On the Periodic Orbits of Refractive Billiard Systems

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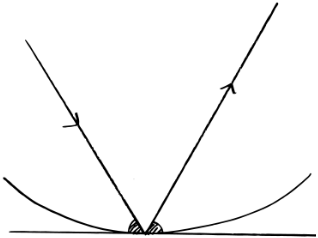
Billiard Trajectories

Mathematical billiards is a dynamical system defined by the following rule: the angle of incidence is equal to the angle of reflection.
Here's a deceptively simple **open** problem:

Problem

Does every triangle have a periodic billiard orbit?

- Every *acute* and *right* triangle has one.
- Every obtuse triangle with obtuse angle < 112.3 degrees has one. This is the current best bound (2018).

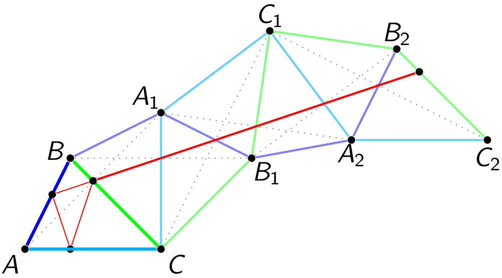


Fagnano's Problem and Solution

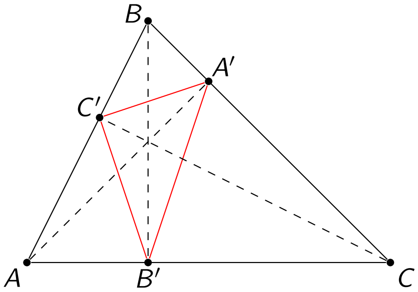
Problem (Fagnano 1775)

Given an acute triangle, what is the inscribed triangle of minimal perimeter?

The **unwrapping argument**:



Fagnano's Problem

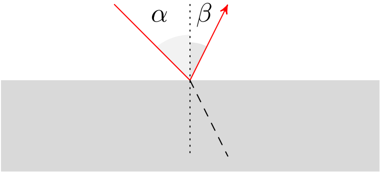


For an acute triangle, the *orthic triangle* is the unique minimizer. It is also a billiard orbit: $\angle C'B'B = \angle A'B'B$, etc. In fact, it is the *only* 3-periodic billiard orbit.

The minimizer of the perimeter is a periodic billiard orbit.

Refractive Billiards

Refractive (Snell) billiards generalize this connection.



Each point on the boundary is assigned a **refraction coefficient** κ such that $\frac{\sin\beta}{\sin\alpha} = \kappa$. Note that we get regular billiards if $\kappa = 1$ for all points.

Periodic billiard orbit	:	Perimeter minimizer
Periodic *refractive* billiard orbit	:	?

Refractive Billiards

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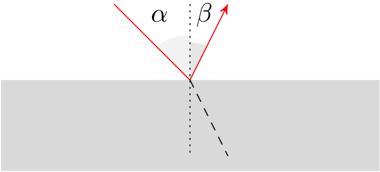

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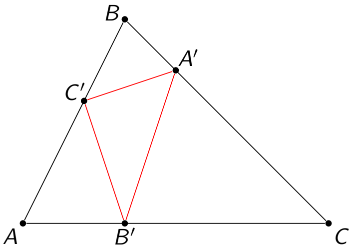
Periodic billiard orbit	:	Perimeter minimizer
periodic *refractive* billiard orbit	:	*Weighted* perimeter minimizer

The Generalized Fagnano Problem



Given $\triangle ABC$ and weights $\lambda_a, \lambda_b, \lambda_c > 0$, which inscribed $\triangle A'B'C'$ minimizes $\lambda_a \cdot |B'C'| + \lambda_b \cdot |C'A'| + \lambda_c \cdot |A'B'|$?

Theorem

The following are equivalent:

1. *The gradient of the weighted perimeter vanishes at* $\triangle A'B'C'$.
2. $\triangle A'B'C'$ *is a counterclockwise refractive billiard orbit with* $\kappa(BC) = \lambda_b/\lambda_c$, $\kappa(CA) = \lambda_c/\lambda_a$, $\kappa(AB) = \lambda_a/\lambda_b$.

