The Dual Canonical Basis of the Spin Representation via the Temperley-Lieb Algebra

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Particle Spin

- Spin is an intrinsic property of particles.
- It describes the interaction of particles with magnetic fields.
- A singular electron has spin(1/2). It has up-spin ($\uparrow$) or down-spin ($\downarrow$). The system has dimension $1 + 1 = 2$.
- Due to quantum superpositioning, the spin of an electron can be viewed as $\mathbb{C}^2$.
- $\downarrow$ is $(0, 1) = v_-$ while $\uparrow$ is $(1, 0) = v_+$.

Example

The vector $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ represents the electron with a $\frac{1}{2}$ chance of $\downarrow$ and a $\frac{1}{2}$ chance of $\uparrow$.

Two-Electron Systems

If there are two electrons, we consider $\mathbb{C}^2 \otimes \mathbb{C}^2$ (The $\otimes$ is called the tensor product).

- There are $2^2 = 4$ possible states for all the electrons, which are $\downarrow \otimes \downarrow$, $\downarrow \otimes \uparrow$, $\uparrow \otimes \downarrow$, and $\uparrow \otimes \uparrow$.
- Alternatively, $v_- \otimes v_-$, $v_- \otimes v_+$, $v_+ \otimes v_-$, and $v_+ \otimes v_+$.
- The two electron system is a vector space because we consider linear combinations of these spin states.

Example

An example of one possible 2-electron spin state is $\frac{1}{\sqrt{2}} \downarrow \otimes \downarrow + \frac{1}{\sqrt{2}} \uparrow \otimes \uparrow$.

n-Electron Systems

Definition

The spin representation of n electrons is $(\mathbb{C}^2)^{\otimes n}$, which encodes the spin states of n-electron systems.

- Dimension 2^n
- Basis $v_\pm \otimes \cdots \otimes v_\pm$
- Contains another basis called Lusztig's dual canonical basis.

Example

When $n = 2$, the dual canonical basis is $v_+ \otimes v_+$, $v_- \otimes v_-$, $v_- \otimes v_+$, and $v_+ \otimes v_- - q^{-1} v_- \otimes v_+$.

- It is a representation of the Temperley-Lieb algebra.

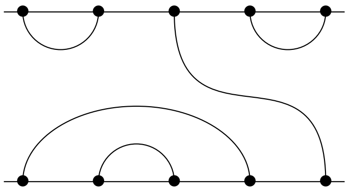
The Temperley-Lieb algebra helps us describe the dual canonical basis.

The Temperley-Lieb Algebra

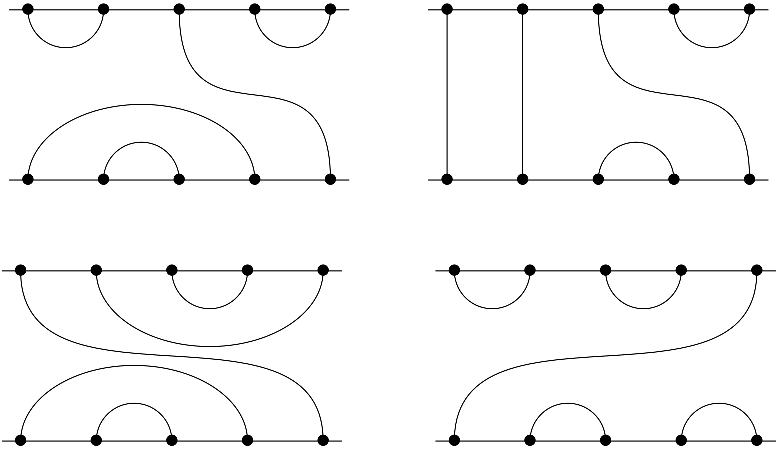
Definition

An *n-diagram* is two parallel lines with n vertices on both the top and bottom line, where the vertices are connected by edges such that:

- the edges are between the parallel lines,
- the edges do not intersect each other, and
- each vertex is the endpoint of exactly one edge.



The Temperley-Lieb Algebra



The Temperley-Lieb Algebra

Definition

An algebra is a vector space with an additional multiplication operation.

Example

Some examples of algebras are

- all $n \times n$ matrices over a field,
- the polynomials in $\mathbb{R}[x]$, and
- the group algebra.

