Full-Twist Presentations for Fundamental Groups of Complexified Hyperplane Arrangements

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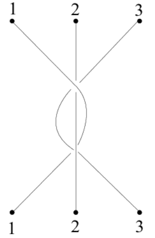
October 19, 2025
MIT PRIMES Conference

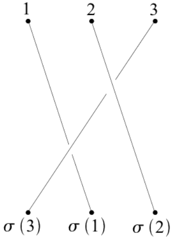
Motivation and Background

- In **1926**, Emil Artin introduced a presentation for the braid group

  $B_n = \langle \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1},\ \sigma_i\sigma_j = \sigma_j\sigma_i\ (|i-j| > 1)\rangle.$
- In **1947**, Artin found a presentation for the pure braid group P_n: the subgroup where each strand returns to its starting point.
- The pure braid group is notoriously difficult to work with.
- This motivates a fundamental question: Can we find a simpler presentation for pure braid groups?



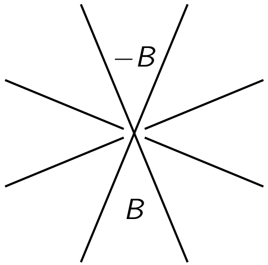
element of the pure braid group



element of the braid group

Central Hyperplane Arrangements

- A *real hyperplane* in $\mathbb{R}^n$ is a linear subspace of dimension $n-1$
- A *central arrangement* $\mathcal{H}$ is a finite collection of real hyperplanes in $\mathbb{R}^n$ all intersecting at the origin.
- Any subset of $\mathcal{H}$ is called a *subarrangement*.
- A subarrangement $\mathcal{A} \subseteq \mathcal{H}$ is said to be *full* if it includes every hyperplane of $\mathcal{H}$ that contains some fixed linear subspace of $\mathbb{R}^n$.
- The *regions* of $\mathcal{H}$ are the connected components of the complement $\mathbb{R}^n \setminus \bigcup_{H\in\mathcal{H}} H$.



Braid Group and Pure Braid Group

Fix a finite Coxeter group W with real hyperplane arrangement $\mathcal{H}$ in $\mathbb{R}^n$. Define $V_{\mathbb{C}}^{\mathrm{reg}} := \mathbb{C}^n \setminus \mathcal{H}_{\mathbb{C}}$.

Definition

The *braid group* of W is defined as

$$B(W) := \pi_1(V_{\mathbb{C}}^{\mathrm{reg}}/W).$$

Definition

The *pure braid group* of W is defined as

$$P(W) := \pi_1(V_{\mathbb{C}}^{\mathrm{reg}}).$$

Main Conjecture: Full-Twist Presentation

The element $\mathbb{c}$ is called the *full twist*, which is defined as

$$\mathbb{c} := e^{2\pi i t} x_0 \text{ for } 0 \leq t \leq 1.$$

Let $\mathrm{Red}_{\mathbb{T}}(\mathbb{c})$ be the set of reduced words in $\mathbb{T}$ for the full twist $\mathbb{c}$ and $[\mathrm{Red}_{\mathbb{T}}(\mathbb{c})]$ as the relation setting all these reduced words equal.

Conjecture

$$P(W) = \langle \mathbb{T} : [\mathrm{Red}_{\mathbb{T}}(\mathbb{c})] \rangle.$$

- Proven for type A, B, D, H_3, and $I_2(m)$
- Can be generalized to central hyperplane arrangements.

