Dynamical Functionals on Ancient ARF and AC Ricci Flows

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Mentor: Isaac Lopez

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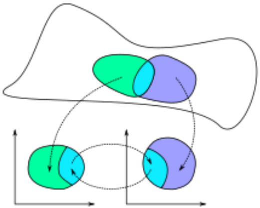
MIT PRIMES Conference

What's a differentiable manifold?



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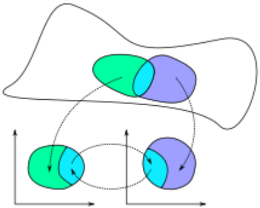

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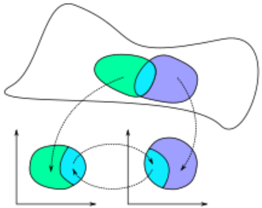

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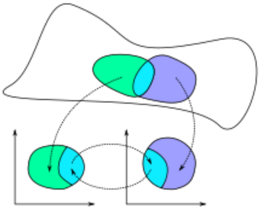

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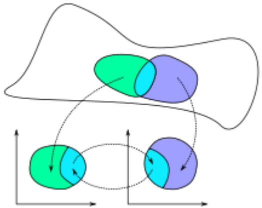

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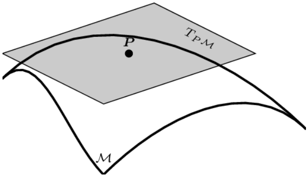
- Examples: Sphere, Torus, Klein Bottle, Mobius Strip, $\mathbb{R}^n$.

Tangent Space



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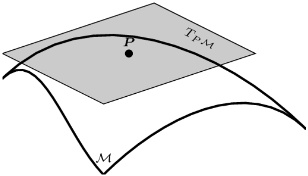

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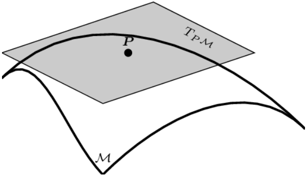

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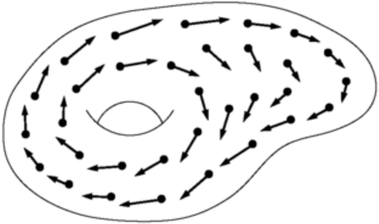
- A vector field assigns each point a tangent vector.
    - Nearby tangent vectors point in similar directions.

A Vector Field



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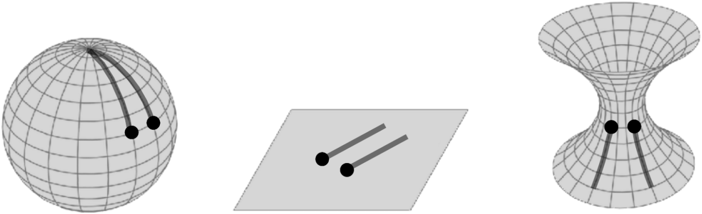
- Length of a curve $\gamma(t)$:

$$\int_0^1 \sqrt{g(\gamma'(t), \gamma'(t))}dt.$$

Ricci Curvature

- Geodesics: locally the shortest path between two points.



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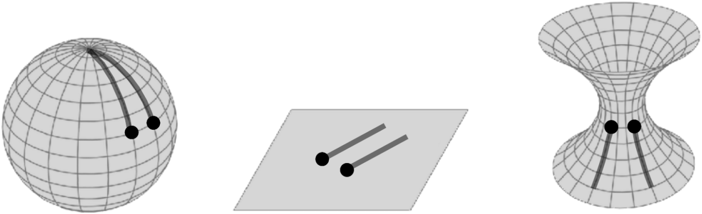
- Geodesics: locally the shortest path between two points.
- Ricci Curvature: how much nearby geodesics deviate from each other.



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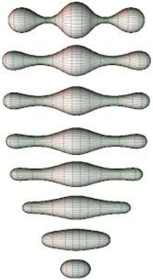

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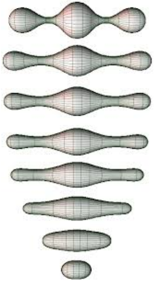
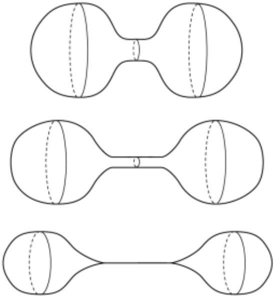


- Makes the manifold rounder while preserving topology.

Singularities



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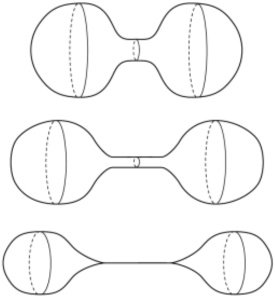

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- Cannot continue Ricci flow when singularity forms.

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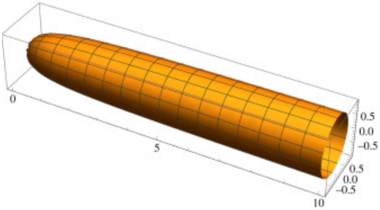

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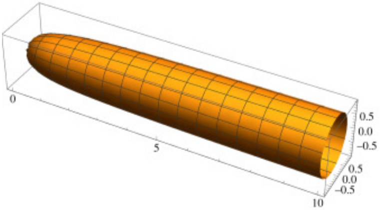

- Idea: cut out the Ricci Soliton before the singularity forms.

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- Only solved Millennium Prize Problem out of 7.

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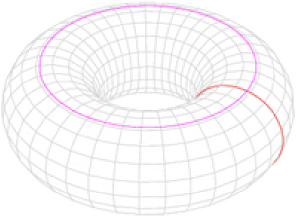

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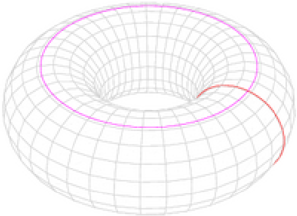

- Proof idea: if we ignore singularities, Ricci flow turns manifolds into spheres.

Perelman's Lambda Functional

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- Would be better if we could write $\lambda(g) = \mathcal{F}(g, f)$ where we know the behavior of f with respect to time.

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- *If there exists a pair of non-positive times (t_1, t_2) such that $t_1 < t_2$ and*

$$\lambda^\infty_{\mathrm{dyn}}(t_1) = \lambda^\infty_{\mathrm{dyn}}(t_2), \tag{2}$$

  then $(g(t))_{t\in[t_1,t_2]}$ is a Ricci-flat, steady gradient Ricci soliton.

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References

[1] Alix Deruelle and Tristan Ozuch. "A Lojasiewicz inequality for ALE metrics". In: *Ann. Sc. Norm. Super. Pisa, to appear* (2020).

[2] Robert Hashlofer and Reto Müller. "Dynamical stability and instability of Ricci-flat metrics". In: *Math. Ann.* 360 (2014), pp. 547–553.

[3] Yu Li. "Ricci flow on asymptotically Euclidean manifolds". In: *Geom. Topol.* 22.3 (2018), 1837–1891. ISSN: 1465-3060. DOI: 10.2140/gt.2018.22.1837. URL: http://dx.doi.org/10.2140/gt.2018.22.1837.

[4] Isaac M. Lopez and Tristan Ozuch. "Ancient and expanding spin ALE Ricci flows". In: *J. Funct. Anal.* 289.111062 (2025).