The Geometry of a Counting Formula for Deformations of the Braid Arrangement

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What is a Hyperplane Arrangement?

Definition: A *hyperplane arrangement* is a finite collection of affine hyperplanes in $\mathbb{R}^n$.

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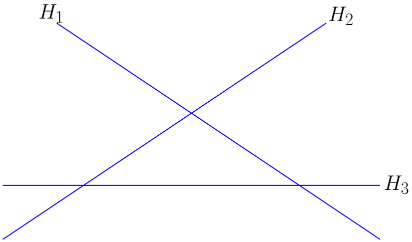

Hyperplane Arrangements

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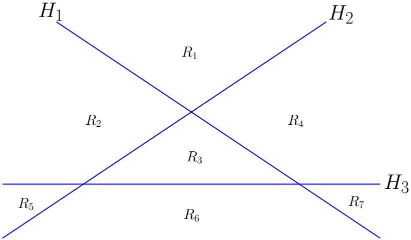
Example: The hyperplanes cut the space into *regions*.



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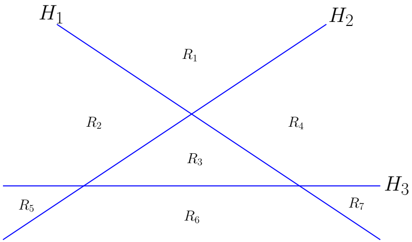

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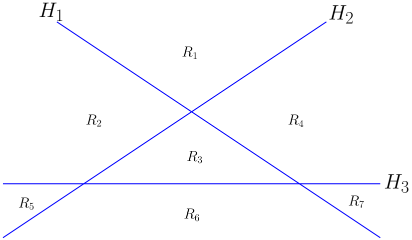

Arbitrary intersection of hyperplanes $\rightarrow$ flat
Intersection of flat and a region $\rightarrow$ face

Deformations of the Braid Arrangement

Definition: Let $\mathbf{S} = (S_{a,b})_{1 \leq a < b \leq n}$ be a collection of finite sets of integers. The $\mathbf{S}$-*braid arrangement* is the collection of the following hyperplanes:

$$\mathcal{A}_{\mathbf{S}} = \{H_{a,b,s} \mid 1 \leq a < b \leq n, s \in S_{a,b}\},$$

where $H_{a,b,s} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_a - x_b = s\}$.

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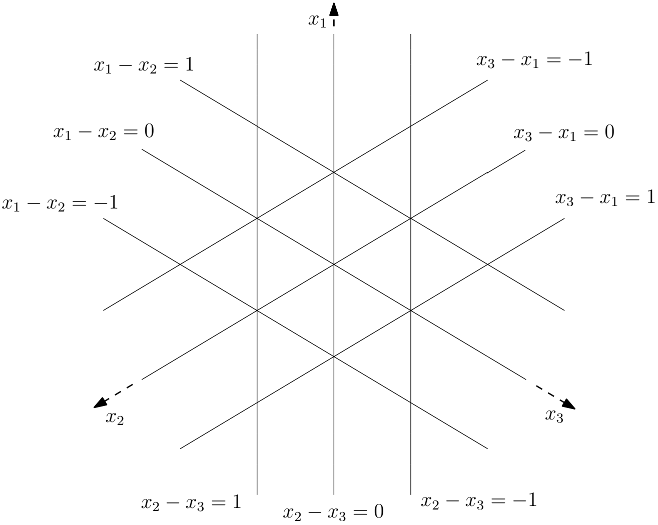
Example: The *m-Catalan arrangement*: $S_{a,b} = [-m..m]$ for all $1 \le a < b \le n$.



The 1-Catalan arrangement in 3 dimensions. We visualize the arrangement by projecting it onto the hyperplane $x_1 + x_2 + x_3 = 0$, since each hyperplane is orthogonal to it.

The Bernardi Formula

Theorem: (Bernardi, 2018) For an **S**-braid arrangement $\mathcal{A}_{\mathbf{S}}$,

$$\#\text{ of regions} = \sum_{(T,B)\in\mathcal{U}_{\mathbf{S}}(n)} (-1)^{n-|B|}$$

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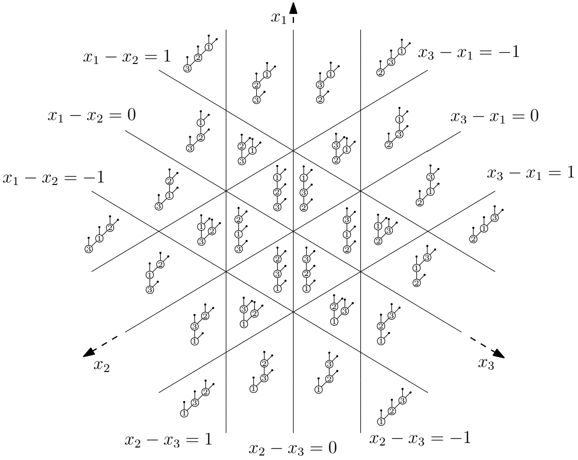
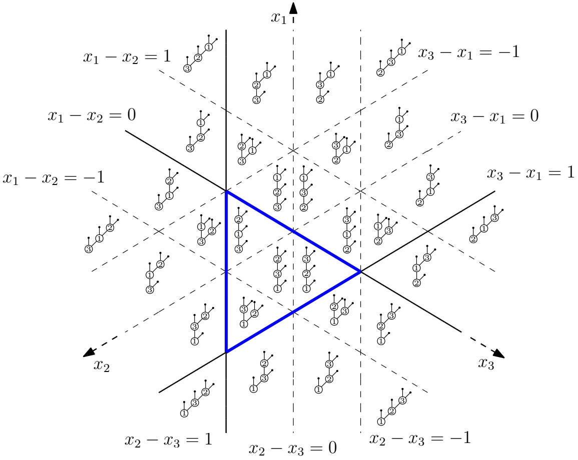
Key points:

- For $m = \max\{|s| \mid s \in S_{a,b}, 1 \leq a < b \leq n\}$, the underlying trees are $(m+1)$-ary trees on n labeled nodes.
- Any **S**-braid arrangement is a subarrangement of the m-Catalan arrangement



Each region of the m-Catalan arrangement can be associated to a unique $(m+1)$-ary tree via a bijection given in Bernardi 2018.



Trees associated with a region R of $\mathcal{A}_{\mathbf{S}}$ form a set of trees T_R. We denote $\mathcal{U}_{\mathbf{S}}(R)$ as the set of $\mathbf{S}$-boxed trees corresponding to T_R.

Region-wise Formula

Question: For a region R of $\mathcal{A}_{\mathbf{S}}$, what is

$$\sum_{(T,B)\in\mathcal{U}_{\mathbf{S}}(R)} (-1)^{n-|B|}?$$

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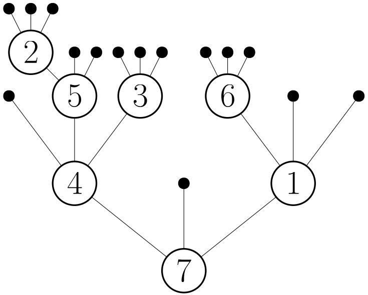
To prove this, we need to understand **S**-boxed trees...

S-Boxed Trees: Definitions

Definition: Let $T \in \mathcal{T}$ be a rooted plane tree with labeled nodes.

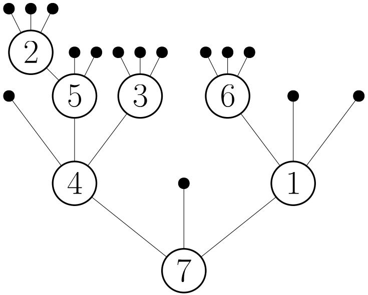
- The *cadet* of a node u is $\text{cadet}(u) =$ rightmost non-leaf child of u (if exists)



S-Boxed Trees: Definitions

Definition: Let $T \in \mathcal{T}$ be a rooted plane tree with labeled nodes.

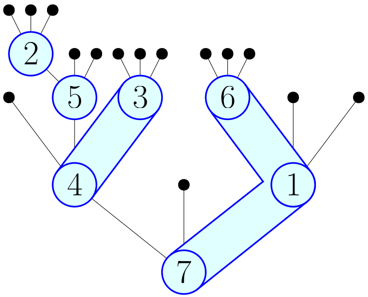
- The *cadet* of a node u is cadet(u) = rightmost non-leaf child of u (if exists)
- A *cadet sequence* is $(v_1, v_2, \ldots, v_k)$ where $v_{i+1} = \text{cadet}(v_i)$



Definition: An ***S**-cadet sequence* is a cadet sequence $(v_1, \ldots, v_k)$ satisfying

$$\sum_{p=i+1}^{j} \mathrm{lsib}(v_p) \notin S^-_{v_i, v_j} \quad \forall i < j.$$

where for $1 \leq a < b \leq n$,
$S^-_{a,b} = \{s \geq 0 \mid -s \in S_{a,b}\}$ and
$S^-_{b,a} = \{s > 0 \mid s \in S_{a,b}\} \cup \{0\}$.



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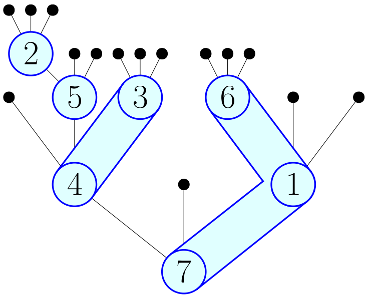
Definition: A pair (T, B) where $T \in \mathcal{T}^{(m)}(n)$ and B is a set of **S**-cadet sequences partitioning the nodes of T is an **S**-*boxed tree.*