Steps to Proving the Conjecture

- Step 1: Identify some viable presentation for the pure braid group $P(W)$

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- Step 1: Identify some viable presentation for the pure braid group $P(W)$
- Step 2: Demonstrate that the presentation we identified is equivalent to one specified in the conjecture

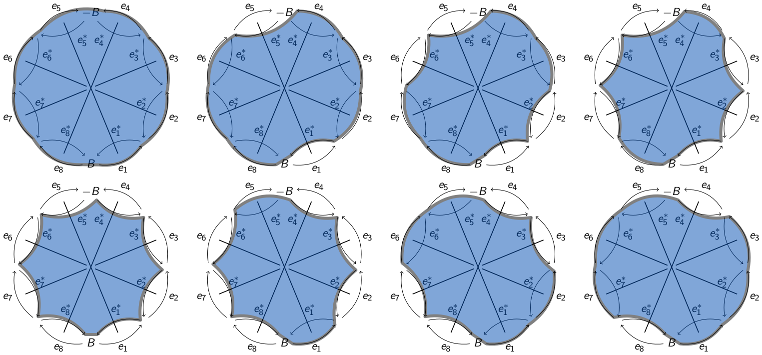
Salvetti Complex: Model for $V_{\mathbb{C}}^{\mathrm{reg}} = \mathbb{C}^n \setminus \mathcal{H}_{\mathbb{C}}$

Fix a base point x_B in the interior of some region B. We only need the 0, 1, and 2-cells for $\pi_1(V_{\mathbb{C}}^{\mathrm{reg}}, x_B)$. Call this complex $P^*(\mathcal{H}, B)$.

0-cells The regions of the hyperplane arrangement.

1-cells For each pair of adjacent regions, connect them with edge e oriented away from B an edge e^* oriented towards B.

2-cells For each rank-two subarrangement $\mathcal{A}$, glue one 2-cell for the regions separated by $\mathcal{A}$.



Salvetti Presentation

Theorem (Salvetti, 87)

Let $\mathcal{H}$ be a hyperplane arrangement, then

$$\pi_1(\mathbb{C}^n \setminus \mathcal{H}, x_B) \cong \pi_1(\mathcal{P}^*(\mathcal{H}, B), B).$$

- Construct a generating set for $\pi_1(\mathcal{P}^*(\mathcal{H}, B), B)$.
- Define $\mathrm{gal}(C)$ as a choice of positive minimal gallery from B to C.
- For $C' \xrightarrow{e} C$, we define the corresponding loop $\mathbb{t}_e$ by:

$$\mathbb{t}_e := \mathrm{gal}(C') e e^* \, \mathrm{gal}(C')^{-1}.$$

- Let $\mathbb{T}_{\mathrm{edge}}$ represent the set of all such loops t_e. This is a valid generating set.

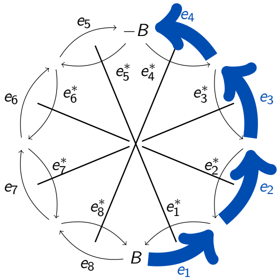
Positive Minimal Gallery

A positive minimal gallery

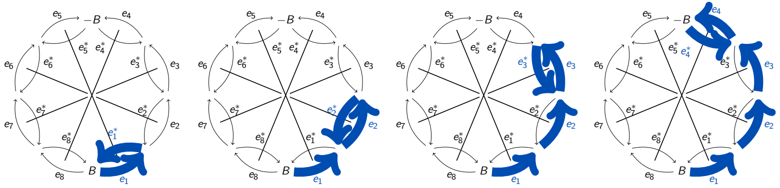
$$\mathbf{b} := b_1, b_2, \ldots, b_N$$

from B to $-B$ specifies a set $\mathbb{T}_{\mathbf{b}} = \{\mathbb{b}_i\}_{i=1}^{N}$ by the formula:

$$\mathbb{b}_i := (b_1 b_2 \cdots b_{i-1}) b_i b_i^* (b_{i-1}^{-1} \cdots b_2^{-1} b_1^{-1}) \text{ for } 1 \leq i \leq N,$$



