The Radon transform and Hecke Algebras

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Outline

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Algebras

Definition

The free vector space on a set A over a field K is denoted by $K[A]$, and is the vector space over K with basis elements e_a for all $a \in A$.

Definition

An associative algebra is a free vector space with associative bilinear product $* : K[A] \times K[A] \to K[A]$.

Example

- The free vector space over a set with a single element is isomorphic to the field itself, as it is a one dimensional vector space.
- The group algebra over a field is a vector space with basis elements e_g for all g in a group G and multiplication $e_g * e_h = e_{gh}$.

Equivalence relations

Definition

An equivalence relation between two sets of elements X and Y is an subset $\sim$ of $X \times Y$.

- This allows us to define equivalence between elements of the set X and Y, namely for $x \in X$, $y \in Y$ and an equivalence relation $\sim$, we say $x \sim y$ iff $(x, y) \in \sim$.

Example

- The diagonal equivalence relation is defined on $X \times X$ and $(x, y) \in \sim$ iff $x = y$.

Invariant equivalence relations

Definition

Given an action by a set G on X and Y, we say that an equivalence relation $\sim$ is invariant under the action by G if for all $x \in X$ and $y \in Y$, $x \sim y \Leftrightarrow gx \sim gy$ for all $g \in G$.

Example

- The diagonal equivalence relation is invariant under actions by any group.

The Radon transform

- Consider two actions of an group G on sets M and N, along with an equivalence relation $\sim$ that is invariant under the action of G.
- We can also define the action of G on $\mathbb{C}[M]$ and $\mathbb{C}[N]$ by the action on the bases.
- Let R be the linear operator from $\mathbb{C}[M]$ to $\mathbb{C}[N]$ defined on the basis of $\mathbb{C}[M]$ as $R(e_m) = \sum_{n \sim m} e_n$.
- Similarly, let R' to be the linear operator from $\mathbb{C}[N]$ to $\mathbb{C}[M]$ defined on the basis as $R'(e_n) = \sum_{m \sim n} e_m$.

Definition

We define $R' \circ R$ as the Radon transform.

- The Radon transform intertwines the action of G on the left, i.e. $R' \circ R(g * m) = g * R' \circ R(m)$.

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Outline

General Linear Group

Definition

The general linear group of dimension n over a field F, denoted $GL_n(F)$, is the group of invertible n by n matrices with entries in the field F.

- We will generally be working with finite fields of order q, denoted by $\mathbb{F}_q$.

Partitions

Definition

A partition of an integer n is a ordered tuple of positive integers such that the sum all integers in the tuple is equal to n.

Example

- $(1, 5, 1)$ is a partition of 7
- $(3, 4)$ is another partition of 7

Levi, unipotent and parabolic subgroups

Definition

The Levi subgroup of type λ, denoted by M_λ, is the product $\prod_{i \in \lambda} GL_i$.

The unipotent subgroup of type λ, known as U_λ, is the group of matrices with arbitrary elements above the blocks of M_λ, ones on the diagonal and zeroes below.

The parabolic subgroup of type λ, denoted as P_λ, is $M_\lambda \cdot U_\lambda = U_\lambda \cdot M_\lambda$.

Finally, we define P_λ^{op} and U_λ^{op} as the subgroups corresponding to the matrices in P_λ and U_λ transposed.

- Note that U_λ is a normal subgroup of P_λ.

Examples

For $\lambda = (2,3,1)$, the corresponding Levi, unipotent, and parabolic subgroups are as follows:

$$M_\lambda = \begin{pmatrix} * & * & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 \\ 0 & 0 & * & * & * & 0 \\ 0 & 0 & * & * & * & 0 \\ 0 & 0 & * & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & * \end{pmatrix} U_\lambda = \begin{pmatrix} 1 & 0 & * & * & * & * \\ 0 & 1 & * & * & * & * \\ 0 & 0 & 1 & 0 & 0 & * \\ 0 & 0 & 0 & 1 & 0 & * \\ 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$P_\lambda = \begin{pmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & * \end{pmatrix}.$$

Unipotent Radon transform

- Let M be the set GL_n/U_λ and N the set GL_n/U_λ^{op}.
- Define an action of GL_n on these two sets by $g * aU_\lambda = gaU_\lambda$ and $g * aU_\lambda^{op} = gaU_\lambda^{op}$.
- Let $g_1 U_\lambda \sim g_2 U_\lambda^{op}$ iff there exists $g \in GL_n$ such that $g \in g_1 U_\lambda$ and $g \in g_2 U_\lambda^{op}$.
- Let $R : \mathbb{C}[M] \to \mathbb{C}[N]$ be the linear operator where $R(e_m) = \sum_{n \sim m} e_n$.
- Similarly, denote $R' : \mathbb{C}[N] \to \mathbb{C}[M]$ to be the linear operator in the other direction defined by $R'(e_n) = \sum_{m \sim n} e_m$.