Sketch of the Argument

Theorem

The two sets below are in bijective correspondence:

1. *Points P in the interior of $\triangle ABC$ satisfying* $|PA| : |PB| : |PC| = \lambda_a : \lambda_b : \lambda_c$,
2. *The set of 3-periodic CCW refractive billiard orbits.*

The bijection sends $P \mapsto$ the pedal triangle of the isogonal conjugate of P.

Example (The Classic Fagnano problem)

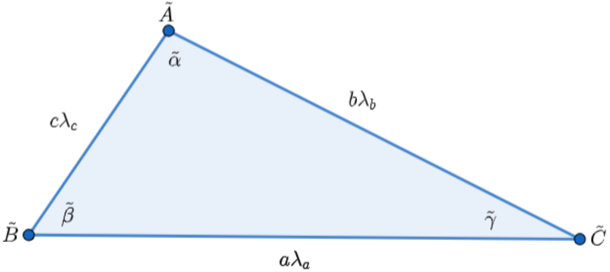
The circumcenter is the unique point with $PA = PB = PC$. Its iso. conj. is the orthocenter, which has the orthic triangle as its pedal triangle.

Sketch of the Argument

It suffices to find the points P in the interior of $\triangle ABC$ satisfying $|PA| : |PB| : |PC| = \lambda_a : \lambda_b : \lambda_c$.

Lemma (Existence)

Let $\triangle ABC$ have side lengths (a, b, c). There exists a point P satisfying $|PA| : |PB| : |PC| = \lambda_a : \lambda_b : \lambda_c$ **somewhere in the plane** *iff the triangle with side lengths $(a\lambda_a, b\lambda_b, c\lambda c)$ exists (possibly degenerate).*



Results

Theorem (Interior & Uniqueness)

A 3-periodic CCW refractive billiard orbit exists iff the triangle with side lengths $(a\lambda_A, b\lambda_B, c\lambda_C)$ *exists and each of the following hold:*

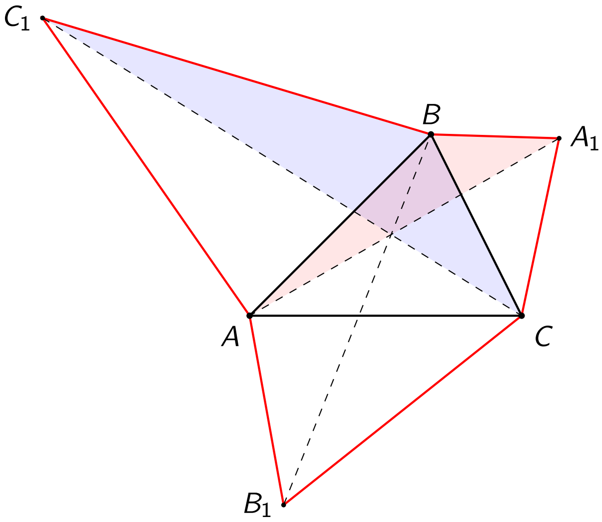
$$\begin{cases} \alpha + \tilde{\alpha} \leqslant \pi, \\ \beta + \tilde{\beta} \leqslant \pi, \\ \gamma + \tilde{\gamma} \leqslant \pi. \end{cases}$$

Moreover, if a 3-periodic orbit exists, it is unique and is the unique minimizer of the weighted perimeter.

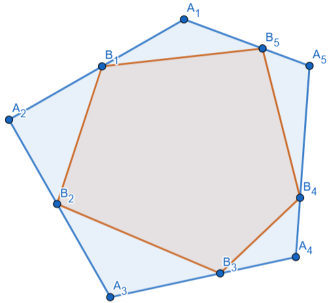
Theorem (Boundary)

If the condition above fails, then P does **not** *exist inside the triangle, and the the minimizer is the shortest "double covering" of an altitude.*

Interior & Uniqueness



What about $n \geq 4$?