The Temperley-Lieb Algebra

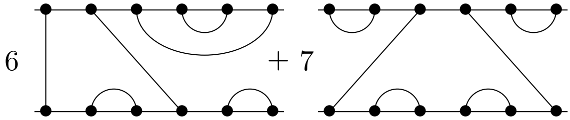
Definition

The Temperley-Lieb Algebra, denoted TL_n, is the algebra over $\mathbb{C}$ generated by n-diagrams.

- If we consider it a vector space, the basis are the diagrams.
- The elements are linear combinations of the diagrams.

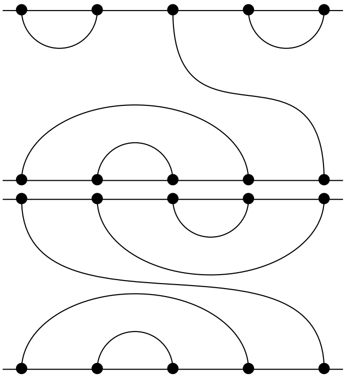
Example

One possible element in TL_6 is



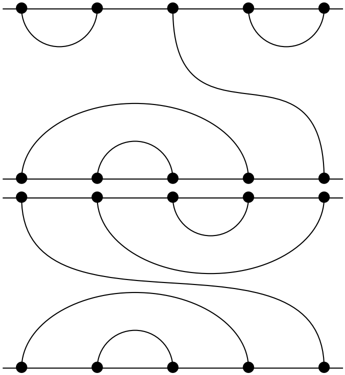
The Temperley-Lieb Algebra

We also need to define multiplication of two diagrams to completely define the algebra TL_n.

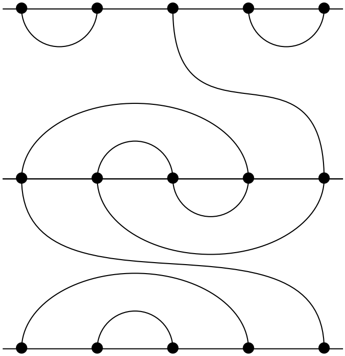


The most natural way is by concatenation.

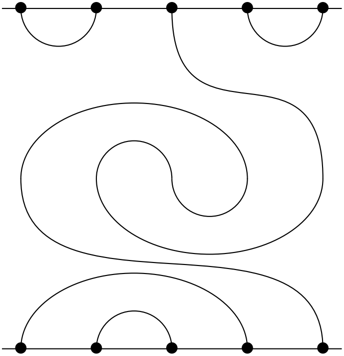
Concatenating Diagrams



Concatenating Diagrams



Concatenating Diagrams



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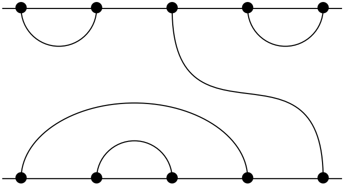
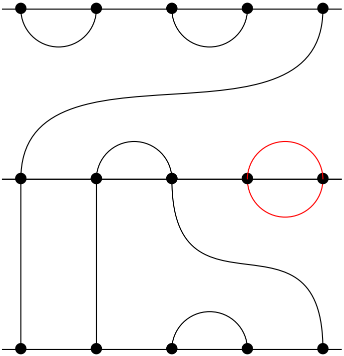


Concatenating Diagrams



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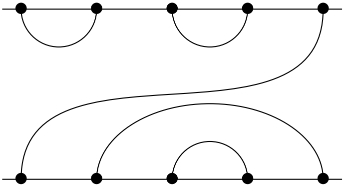


Action of Temperley-Lieb Algebra

I mentioned before that the spin representation $(\mathbb{C}^2)^{\otimes n}$ is a representation of TL_n.

- This means that any element in TL_n can act on any element of $(\mathbb{C}^2)^{\otimes n}$ to obtain a new element $(\mathbb{C}^2)^{\otimes n}$.
- This action must be linear, so we only need to define it on the basis.

Action of Temperley-Lieb Algebra

Definition

The action of a n-diagram on $(\mathbb{C}^2)^{\otimes n}$ is defined based on every "feature" that appears:

- For each arc facing down connecting the ith and jth vertices, we apply

$$v_+ \otimes v_+ \mapsto 0, v_+ \otimes v_- \mapsto -q, v_- \otimes v_+ \mapsto 1, v_- \otimes v_- \mapsto 0$$

  to the ith and jth components, representing annihiliation.
- For each arc facing up connecting the i and $i+1$th vertices, we apply

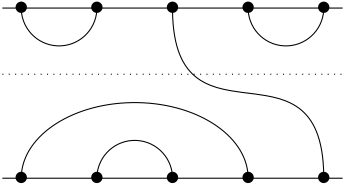
$$\delta \colon 1 \mapsto v_+ \otimes v_- - q^{-1} v_- \otimes v_+$$

  between the i and $i+1$th components, representing creation.
- For each line, we apply the identity.

