Semisimplifications of α_p-equivariant GL_n-modules (Mentor: Dr. Arun S. Kannan)

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Outline

1. Lie Algebras and Representations
2. Lie Superalgebras
3. Semisimplification
4. Acknowledgements

What is a Lie Algebra?

Definition

A Lie algebra over a field F is a F-vector space $\mathfrak{g}$ with a bilinear map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ (the *bracket*) such that for all $x, y, z \in \mathfrak{g}$:

1. $[x, y] = -[y, x]$ (antisymmetry),
2. $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ (Jacobi identity).

Example

We write $\mathfrak{gl}(n, F)$ for the vector space of all $n \times n$ matrices over F with the Lie bracket defined by $[x, y] = xy - yx$ (standard matrix multiplication).

Example

We write $\mathfrak{sl}(n, F)$ for the subspace of $\mathfrak{gl}(n, F)$ with matrices of trace 0.

Representations

A representation of an algebraic object is a way to study its structure using linear algebra.

Definition

Let L be a Lie algebra over a field F. A representation of L is a Lie algebra homomorphism $\varphi : L \to \mathfrak{gl}(V)$ where V is a finite-dimensional vector space over F.

Let $L = \mathfrak{sl}(2, \mathbb{C})$.

Example (The trivial representation)

If $V = F$ then the trivial representation says that all elements of L get sent to the zero map, $\varphi(l) = 0$ for all $l \in L$.

Example (The natural representation)

If L is a Lie subalgebra of $\mathfrak{gl}(V)$ then the map $\varphi(l) = l$ (i.e., the inclusion map) is a representation. One way this can happen is if $\mathfrak{gl}(V)$ is $\mathfrak{gl}(2, \mathbb{C})$.

Representations

Example (The adjoint representation)

The adjoint map is $\mathrm{ad} : L \to \mathfrak{gl}(L)$ defined by $(\mathrm{ad}\ x)(y) = [x, y]$.

Modules

- Since our representations are of the form $L \to \mathfrak{gl}(V)$, we can think of them as L "acting on" the vector space V to produce another vector in V.
- In this manner V becomes a module for L.
- Modules and representations encode the same information in different ways.
- If a module has no nontrivial *submodules* we say it is simple.
- Simple modules are analogous to prime numbers.
- Since simple modules are the building blocks of modules, we are interested in their structure.

Modules

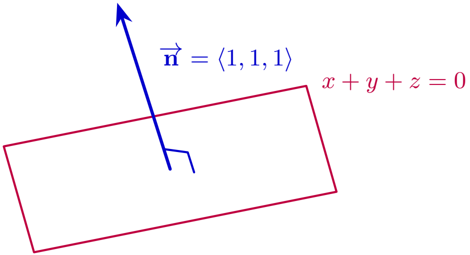
- We work over characteristic p
- Simple modules for $\mathfrak{gl}(n)$ are indexed by $\lambda \in (\mathbb{Z}/p\mathbb{Z})^n$.
- For $\mathfrak{sl}(n)$ it will be convenient to choose a representative $\lambda_1 = p - 1$.
- We call the simple modules $L(\lambda)$

Definition (Semisimple Module)

A module is **semisimple** if it can be written as a direct sum of simple modules.

Semisimplicity

- Not all modules are semisimple!
- How does S_3, the permutation group, act on $\mathbb{R}^3$?



- In characteristic 0 we have $\mathbb{R}^3 = \text{plane} \oplus \text{normal vector}$.
- In characteristic 3, the normal vector lies on the plane, $1 + 1 + 1 = 0$.
- In characteristic 3, $\mathbb{R}^3$ has submodules but is not semisimple.

Definition (Super vector space)

A super vector space is a $\mathbb{Z}_2$ graded vector space, $V = V_0 \oplus V_1$. Vectors that lie purely in V_0 or V_1 are called *homogeneous*. The *parity* of a homogeneous element, x, is dependent on which vector space it lies in:

$$|x| = \begin{cases} 0 & x \in V_0 \\ 1 & x \in V_1 \end{cases}$$

Naturally, a vector of parity 0 is called *even* and a vector of parity 1 is called *odd*.

Lie superalgebras

- The Lie superalgebra $\mathfrak{gl}(m|n)$ can be identified by $(m+n) \times (m+n)$ matrices.

$$\begin{array}{c} \\ 1, \dots, m \\ \bar{1}, \dots, \bar{n} \end{array} \begin{array}{c} \begin{array}{cc} 1, \dots, m & \bar{1}, \dots, \bar{n} \end{array} \\ \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \end{array}$$

- If $n = 0$ then the Lie superalgebra becomes the Lie algebra $\mathfrak{gl}(m)$. Thus the superalgebra is a "generalization" of Lie algebras.
- Difficulty arises due to the interaction between even and odd elements.
- We have that $|i| = 0$ and $|\bar{i}| = 1$ (the $\mathbb{Z}/2\mathbb{Z}$ grading).
- For a basis element e_{ij} we have $|e_{ij}| \equiv |i| + |j| \pmod 2$

$$\left[\begin{array}{c|c} \text{Even} & \text{Odd} \\ \hline \text{Odd} & \text{Even} \end{array}\right] \tag{1}$$

Lie superalgebras

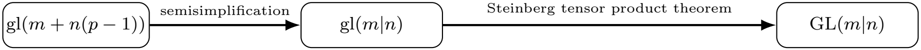
- Simple modules for $\mathfrak{gl}(m|n)$ are indexed by $(\mathbb{Z}/p\mathbb{Z})^{m+n}$.
- The supertrace of a matrix is is defined as $\mathrm{tr}(A) - \mathrm{tr}(D)$ where tr is the trace operation we are used to.
- Just like how $\mathfrak{sl}(n)$ exists for regular Lie algebras, we can define $\mathfrak{sl}(m|n)$ in a similar manner:

Definition

We write $\mathfrak{sl}(m|n)$ for the subspace of $\mathfrak{gl}(m|n)$ with supertrace 0.