Given a convex n-gon $A_1 A_2 \dots A_n$, indices mod n and *weights* $\lambda_1, \lambda_2, \dots \lambda_n > 0$, which $B_1 B_2 \dots B_n$ minimizes the *weighted perimeter*

$$\sum_{i=1}^{n} \lambda_i \cdot |B_{i-1} B_i|?$$

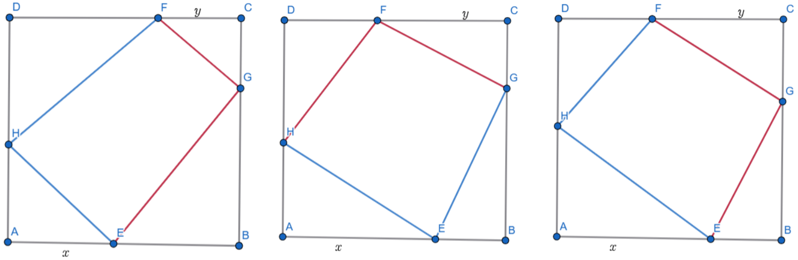
$n = 3$ had a nice geometric solution, but it's harder for $n \geq 4$.

$n \geq 4$ is tough.

Idea

Create a discrete dynamical system that has the weighted perimeter as an *energy function*, that decreases with each step.

An example is the unit square for $n = 4$. We can define $T : [0,1] \times [0,1] \to [0,1] \times [0,1]$ by alternating pairs of sides:

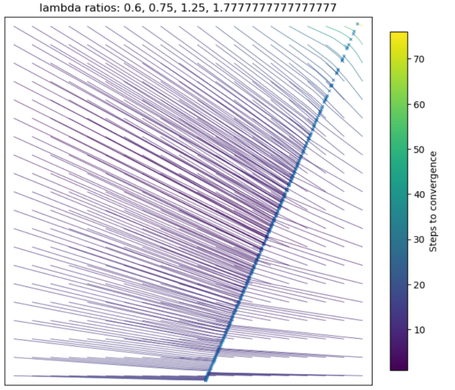


Then, the weighted perimeter strictly decreases. A quadrilateral is a refractive billiard orbit if and only if it is a fixed point of this map.

Why is it tough?

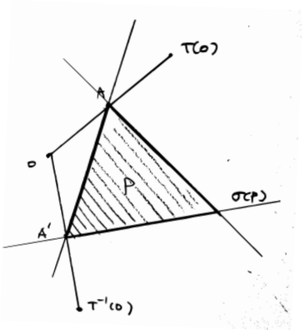
Using this, we can run numerical experiments. Even the local stability of this system is very sensitive to the weights.

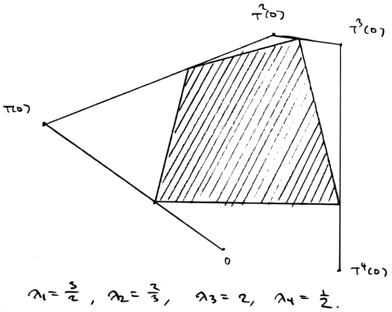
Example (Orbits are **not** unique.)



Polygonal Refractive Outer Billiards

Let P be a convex polygon. For a point o, we call the leftmost vertex of P from o's perspective the *head*. $T(o)$ is the reflection of o about its head.





For the **refractive** version, we have *refractive indices* $\lambda_1, \lambda_2, \ldots, \lambda_n > 0$ that multiply to 1. $T^k(o)$ is the reflection of $T^{k-1}(o)$ about its head, followed by a scaling by λ_k about its head.

Gutkin & Simyani (1992)

Theorem (Gutkin and Simanyi 1992)

If P is **quasi-rational** *then the orbits of T are bounded. If P is* **rational** *the orbits of T are periodic.*

Definition (Rational)

A polygon P is called *rational* if the vertices of P belong to a lattice.

Definition (Quasi-rational)

Take $Q = A_1 \ldots A_{2m+1}$ to be a **necklace polygon** of P. Then there exist $2m$ positive real numbers r_i satisfying $1 \leq i \leq 2m$ such that

$$\overrightarrow{A_i A_{i+1}} = r_i \vec{a}_i,$$

with $r_{m+i} = r_i$. A polygon P is called *quasi-rational* if $r_1, \ldots, r_m$ are rational up to a common factor, i.e., $(r_1 : r_2 : \cdots : r_m) \in \mathbb{QP}^{m-1}$.

