On the e-positivity of the Chromatic Symmetric Functions of Left-melting Clique Chains

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October 19, 2025
MIT PRIMES Conference

Graphs

Definition

A graph G is a pair (V, E), where V is a set of vertices and E is a set of tuples of vertices (i, j) with $i, j \in V$. We say vertices i and j are adjacent if $(i, j) \in E$.

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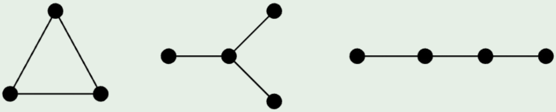
Example



Graph Colorings

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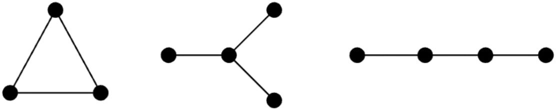
Definition

A proper coloring on a graph G is a function $\kappa : V \to \mathbb{N}$ such that if $(i, j) \in E$, then $\kappa(i) \neq \kappa(j)$.

Graph Colorings

How many proper colorings with at most 4 colors are there?



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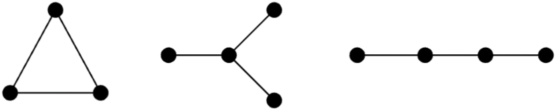

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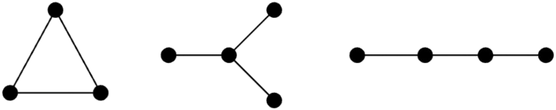

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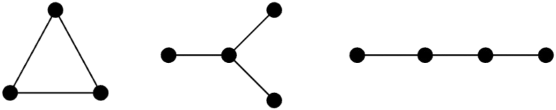

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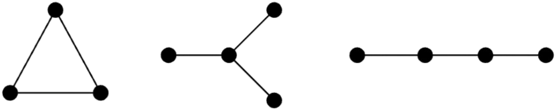

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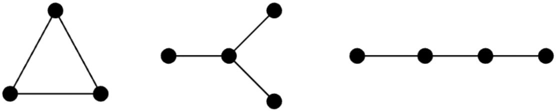

There are $k \cdot (k-1) \cdot (k-2) = k(k-1)(k-2)$ ways to color K_3.

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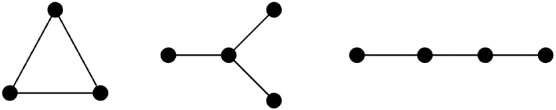

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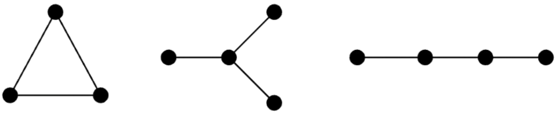

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The chromatic polynomial of any tree with n vertices is the same.

$$\chi_G(k) = k(k-1)^{n-1}$$

Chromatic Symmetric Functions

Definition

The *chromatic symmetric function* of G is the formal power series in variables $x = (x_1, x_2, x_3, \ldots)$ given by

$$X_G(x) = \sum_{\kappa} \left(\prod_{v \in V} x_{\kappa(v)} \right),$$

where the sum ranges over all proper colorings κ.

Chromatic Symmetric Functions

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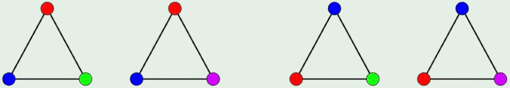
$$X_{K_3} = x_1x_2x_3 + x_1x_2x_4 + \cdots + x_2x_1x_3 + x_2x_1x_4 + \cdots = 6\sum_{i<j<k} x_ix_jx_k.$$



Chromatic Symmetric Functions

If $x = (1, 1, 1, 1, \ldots, 0, 0, \ldots)$ for k number of 1's then $X_G(x) = \chi_G(k)$.

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Conjecture (Stanley, 1995)

If T_1, T_2 are two non-isomorphic trees, then $X_{T_1} \neq X_{T_2}$.

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Example

For G the claw graph,

$$X_G = \sum_{i \neq j} x_i^3 x_j + 6 \sum_{j<k;\, i \neq j,k} x_i^2 x_j x_k + 24 \sum_{i<j<k<\ell} x_i x_j x_k x_\ell.$$

Bases

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The *k-th elementary symmetric function* is

$$e_k = \sum_{i_1 < \cdots < i_k} x_{i_1} \ldots x_{i_k}$$

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Example

Consider $\lambda = 211$. Then, we have

$$e_{211} = e_2e_1e_1 = (x_1x_2 + x_1x_3 + x_2x_3 + \ldots)(x_1 + x_2 + x_3 + \ldots)^2.$$

Bases

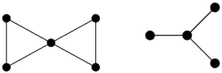
Example

Let G be the bowtie graph. Then, the chromatic symmetric function of G is

$$X_G = 4e_{32} + 12e_{41} + 20e_5.$$

The chromatic symmetric function of the claw graph is

$$X_{K_{1,3}} = e_{211} - 2e_{22} + 5e_{31} + 4e_4.$$



Positivity

Definition

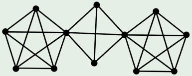
A graph G is said to be *e-positive* if the expansion of the chromatic symmetric function of G has all non-negative coefficients in the e-basis.

Past work

Theorem (Tom, 2024)

Chains of cliques are e-positive.

Example



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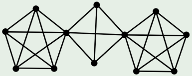
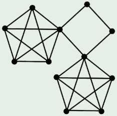


Theorem (Tom-Vailaya, 2024)

Adjacent chains of cycles and cliques are e-positive.

Example

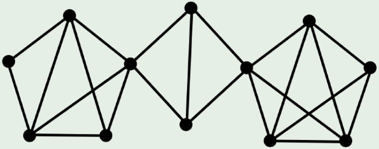


Main result

Theorem (Li-Tom, 2025)

Left-melting clique chains are e-positive.

Example



Acknowledgments

I would like to thank:

My mentor Dr. Foster Tom.

The MIT PRIMES-USA Program for making this research possible.

My family.

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