Szemerédi–Trotter Theorem Sharp Constructions

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Talk Overview

1. Introduction to incidence geometry and definitions
2. Introduction to the Szemerédi-Trotter theorem
3. Sharp constructions
4. Square grid sharp construction
5. Latest sharp constructions

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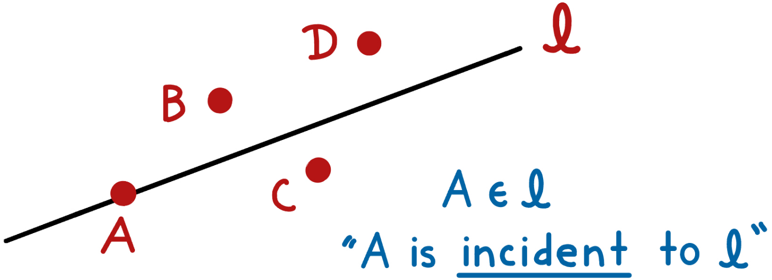
Introduction to Incidence Geometry

- Incidence Geometry focuses on the study of geometric objects (points, lines, and other shapes) and their relationships.
- When a point lies on a line, we call it **incident** to that line.



- The Szemerédi-Trotter theorem provides insight into the number of incidences between a set of points a set of lines

Important Definitions

Definition (r-rich)

When a point lies on at least r-many lines, we call it **r-rich**.

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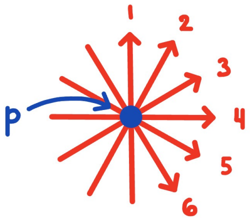

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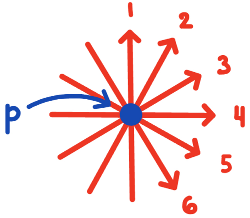


point p is "6 rich"

point p is "5 rich"

Important Definitions

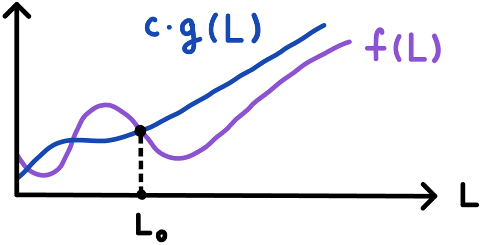
Definition (r-rich)

When a point lies on at least r-many lines, we call it **r-rich**.

Definition (Asymptotic upper bound notation)

Let $f, g : \mathbb{R} \to \mathbb{R}$. We write $f(L) \lesssim g(L)$ if there exist constants $c, L_0 \in \mathbb{R}^+$ such that

$$|f(L)| \leq c\,|g(L)| \quad \text{for all } L > L_0.$$



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Szemerédi-Trotter Theorem Corollary

Theorem (Szemerédi-Trotter Theorem Corollary)

For a set of lines L, let $P_r(L)$ be the set of r-rich points in $\mathbb{R}^2$. Then,

$$|P_r(L)| \lesssim \left(|L|^2 \ r^{-3} + |L| \ r^{-1}\right).$$

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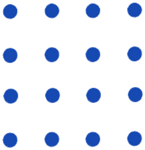
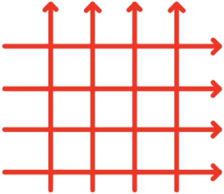


set P



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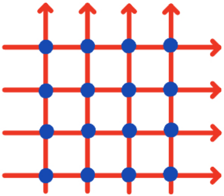

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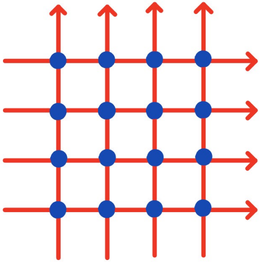

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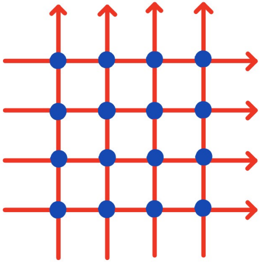

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Then, $|P_2(L)| = 16$.

Understanding Szemerédi-Trotter Theorem Corollary

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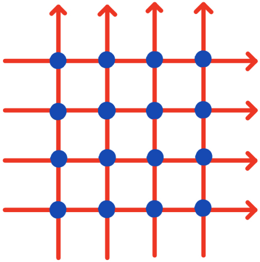

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$$|P_2(L)| \lesssim |L|^2 r^{-3} + |L| r^{-1}$$

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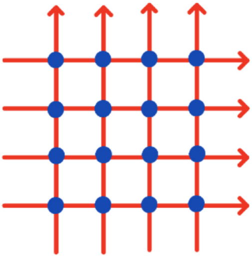

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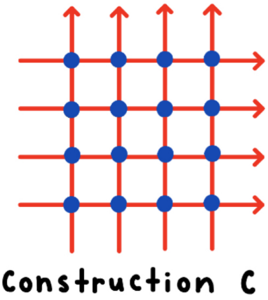

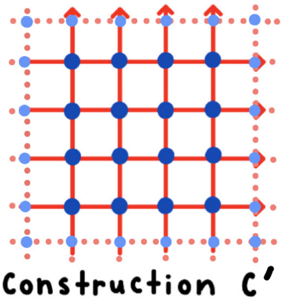

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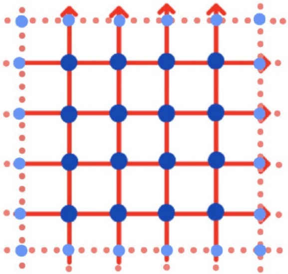

Construction C'

Let $r = 2$.

