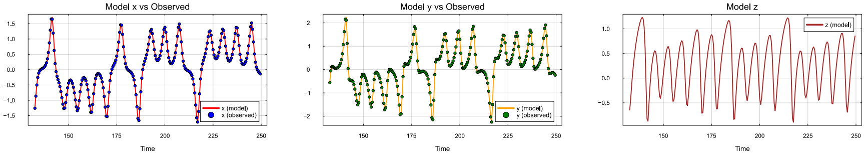
True System:

$$\dot{x} = y - x$$
$$\dot{y} = 2.8x - 0.1xz - 0.1y$$
$$\dot{z} = 0.1xy - 0.267z$$

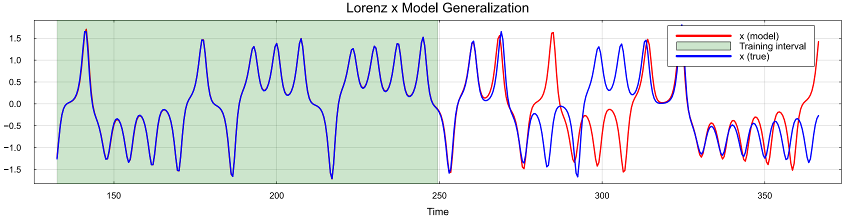
$$\downarrow$$

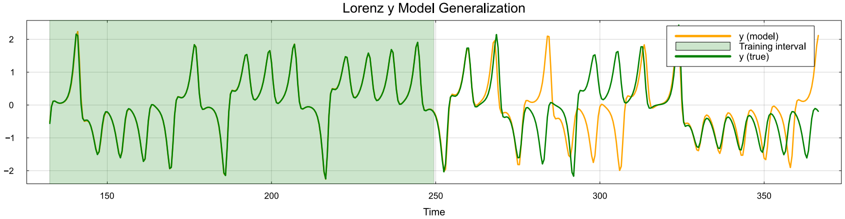
L-SINDy System:

$$\dot{x} = 0.9888y - 0.9876x$$
$$\dot{y} = -0.09137y + 1.613xz$$
$$\dot{z} = 0.4558 - 0.2641z - 0.6051xy$$



Lorenz System

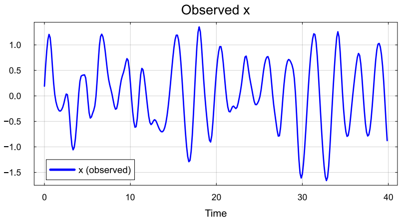


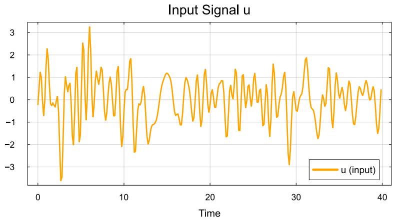


Bouc-Wen Hysteresis

True System: $\ddot{x} = 2.5u - 0.05z - 2.5x - 0.05\dot{x}$

$$\dot{z} = 50\dot{x} - 0.8|\dot{x}|z + 1.1\dot{x}|z|$$



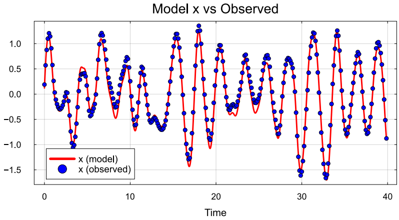


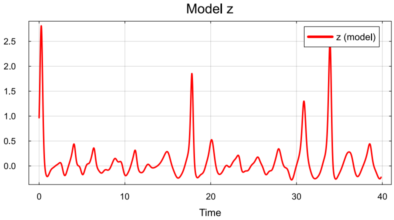
Schoukens, M., & Noël, J. P. (2017).

Three Benchmarks Addressing Open Challenges in Nonlinear System Identification.

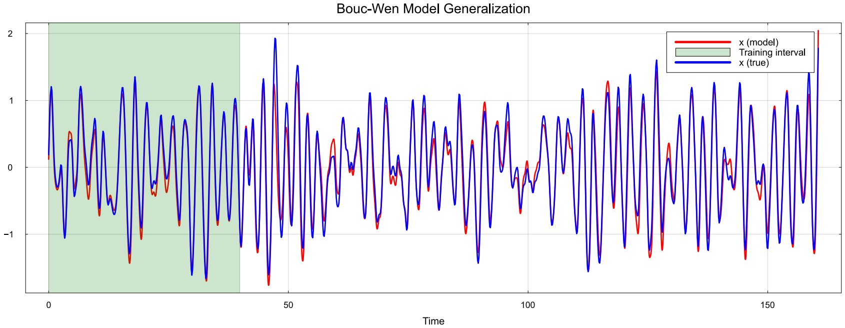
Bouc-Wen Hysteresis

$$\textbf{True System:}\quad \ddot{x} = 2.5u - 0.05z - 2.5x - 0.05\dot{x}$$
$$\dot{z} = 50\dot{x} - 0.8|\dot{x}|z + 1.1\dot{x}|z|$$
$$\downarrow$$
$$\textbf{L-SINDy System:}\quad \ddot{x} = 2.354u - 2.766z - 5.578x$$
$$\dot{z} = -9.328z + 0.987\dot{x} + 4.347\dot{x}z - 0.325\dot{x}^2$$

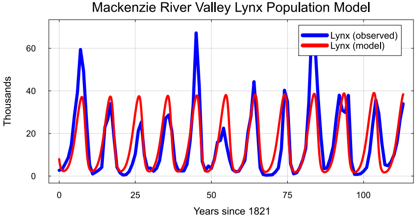


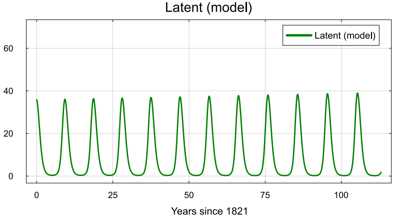


Bouc-Wen Hysteresis



Conclusion





Advantages of the L-SINDy Algorithm:

- Latent state estimation step → **significantly faster training**
- Prediction Error Method → **strong robustness to noise**
- Novel sensitivity-based sparsification → **accurate, parsimonious models**

Acknowledgments

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- The PRIMES-USA staff for providing this amazing opportunity for high school students to conduct original research through the PRIMES program
- My friends and family for their encouragement and support throughout this journey

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End of Presentation

THANK YOU!