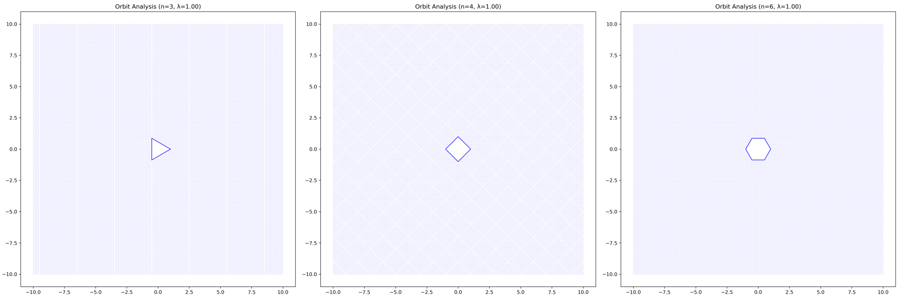
Examples

Example

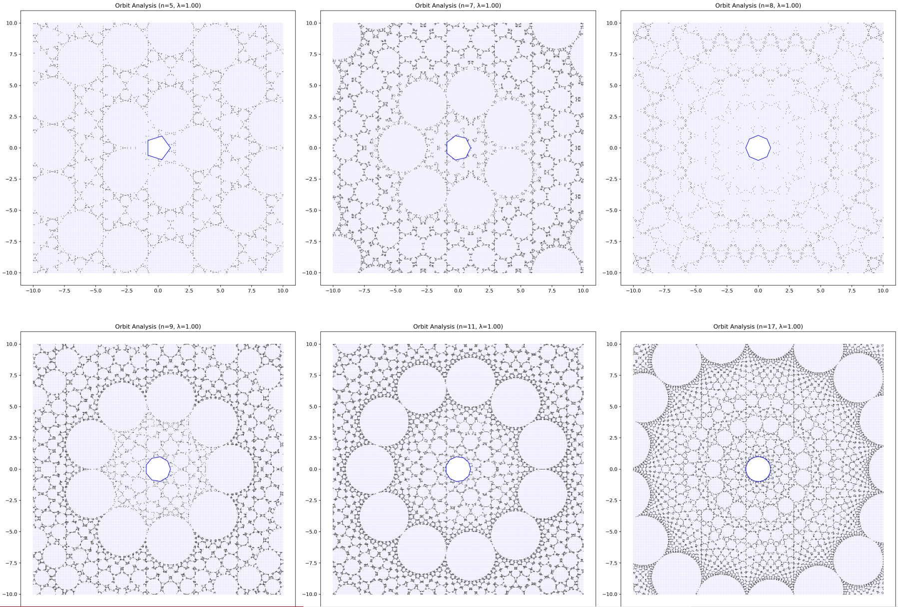
A regular n-gon is:

1. **Rational** when $n = 3, 4, 6$
2. **Quasi-rational** for all n.

Blue if it returns before $\approx$ 20k iterations; black otherwise. $n = 3, 4, 6$ is all blue!



Examples



Orbit Analysis (n=5, λ=1.00)
Orbit Analysis (n=7, λ=1.00)
Orbit Analysis (n=8, λ=1.00)
Orbit Analysis (n=9, λ=1.00)
Orbit Analysis (n=11, λ=1.00)
Orbit Analysis (n=17, λ=1.00)
10.0
7.5
5.0
2.5
0.0
−2.5
−5.0
−7.5
−10.0

Results

Here's our main result regarding refractive outer billiards:

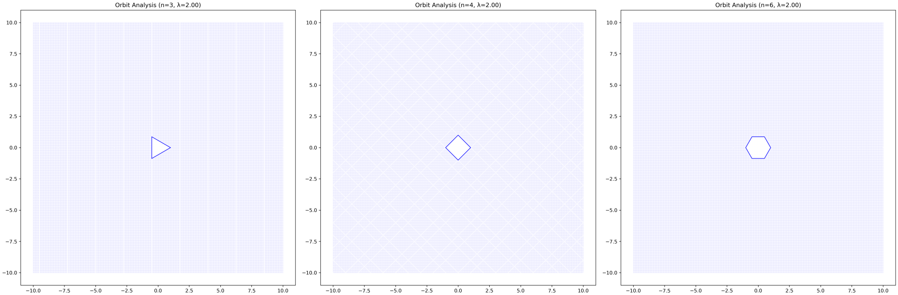
Theorem (Park 2025)