Definition

We define $R' \circ R$ as the unipotent Radon transform.

Problem of Interest

- Since the Radon transform is linear, we can write it as a matrix of dimension $|GL_n/U_\lambda|$.
- The unipotent Radon transform intertwines the action of M_λ on the right, i.e. $R' \circ R(a * m) = R' \circ R(a) * m$.
- The unipotent Radon transform can be written as

$$R' \circ R(gU_\lambda) = \sum_{u \in U_\lambda} \sum_{u^{op} \in U_\lambda^{op}} guu^{op} U_\lambda.$$

Problem

What are the eigenvalues of the unipotent Radon transform?

Outline

The induction argument

Theorem

All eigenvalues when $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$ *can be found by multiplying an eigenvalue from* $\lambda = (n - \lambda_k, \lambda_k)$ *and an eigenvalue from* $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_{k-1})$.

Corollary

Thus, the only λ *that we need to consider to classify all possible eigenvalues for arbitrary* λ *are* $\lambda = (k, n - k)$.

Miscellaneous Results

Proposition

The matrix for the Radon transform is symmetric with integer coefficients.

Corollary

All eigenvalues for the Radon transform are real algebraic integers.

Proposition

- The largest eigenvalue for the Radon transform is $|U_\lambda|^2$.
- The eigenvalues for GL_2 are q^2 with multiplicity $q-1$, q with multiplicity $(q-1)(q^2-q-2)$, and 1 with multiplicity $(q-1)q$.

Outline

Radon transform

Recall that

$$R' \circ R(gU_\lambda) = \sum_{u \in U_\lambda} \sum_{u^{op} \in U_\lambda^{op}} guu^{op} U_\lambda.$$

The $\sum_{u,u^{op}} uu^{op} U_\lambda$ part of the formula leads us to consider the algebra of double cosets $U_\lambda g U_\lambda$.

Hecke Algebras

- A Hecke Algebra $\mathcal{H}(G, H)$ is an algebra of double cosets HgH for a subgroup H of a finite group G.
- Multiplication of two elements Hg_1H and Hg_2H in this algebra is defined as the cosets that make up $a \cdot b$, for $a \in Hg_1H$ and $b \in Hg_1H$.
- Multiplication is also normalized by a factor of $\frac{1}{|H|}$ in order to make the coset HeH act as the identity.

Radon transform in Hecke Algebras

- The Hecke algebra $\mathcal{H}(GL_n, U_\lambda)$ acts on $\mathbb{C}[GL_n/U_\lambda]$ on the right.
- The Radon transform is given by multiplication of an element from $\mathcal{H}(GL_n, U_\lambda)$.

Proposition

All the eigenvalues of the unipotent Radon transform in $\mathbb{C}[GL_n/U_\lambda]$ are also eigenvalues of the Radon transform in $\mathcal{H}(GL_n, U_\lambda)$.

Yokonuma Hecke Algebra

Definition

The Yokonuma Hecke Algebra $\mathcal{H}(GL_n, U_\lambda)$ describes multiplication of double cosets of U_λ for $\lambda = (\underbrace{1, \ldots, 1}_{n \text{ times}})$. It has generators g_i and t_i, and the following relations:

- $t_i t_j = t_j t_i$;
- $t_j g_i = g_i t_{js_i}$;
- $t_j^{q-1} = 1$;
- $g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$;
- $g_i g_j = g_j g_i \quad (|i - j| \geq 1)$;
- $g_i^2 = q + \sum_{k=1}^{q-1} t_i^k t_{i+1}^{-k+(q-1)/2} g_i$.

Yokonuma-Hecke algebra and the case $\lambda = (k, n-k)$

Proposition

Let $(R' \circ R)_\lambda$ be the unipotent Radon transform for $\lambda = (k, n-k)$ and let Y_n be the unipotent Radon transform for $\lambda = (\underbrace{1, \ldots, 1}_{n \text{ times}})$ then

$$Y_n = Y_k \circ Y_{n-k} \circ (R' \circ R)_\lambda$$

and $Y_k, Y_{n-k}, (R' \circ R)_\lambda$ pairwise commute.

That helps us to find some eigenvalues of $(R' \circ R)_\lambda$.

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