Example of Positive Minimal Gallery



If $\mathbf{b} = e_1 e_2 e_3 e_4$, $\mathbb{T}_{\mathbf{b}}$ contains the loops

$$\begin{aligned} \mathbb{b}_1 &= \mathbb{t}_{e_1} = e_1 e_1^*, \\ \mathbb{b}_2 &= \mathbb{t}_{e_2} = e_1 e_2 e_2^* e_1^{-1}, \\ \mathbb{b}_3 &= \mathbb{t}_{e_3} = e_1 e_2 e_3 e_3^* e_2^{-1} e_1^{-1}, \text{ and} \\ \mathbb{b}_4 &= \mathbb{t}_{e_4} = e_1 e_2 e_3 e_4 e_4^* e_3^{-1} e_2^{-1} e_1^{-1}. \end{aligned}$$

Forming the Generators

Theorem (Salvetti, 87)

For any positive minimal gallery $\boldsymbol{b}$ *from* B *to* $-B$, $\mathbb{T}_b$ *is a generating set of* $\pi_1(\mathcal{P}^*(\mathcal{H}, B), B)$. *Specifically, if* $C \xrightarrow{e} C'$ *with* $H_e = H_{b_k}$, *then*

$$\mathbb{t}_e = \left(\prod_{k>i\geq 1, H_i \notin S(c)} \mathbb{b}_i \right)^{-1} \mathbb{b}_k \left(\prod_{k>i\geq 1, H_i \notin S(c)} \mathbb{b}_i \right).$$

One can see that the conjugation step is the most difficult step in calculating the relations for $\pi_1(V_{\mathbb{C}}^{\mathrm{reg}}, x_B)$.

Rank-Two Relations

Theorem (Salvetti, 87)

Fix a positive minimal gallery $\mathbf{b}$ *from* B *to* $-B$ *in* $\mathcal{P}^*(\mathcal{H}, B)$. *For each full rank two subarrangement* $\mathcal{A}$ *of* $\mathcal{H}$, *we can choose one two-cell in* $\mathcal{P}^*(\mathcal{H}, B)$ *with edges labeled*

$$e_1, e_2, \ldots, e_M, \quad e_{M+1}, \ldots, e_{2M}$$

where $\{H_{e_i}\}_{i=1}^{M} = A$ *and* $H_{e_1} <_b \cdots <_b H_{e_M}$.
Denote $[\mathcal{A}]_{T_b}$ *as representing the relations*

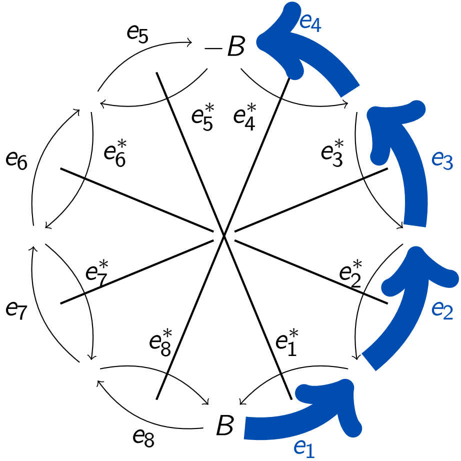
$$\mathbb{t}_{e_M} \cdots \mathbb{t}_{e_2}\mathbb{t}_{e_1} = \mathbb{t}_{e_1}\mathbb{t}_{e_M} \cdots \mathbb{t}_{e_2} = \cdots = \mathbb{t}_{e_{M-1}} \cdots \mathbb{t}_{e_2}\mathbb{t}_{e_1}\mathbb{t}_{e_M}.$$

Then

$$\pi_1(\mathcal{P}^*(\mathcal{H}, B), B) = \langle \mathbb{T}_b : [\mathcal{A}]_{T_b} \rangle.$$

where the relations range over all full rank-two subarrangements $\mathcal{A} \subset \mathcal{H}$.