Action of the Temperley-Lieb Algebra

Example

The following diagram acts on $v_- \otimes v_- \otimes v_+ \otimes v_+ \otimes v_+$ in $(\mathbb{C}^2)^{\otimes 5}$.



We have that

- $(\mathbb{C}^2)^{\otimes 5} \to (\mathbb{C}^2)^{\otimes 1} \to (\mathbb{C}^2)^{\otimes 5}$
- $v_- \otimes v_- \otimes v_+ \otimes v_+ \otimes v_+ \to v_+ \to \delta \otimes v_+ \otimes \delta$

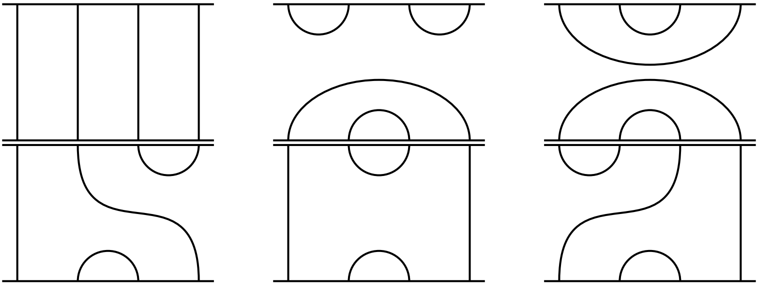
Results

Theorem

There is a bijection between the diagrammatic basis of TL_n *and Lusztig's dual canonical basis of* $(\mathbb{C}^2)^{\otimes n}$ *given by the action of diagrams on* $\underbrace{v_- \otimes \cdots \otimes v_-}_{k} \otimes \underbrace{v_+ \otimes \cdots \otimes v_+}_{n-k}$ *for all* k*.*

- The dual canonical basis is an important basis of the spin representation.
- The bijection is constructive: we can easily see which diagrams correspond to which elements of the dual canonical basis.

The six diagrams for $n = 4$ acting on $v_- \otimes v_- \otimes v_+ \otimes v_+$ that give the dual canonical basis.



The arcs on the bottom in all of these diagrams cross the vertical line between the second and third vertices.

Other Results

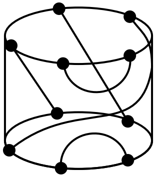
- We give an explicit, non-inductive formula for the dual canonical basis.
- We look at other known properties of the dual canonical basis through this perspective, simplifying things.
- We give a new axiomatic definition of the canonical basis (which is another important basis of the spin representation).

To prove our results, we also look at other objects such as the Hecke algebra and spherical and aspherical modules.

Future Work

Currently, studying the analog of the spin representation in the affine setting.

- There is no axiomatic definition of the dual canonical basis.
- There is still an analog of the Temperley-Lieb algebra, the affine Temperley-Lieb algebra.
- Know that the approach via $\mathrm{TL}_{\mathrm{aff}}$ works for a specific case.



Acknowledgements

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Result Clarification

Theorem

There is a bijection between the diagrammatic basis of TL_n *and Lusztig's dual canonical basis of* $(\mathbb{C}^2)^{\otimes n}$ *given by the action of diagrams on* $\underbrace{v_- \otimes \cdots \otimes v_-}_{k} \otimes \underbrace{v_+ \otimes \cdots \otimes v_+}_{n-k}$ *for all* k*.*

- This statement is a simplification; it isn't completely accurate, as the dimension of TL_n isn't 2^n.
- The bijection is $\bigoplus_{0 \le k \le n} \mathrm{Ind}_{\mathrm{TL}_k \otimes \mathrm{TL}_{n-k}}^{\mathrm{TL}_n} \mathbb{C}_{\mathrm{triv}} \cong (\mathbb{C}^2)^{\otimes n}$.
- Diagrams whose only arcs on the bottom cross the vertical line between the kth and $k+1$th vertices form a basis of $\mathrm{Ind}_{\mathrm{TL}_k \otimes \mathrm{TL}_{n-k}}^{\mathrm{TL}_n} \mathbb{C}_{\mathrm{triv}}$.