Then, $|P_2(L)| = |P|$

$$|P_2(L)| \lesssim |L|^2 r^{-3} + |L| r^{-1}$$

$$|P| \lesssim \left(2|P|^{\frac{1}{2}}\right)^2 r^{-3} + 2|P|^{\frac{1}{2}} r^{-1}$$

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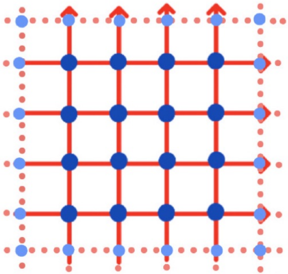

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$$|P| \lesssim 4|P| r^{-3} + 2|P|^{\frac{1}{2}} r^{-1}$$

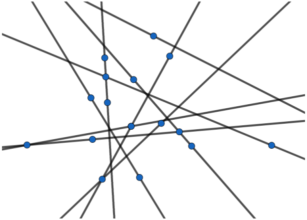
Theorem (Szemerédi-Trotter Theorem)

Let P be a set of points in $\mathbb{R}^2$ and L a set of lines in $\mathbb{R}^2$. Let the set of incidences $I(P, L)$ between P and L be defined as

$$I(P, L) = \{(p, \ell) \in P \times L \mid p \in \ell\}.$$

Then,

$$|I(P, L)| \lesssim (|L|^{2/3}|P|^{2/3} + |L| + |P|).$$



Talk Overview

Sharp Constructions

Definition (Sharp construction)

Suppose,

$$|I(P, L)| \lesssim f(|P|, |L|),$$

where $|I(P, L)|$ is the number of incidences.

A **sharp construction** is a construction of points and lines for which

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Notes:

- Show that the theorem's bound cannot be improved asymptotically.
- Multiple sharp constructions can exist for the same theorem.

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The Square Grid Construction

Recall the Szemerédi-Trotter theorem:

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Define the set of slopes as the first r elements from the ordered list of fractions in lowest terms: $0, 1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \ldots$.

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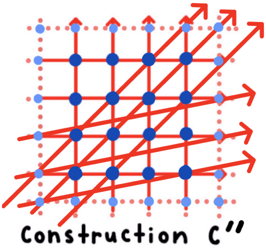

P := N × N grid

L := set of lines incident to a point in P, where each line has slope from the list of r slopes

The Square Grid Construction

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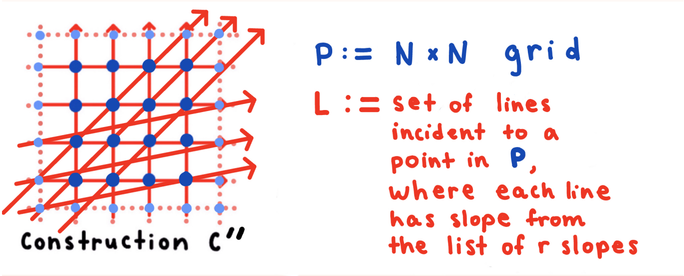

Construction C'' yields $|I(P,L)| \sim (|P|^{2/3}|L|^{2/3})$ incidences. Hence, C'' is a sharp construction.

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Latest Sharp Constructions

Earlier sharp constructions:

- Used point sets that were products of **arithmetic progressions**.

Modern sharp constructions:

- Use point sets that are **products of GAPs**.
- Point set for Silier-Guth construction: $A_N \times A_N$, where

$$A_N = \left\{ x_1 + x_2\sqrt{k} \mid x_1, x_2 \in [-\sqrt{N}, \sqrt{N}] \right\}.$$

- Point set for our research: $A_N \times A_N$, where

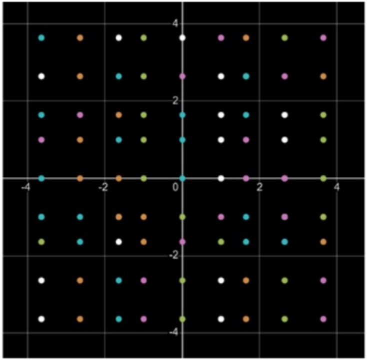
$$A_N = \left\{ x_1 + x_2\sqrt[3]{k} \mid x_1, x_2 \in [-\sqrt{N}, \sqrt{N}] \right\}.$$

Generalized Arithmetic Progression Example

Let the point set P consist of all points of the form $(a, b) \in A_N^2$ where

$$A_N = \left\{ x_1 + x_2\sqrt{k} \mid x_1, x_2 \in [-\sqrt{N}, \sqrt{N}] \right\}.$$

The point set P is illustrated for $k = 7, N = 3$ below.



Construction for $k = 7$ and $N = 3$.

Thank you

Thank you to my mentor Jiahui Yu,
Prof. Etingof, Dr. Gerovitch, Dr. Khovanova,
and my parents!

Questions?

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