Example



$$\pi_1(V_{\mathbb{C}}^{\mathrm{reg}}, x_B) = \left\langle \mathbb{b}_1, \mathbb{b}_2, \mathbb{b}_3, \mathbb{b}_4 : \begin{array}{ll} \mathbb{b}_4\mathbb{b}_3\mathbb{b}_2\mathbb{b}_1 = & \mathbb{b}_1\mathbb{b}_4\mathbb{b}_3\mathbb{b}_2 \\ = \mathbb{b}_2\mathbb{b}_1\mathbb{b}_4\mathbb{b}_3 & = \mathbb{b}_3\mathbb{b}_2\mathbb{b}_1\mathbb{b}_4 \end{array} \right\rangle .$$

Algorithmic Procedure for Step 1

- Identify every single full rank 2 subarrangement $\mathcal{A}$.
- Select a positive minimal gallery **b** to minimize the number of conjugation.
- Determine each of the rank 2 relations $[\mathcal{A}]_{\mathcal{T}_b}$.
- Combine the relations $[\mathcal{A}]_{\mathcal{T}_b}$ to form a presentation of $P(W)$.
- For finite Coxeter arrangement, the optimal choice for **b** is given by the c-sorting word for the long element $\mathbf{w}_o(\mathbb{c})$.

Noncrossing Relations

Theorem

Let $\mathcal{H}$ be a finite Coxeter arrangement and c a Coxeter element of W. For the generators $\mathbb{T}_c := \mathbb{T}_{\mathbf{b}}$ determined by the c-sorting word $\mathrm{w}_\circ(\mathrm{c})$, *each c-noncrossing rank-2 subarrangement $\mathcal{A}$ of $\mathcal{H}$ yields a relation*

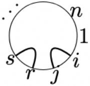
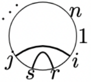
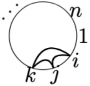
$$[\mathcal{A}]_{\mathbb{T}_c} = [\mathrm{Red}_{\mathbb{T}_c}(\mathbb{c}_{\mathcal{A}})].$$

- If $\mathcal{A} = \{H_{i_1} <_c \cdots <_c H_{i_k}\}$, then

$$\mathbb{b}_{i_k} \cdots \mathbb{b}_{i_1} = \mathbb{b}_{i_1} \mathbb{b}_{i_k} \cdots \mathbb{b}_{i_2} = \cdots = \mathbb{b}_{i_{k-1}} \cdots \mathbb{b}_{i_1} \mathbb{b}_{i_k}.$$

- Relations from c-noncrossing subarrangements appear naturally from the c-sorting gallery.
- c-crossing subarrangements require explicit computation (as shown next).

Computation for Type A

	Type A_{n-1} c-Noncrossing Full Rank-Two Subarrangements	
Label	$\mathcal{N}_c(A_{n-1}, A_1 \times A_1)$	$\mathcal{N}_c(A_{n-1}, A_2)$
Subarrangement	$\{H_{ij}, H_{rs}\}$	$\{H_{ij}, H_{ik}, H_{jk}\}$
Conditions	$i<j<r<s$ or $i<r<s<j$	$i<j<k$
Picture	 	
Relation	$[(ij)(rs)]$	$[(ij)(ik)(jk)]$

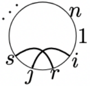
	Type A_{n-1} c-Crossing Full Rank-Two Subarrangements	
Label	$\mathcal{C}_c(A_{n-1}, A_1 \times A_1)$	
Subarrangement	$\{H_{ij}, H_{rs}\}$	
Conditions	$i<r<j<s$	
Picture		
Relation	$[(ij)(rs)^{(js)}]$	

FIGURE 1. The full rank-two subarrangements of type A_{n-1}.