Let $\hat{T}$ be the refractive outer billiard map about a convex polygon P. If P is quasi-rational, then the orbits of $\hat{T}$ are bounded. If P is rational and **each of $\lambda_1 \ldots \lambda_n$ are rational**, *then the orbits of the refractive dual billiards map $\hat{T}$ are periodic.*

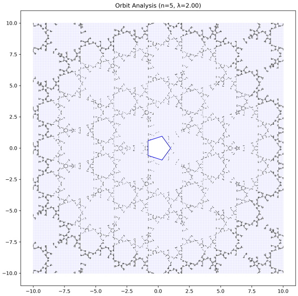
Quasi-rational	$\Longrightarrow$	Bounded
Rational + each λ rational	$\Longrightarrow$	periodic

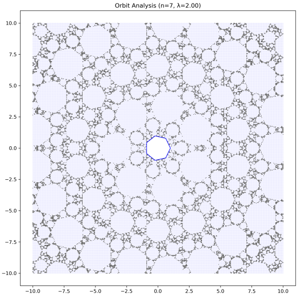
Examples

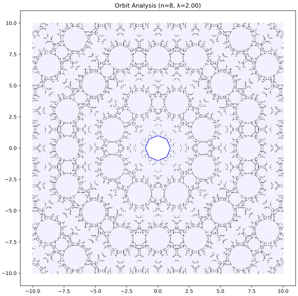
For simplicity, we consider the orbit with $\lambda_1 = 2, \lambda_2 = 1/2$.

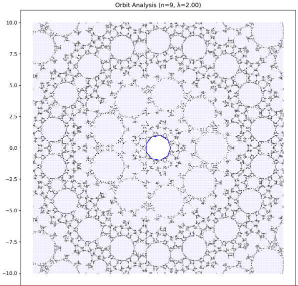


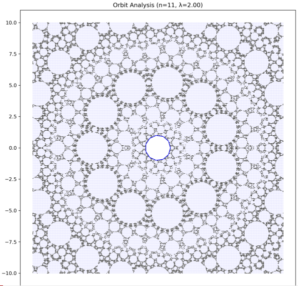
Examples

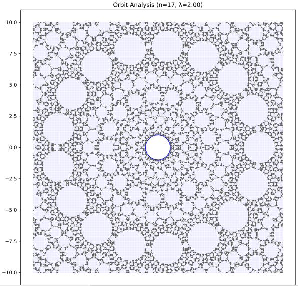

Orbit Analysis (n=5, λ=2.00)
10.0 7.5 5.0 2.5 0.0 −2.5 −5.0 −7.5 −10.0
−10.0 −7.5 −5.0 −2.5 0.0 2.5 5.0 7.5 10.0


Orbit Analysis (n=7, λ=2.00)
10.0 7.5 5.0 2.5 0.0 −2.5 −5.0 −7.5 −10.0
−10.0 −7.5 −5.0 −2.5 0.0 2.5 5.0 7.5 10.0


Orbit Analysis (n=8, λ=2.00)
10.0 7.5 5.0 2.5 0.0 −2.5 −5.0 −7.5 −10.0
−10.0 −7.5 −5.0 −2.5 0.0 2.5 5.0 7.5 10.0


Orbit Analysis (n=9, λ=2.00)
10.0 7.5 5.0 2.5 0.0 −2.5 −5.0 −7.5 −10.0


Orbit Analysis (n=11, λ=2.00)
10.0 7.5 5.0 2.5 0.0 −2.5 −5.0 −7.5 −10.0


Orbit Analysis (n=17, λ=2.00)
10.0 7.5 5.0 2.5 0.0 −2.5 −5.0 −7.5 −10.0

Acknowledgments

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References

[1] Eugene Gutkin and Nandor Simanyi. "Dual polygonal billiards and necklace dynamics". In: *Communications in Mathematical Physics* 143 (1992), pp. 431–449.