The main idea

- The Character problem is concerned with classifying all simple representations of $GL(m|n)$.
- The problem has been solved for the group $GL(n)$ in characteristic zero but not in positive characteristic.
- Therefore the problem for $GL(m|n)$ is even harder since its a generalization.



- The process of semisimplification sends representations to representations.

Semisimplification

- Setup: Consider a $\mathfrak{gl}(n)$ module, M.
- Pick some element of $\mathfrak{gl}(n)$, call it t. This element t must be "nilpotent" which means $t^p \cdot m = 0$ for all $m \in M$.
- We can decompose M with respect to t into several Jordan blocks, J_i.

$$J_i : a \to t \cdot a \to t^2 \cdot a, \ldots, t^{i-1} \cdot a$$

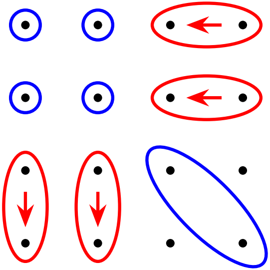
  where $t^i \cdot a = 0$.
- We developed an algorithm for decomposition into Jordan blocks and proved that it works.

Semisimplification

- From now on we work in $p = 3$
- After semisimplification the J_1 and J_2 blocks become simple objects, L_1 and L_2. The J_3 vanish (intuitively this is because $3 = 0$)
- The J_2 merge to become odd vectors and the J_1 become even vectors.

Semisimplification

- How does $\mathfrak{gl}(4)$ semisimplify into $\mathfrak{gl}(2|1)$ under $t = -e_{43}$?



$4J_1$	$2J_2$
$2J_2$	$J_1 \oplus J_3$

L_1	L_1	L_2
L_1	L_1	L_2
L_2	L_2	L_1

Semisimplification

- Intuitively the semisimplified module should be a function of λ.
- If $\lambda = (a, b, c, d)$ then let $\overline{\lambda} = (a, b \mid c + d)$.
- Sometimes $L(\overline{\lambda})$ appears as a submodule in the semisimplification.
- The semisimplified module falls into one of 4 classes:

	$L(\overline{\lambda})$ **appears**	$L(\overline{\lambda})$ **does not appear**
Simple	$L(2, 2, 2, 2)$	$L(2, 2, 0, -2)$
Semisimple	$L(2, 2, 0, -1)$	$L(2, 0, 0, -2)$

- There is a special case of the simple module in which $L(\overline{\lambda})$ does not appear, called the zero map. This happens for $L(2, 0, -2, -4)$.

Semisimplification

- An example of $L(\overline{\lambda})$ appearing in a semisimple module.
- Consider $L(2, 2, 0, -1)$
- The semisimplification is $\overline{L(\lambda)} = L(2, 2 \mid -1) \oplus L(1, 0 \mid 2)$.

Semisimplification

- In all cases calculated we observed that the semisimplification of a simple module is always semisimple.
- This method can be generalized to higher rank and characteristic.
- We noticed that the module $L(p-1, 0, -(p-1), \ldots, -(n-2)(p-1))$ semisimplifies into the zero map.
- The semisimplification functor behaves like a characteristic p version of the Duflo-Serganova functor.

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Semisimplification

- An example of $L(\overline{\lambda})$ appearing with the semisimplification being simple.
- The **trivial representation** for $\mathfrak{sl}(4)$ is $L(2,2,2,2)$.
- After semisimplification we get $\overline{L(\lambda)} = L(2,2 \mid 1)$.
- As expected, adding the last two entries of λ works:

$$(2,2,2,2) \to (2,2,4) \to (2,2,1) \to (2,2 \mid 1).$$

Semisimplification

- An example of $L(\overline{\lambda})$ appearing with the semisimplification being simple.
- Consider the **natural representation**, $L(2, 1, 1, 1)$.
- We get $\overline{L(\lambda)} = L(2, 1 \mid 2)$.

$$(2, 1, 1, 1) \to (2, 1, 2) \to (2, 1 \mid 2).$$

Semisimplification

- An example of $L(\overline{\lambda})$ not appearing with the semisimplification being simple.
- Consider $L(2, 2, 0, -2)$
- We get $\overline{L(\lambda)} = L(1, -1 \mid 2)$. Notice how $L(2, 2 \mid -2)$ is not present.

Semisimplification

- An example of $L(\overline{\lambda})$ not appearing with the semisimplification being semisimple.
- Consider $L(2, 0, 0, -2)$
- We have $\overline{L(\lambda)} = L(-1, 2 \mid -1) \oplus L(2, -1 \mid -1)$.

Semisimplification

- There is one case in which the semisimplified module is the zero map.
- Consider $L(2, 0, -2, -4)$
- We have $L(2, 0, -2, -4) = 243J_3$.
- Since the J_3 vanish after semisimplification, we can conclude that the semisimplified module is the zero map.