Presentation for Type A (Step 1)

Theorem

Let $c = s_{n-1}s_{n-2}\cdots s_1$. *Then*

$$P(A_{n-1}) = \left\langle\; \mathbb{T}_c \;\middle|\; \begin{array}{cr} [(\mathbb{i}\mathbb{j})(\mathbb{r}\mathbb{s})] & \textit{if } \{H_{ij}, H_{rs}\} \in \mathcal{N}_c(A_{n-1}, A_1 \times A_1) \\ [(\mathbb{i}\mathbb{j})(\mathbb{i}\mathbb{k})(\mathbb{j}\mathbb{k})] & \textit{if } \{H_{ij}, H_{ik}, H_{jk}\} \in \mathcal{N}_c(A_{n-1}, A_2) \\ [(\mathbb{i}\mathbb{j})(\mathbb{r}\mathbb{s})^{(\mathbb{j}\mathbb{s})}] & \textit{if } \{H_{ij}, H_{rs}\} \in \mathcal{C}_c(A_{n-1}, A_1 \times A_1) \end{array} \;\right\rangle.$$

Step 2

Firstly, we need to rewrite the relation to eliminate the conjugation, i.e. make all the relations positive.

Theorem

The pure braid group $P(A_{n-1})$ has the positive presentation:

$$P(A_{n-1}) = \left\langle \; \mathbb{T}_c \;\middle|\; \begin{array}{cr} [(\mathbb{ij})(\mathbb{rs})] & \textit{if } \{H_{ij}, H_{rs}\} \in \mathcal{N}_c(A_{n-1}, A_1 \times A_1) \\ [(\mathbb{ij})(\mathbb{ik})(\mathbb{jk})] & \textit{if } \{H_{ij}, H_{ik}, H_{jk}\} \in \mathcal{N}_c(A_{n-1}, A_2) \\ x = y & \textit{if } \{H_{ij}, H_{rs}\} \in \mathcal{C}_c(A_{n-1}, A_1 \times A_1) \end{array} \right\rangle$$

where $x = y := (\mathbb{ij})(\mathbb{is})(\mathbb{rs})(\mathbb{js}) = (\mathbb{is})(\mathbb{rs})(\mathbb{js})(\mathbb{ij})$.

Then, it can be shown that each of these relations can be completed to reduced words for the full twist, confirming our conjecture.

Type B Noncrossing

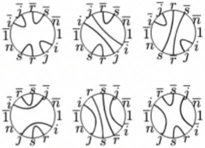
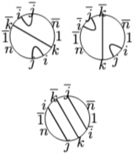
Label	Type B_n c-Noncrossing Full Rank-Two Subarrangements			
	$\mathcal{N}_c(B_n, A_1\times A_1, 1)$	$\mathcal{N}_c(B_n, A_1\times A_1, 2)$	$\mathcal{N}_c(B_n, A_2)$	$\mathcal{N}_c(B_n, B_2)$
Subarrangement	$\{H_{ij}, H_{rs}\}$	$\{H_{ij}, H_{k\bar{k}}\}$	$\{H_{ij}, H_{ik}, H_{jk}\}$	$\left\{\begin{matrix} H_{ij}, H_{i\bar{i}}, \\ H_{\bar{i}j}, H_{j\bar{j}} \end{matrix}\right\}$
Conditions	$\lvert i\rvert<\lvert j\rvert<\lvert r\rvert<\lvert s\rvert$ and $\begin{matrix} I=J=R=S \\ \bar{I}=J=\bar{R}=S \text{ or} \\ I=J=\bar{R}=S \end{matrix}$ $\lvert i\rvert<\lvert r\rvert<\lvert s\rvert<\lvert j\rvert$ and $\begin{matrix} I=J=R=S \\ \bar{I}=J=\bar{R}=S \text{ or} \\ \bar{I}=J=R=S \end{matrix}$	$\begin{matrix}\lvert i\rvert<\lvert j\rvert<\lvert k\rvert \\ I=J\end{matrix}$ $\begin{matrix}\lvert k\rvert<\lvert i\rvert<\lvert j\rvert \\ I=J\end{matrix}$ $\begin{matrix}\lvert i\rvert<\lvert k\rvert<\lvert j\rvert \\ I=\bar{J}\end{matrix}$	$\lvert i\rvert<\lvert j\rvert<\lvert k\rvert$ and $\begin{matrix} I=J=K \\ I=\bar{J}=\bar{K} \text{ or} \\ I=J=\bar{K} \end{matrix}$	None
Picture				
Relation	$[((i\,j))((r\,s))]$	$[((i\,j))((k\,\bar{k}))]$	$[((j\,k))((i\,k))((i\,j))]$	$[((j\,\bar{j}))((\bar{i}\,j))((i\,\bar{i}))((i\,j))]$

