Nevanlinna Theory of Random Meromorphic Functions

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Outline

1. Riemann Surfaces
2. Nevanlinna Theory
3. Results

What is a holomorphic function?

Definition

A function $f : \mathbb{C} \to \mathbb{C}$ is holomorphic if its derivative, defined by

$$f'(z) := \lim_{h \to 0} \frac{f(z+h) - f(z)}{h},$$

exists for all z.

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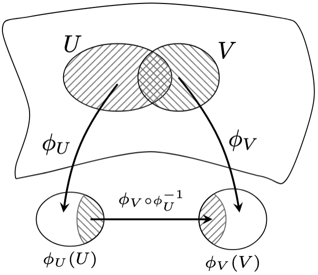
- $f(z) = P(z)$ for polynomials P
- $f(z) = \sin z$
- $f(z) = e^z$

What is a Riemann surface?

Definition

A Riemann surface is a surface that can be covered by coordinate patches such that

- Each coordinate patch U has a coordinate map $\phi_U : U \to \mathbb{C}$
- The transition maps $\phi_V \circ \phi_U^{-1}$ are holomorphic



Examples

The complex plane $\mathbb{C}$ is a Riemann surface.

Examples

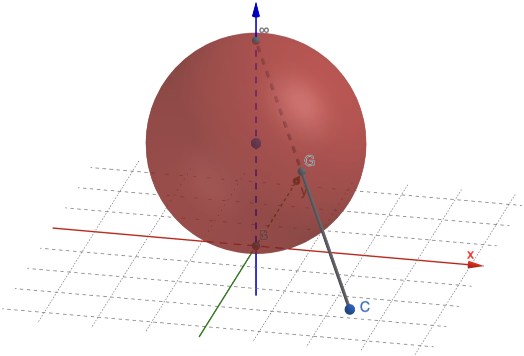
The unit disk

$$\mathbb{D} := \{|z| < 1\}$$

is a Riemann surface.

Examples

We project each point of $\mathbb{C}$ onto a sphere. The top point corresponds to ∞.



The Riemann sphere $\overline{\mathbb{C}}$ is a Riemann surface.

Meromorphic functions

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- $f(z) = \frac{1}{z}$ is meromorphic with a pole at 0.
- The function
$$f(z) = e^{\frac{1}{z}}$$
is not meromorphic.

Main Idea

A meromorphic function wraps a topological plane around a sphere.

Uniformization Theorem

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Any simply connected open Riemann surface is isomorphic to $\mathbb{C}$ or $\mathbb{D}$.

- Simply connected open Riemann surface = looks like a topological plane.

Surfaces spread over the sphere

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- Type problem: is X isomorphic to $\mathbb{C}$ or $\mathbb{D}$?

Nevanlinna characteristic

Let $f : \mathbb{C} \to \overline{\mathbb{C}}$ be meromorphic.

Definition

For $t \geq 0$, the average covering number $A(t)$ is the average number of preimages of a point of $\overline{\mathbb{C}}$ contained in the disk $\{|z| \leq t\}$.

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Definition

The Nevanlinna characteristic of f is defined by

$$T_f(r) = \int_0^r \frac{A(t)}{t} dt.$$

Order of growth

Definition

The order of growth λ of a meromorphic function f is defined by

$$\lambda := \inf\{k > 0 \mid T_f(r) = O(r^k) \text{ as r} \to \infty\}$$

Examples

Example

The function $f(z) = z$ has order of growth 0.

We have $A(t) \leq 1$, so

$$T_f(r) = \int_0^r \frac{A(t)}{t} dt \lesssim \int_1^r \frac{1}{t} dt = \log r.$$

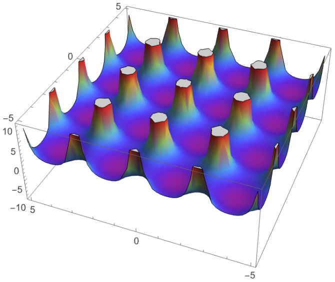
Examples

Example

A doubly periodic meromorphic function $\wp(z)$ satisfying $\wp(z) = \wp(z+1)$ and $\wp(z) = \wp(z+i)$ has order of growth 2.

We have $A(t) \sim t^2$, so

$$T_f(r) = \int_0^r \frac{A(t)}{t} dt \sim \int_0^r \frac{t^2}{t} dt = 2r^2$$



Why do we care?

- Polynomials of high degree have more roots and grow faster than polynomials of low degree.
- Is this true for meromorphic functions?

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Define the root counting function

$$n_f(r,a) = \#\{|z| \leq r : f(z) = a\}$$

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For $f(z) = z$ (order of growth 0),

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Define the root counting function

$$n_f(r, a) = \#\{|z| \leq r : f(z) = a\}$$

For $f(z) = z$ (order of growth 0),

$$n_f(r, a) \leq 1.$$

For $f(z) = \wp(z)$ (order of growth 2),

$$n_f(r, a) \sim r^2.$$

Why do we care?

Theorem (Picard-Borel)

If $T_f(r) = O(r^k)$, then

$$n_f(r, a) = O(r^k).$$

The converse holds for all a with at most two exceptions.

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Theorem (Picard-Borel)

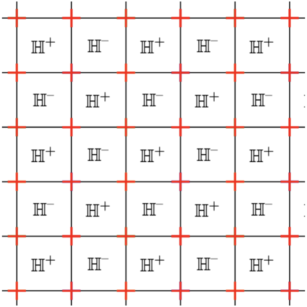
If $T_f(r) = O(r^k)$, then
$$n_f(r, a) = O(r^k).$$
The converse holds for all a with at most two exceptions.

- Picard's Little Theorem: a meromorphic function that omits three values is constant.

Square Grid Surfaces

Definition

A square grid surface is a surface spread over the sphere consisting of a grid of hemispheres of the Riemann sphere glued together at the boundary.



Definition

A random square grid surface is a square grid surface whose vertices are chosen independently and at random.

Results

- Want to study 'typical' behavior for a class of functions.
- Choose a random function and see what holds with high probability.

Theorem (I.)

A random square grid surface

- *is almost surely isomorphic to* $\mathbb{C}$
- *almost surely corresponds to a function with order of growth at least 2 and lower order of growth at most 2.*

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Questions?