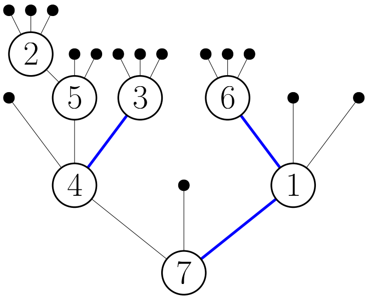


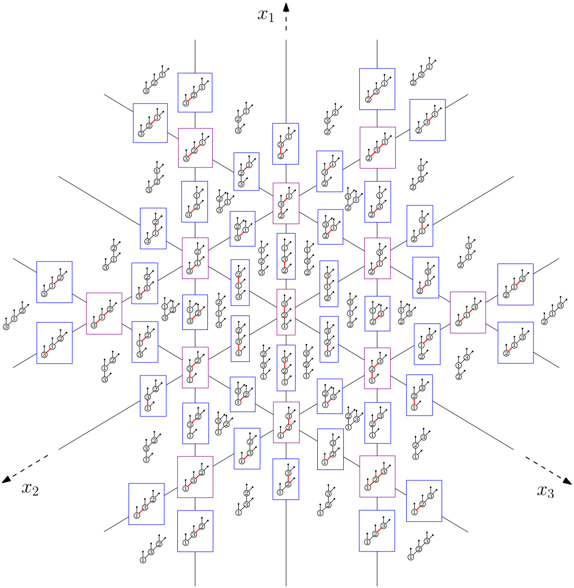
Marked Trees

Recent development (Bernardi, 2024): Bijection between faces of the m-Catalan arrangement and marked (m, n)-trees.

Definition: A *marked (m, n)-tree* is a pair (T, μ) where:

- $T \in \mathcal{T}^{(m)}(n)$
- μ is a set of cadet edges of T
- If $e = \{j, 0\text{-child}(j)\} \in \mu$, then $j < 0\text{-child}(j)$.

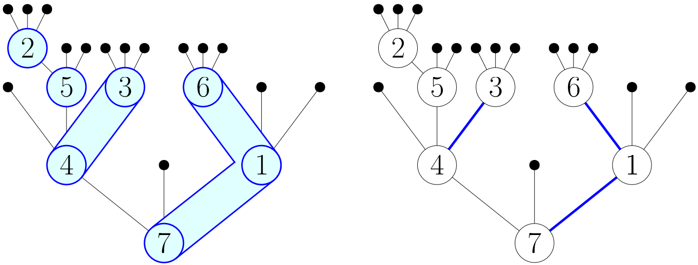




Bernardi's bijection for faces of the Catalan arrangement.

Observations!

We notice that **S**-boxed trees and marked trees look similar!



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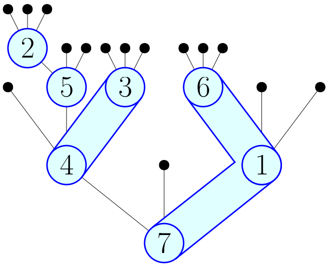

In fact, there is a bijection between **S**-boxed trees and the set of marked trees corresponding to faces avoiding $\mathcal{A}_{\mathbf{S}}$ hyperplanes.

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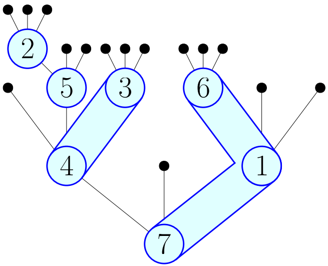

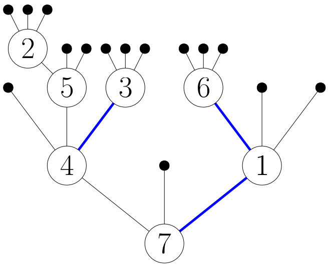

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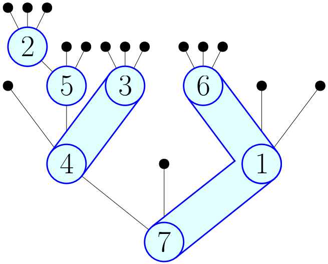

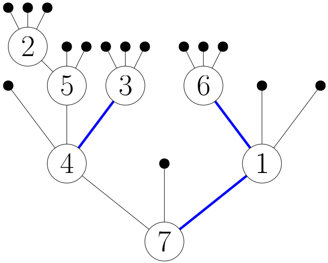

In fact, there is a bijection between **S**-boxed trees and the set of marked trees corresponding to faces avoiding $\mathcal{A}_{\mathbf{S}}$ hyperplanes. We can now reinterpret the Bernardi formula geometrically in terms of faces.

Completing the Proof

Main Result: For any region R of $\mathcal{A}_{\mathbf{S}}$:

$$\sum_{(T,B)\in\mathcal{U}_{\mathbf{S}}(R)} (-1)^{n-|B|} = 1.$$

Proof:

- By the bijection:

$$\sum_{(T,B)\in\mathcal{U}_{\mathbf{S}}(R)} (-1)^{n-|B|} = \sum_{F\in\mathcal{F}_{\mathbf{S}}(R)} (-1)^{n-\dim(F)}.$$

- The faces in $\mathcal{F}_{\mathbf{S}}(R)$ partition region R.
- This sum equals the Euler characteristic of R.
- Since R is contractible: $\chi(R) = 1$. $\square$

Thank You!

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