FIGURE 2. The full rank-two c-noncrossing subarrangements of type B_n. We use capital letters to mean the sign of the corresponding letter (for example, $I = \text{sign}(i)$ and $\bar{I} = -\text{sign}(i)$). Observe that i, j, r, s, and k do not necessarily stand for positive numbers, and that $\bar{i}$, $\bar{j}$, $\bar{r}$, $\bar{s}$, and $\bar{k}$ do not necessarily stand for negative numbers.

Type B Crossing

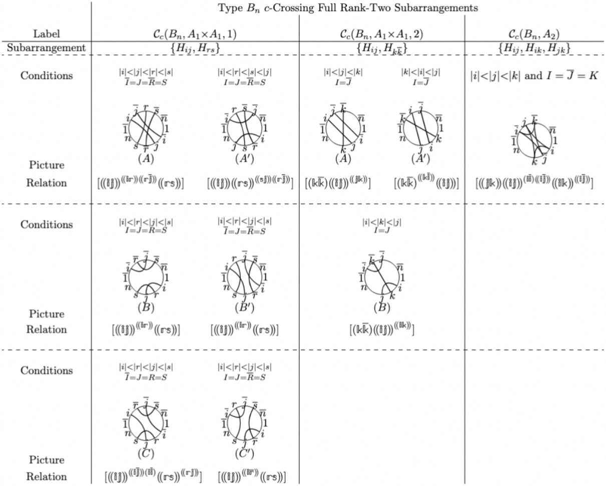


FIGURE 3. The full rank-two c-crossing subarrangements of type B_n. We use capital letters to mean the sign of the corresponding letter (for example, $I = \mathrm{sign}(i)$ and $\overline{I} = -\mathrm{sign}(i)$). Observe that i, j, r, s, and k do not necessarily stand for positive numbers, and that $\bar{i}$, $\bar{j}$, $\bar{r}$, $\bar{s}$, and $\bar{k}$ do not necessarily stand for negative numbers.

Type D Noncrossing

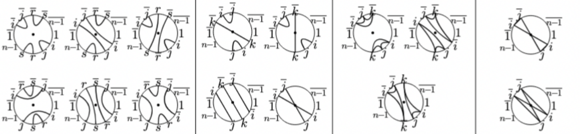
Type D_n c-Noncrossing Full Rank-Two Subarrangements				
Label	$\mathcal{N}_c(D_n, A_1\times A_1, 1)$	$\mathcal{N}_c(D_n, A_1\times A_1, 2)$	$\mathcal{N}_c(D_n, A_2, 1)$	$\mathcal{N}_c(D_n, A_2, 2)$
Subarrangement	$\{H_{ij}, H_{rs}\}$	$\{H_{ij}, H_{kn}\}, \{H_{in}, H_{j\overline{n}}\}$	$\{H_{ij}, H_{ik}, H_{jk}\}$	$\{H_{in}, H_{ij}, H_{jn}\}$
Conditions	$\|i\|<\|j\|<\|r\|<\|s\|$ and $\begin{smallmatrix} I=J=R=S \\ \overline{I}=J=\overline{R}=S \\ I=J=\overline{R}=S \end{smallmatrix}$ or $\|i\|<\|r\|<\|s\|<\|j\|$ and $\begin{smallmatrix} I=J=R=S \\ \overline{I}=J=\overline{R}=S \\ \overline{I}=J=R=S \end{smallmatrix}$	$\begin{smallmatrix} \|i\|<\|j\|<\|k\| \\ I=J \end{smallmatrix}$ $\begin{smallmatrix} \|k\|<\|i\|<\|j\| \\ I=J \end{smallmatrix}$ $\begin{smallmatrix} \|i\|<\|k\|<\|j\| \\ I=\overline{J} \end{smallmatrix}$ $\begin{smallmatrix} \|i\|<\|j\| \\ I=J \end{smallmatrix}$	$\|i\|<\|j\|<\|k\|$ and $\begin{smallmatrix} I=J=K \\ I=\overline{J}=\overline{K} \\ I=J=\overline{K} \end{smallmatrix}$ or	$\|i\| < \|j\|$ and $\begin{smallmatrix} I=J \\ \overline{I}=J \end{smallmatrix}$ or
Picture				
Relation	$[((ij))((rs))]$	$[((ij))((kn))], [((ij))((j\overline{n}))]$	$[((jk))((ik))((ij))]$	$[((in))((ij))((jn))]$

FIGURE 4. The full rank-two c-noncrossing subarrangements of type D_n. The point in the center corresponds to n and $\overline{n}$. We use capital letters to mean the sign of the corresponding letter (for example, $I = \mathrm{sign}(i)$ and $\overline{I} = -\mathrm{sign}(i)$). Observe that i, j, r, s, and k do not necessarily stand for positive numbers, and that $\overline{i}$, $\overline{j}$, $\overline{r}$, $\overline{s}$, and $\overline{k}$ do not necessarily stand for negative numbers. Also i, j, r, s, and k are never equal to n or $\overline{n}$. The letter n is denoted explicitly if they are used.

Type D Crossing

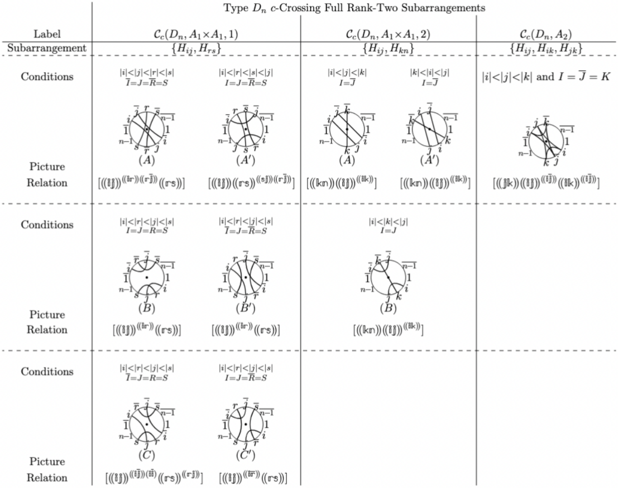


FIGURE 5. The full rank-two c-crossing subarrangements of type D_n. The center point for each circle represents n and $\overline{n}$. We use capital letters to mean the sign of the corresponding letter (for example, $I = \text{sign}(i)$ and $\overline{I} = -\text{sign}(i)$). Observe that i, j, r, s, and k do not necessarily stand for positive numbers, and that $\overline{i}$, $\overline{j}$, $\overline{r}$, $\overline{s}$, and $\overline{k}$ do not necessarily stand for negative numbers. Also i, j, r, s, and k are never equal to n or $\overline{n}$. The letter n is denoted explicitly if they are used.

Future Work

- Prove the conjecture for other exceptional cases such as F_4, H_4, E_6, ...
- Current progress for F_4: For the 67 crossing subarrangements, we have rewritten 60 of these relations positively
- Find methods to write the non-positive relations as positive relation in a systematic manner
- Generalize the conjecture to central hyperplane arrangements

Acknowledgments

- Nathan Williams
- Neha Goregaokar
- MIT-PRIMES