Improved Bounds for Novelty Games

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What Is a Novelty Game?

$$
\begin{array}{lcl}
P_1 & \boxed{\quad X_{11} \quad \cdots \quad X_{1k} \quad} & \longrightarrow Y_1 \\
 & & \\
P_2 & \boxed{\quad X_{21} \quad \cdots \quad X_{2k} \quad} & \longrightarrow Y_2 \\
\vdots & \vdots & \quad\;\; \vdots \\
P_p & \boxed{\quad X_{p1} \quad \cdots \quad X_{p,k-1} \quad X_{p,k} \quad} & \longrightarrow Y_p
\end{array}
$$

- p players, k inputs for each player
- $X_{i,j}, Y_i \in [N]$

The novelty game was proposed in 2024 by Lechine and Seiller [3].

How Do We Win?

Definition (Novelty Property)

At least one output is a new number not appearing among the inputs:

$$\{ Y_1, Y_2, \ldots, Y_p \} \setminus \{ X_{i,j} : 1 \le i \le p,\ 1 \le j \le k \} \neq \varnothing.$$

We win if this *novelty property* holds for **ALL** possible inputs.

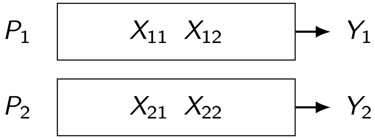
Definition

We denote bounds by

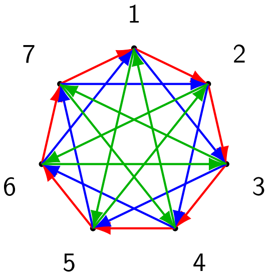
$$\mathfrak{B}(p,k) \;:=\; \min\{ N \in \mathbb{N} \;:\; \exists \text{ a winning strategy for } (p,k,N) \}.$$

Our goal is to reduce $\mathfrak{B}(p,k)$.

Connection to Paradoxical Tournaments



$X_{i,j}, Y_i \in [7]$
$p = 2, k = 2, N = 7$



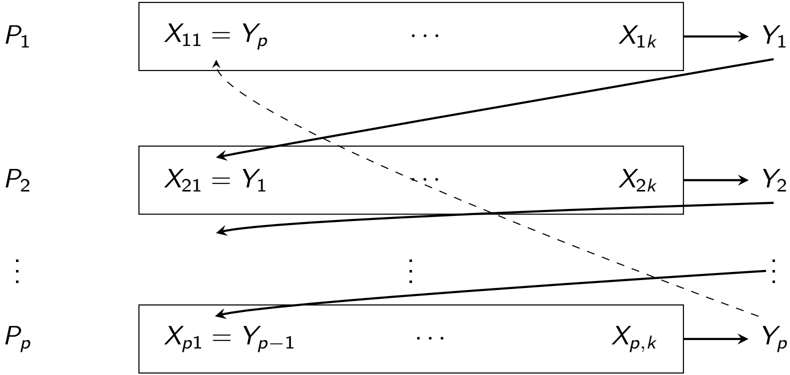
Lemma (Strategies $\leftrightarrow$ digraphs)

A successful strategy for (p, k, N) novelty games exists if and only if there is a directed graph G on N vertices such that G has no directed cycle of length $\leq p$.

The $(2, k, N)$ novelty game coincides with the classic k-paradoxical tournament problem [1, 2, 4].

Cycles in Novelty Games

The below is a cycle of length p for the (p, k) novelty game. The novelty property does **not** hold for this case.



- Parents, grandparents, ancestors

Ancestor-Bookkeeping Strategies

An *ancestor-bookkeeping* strategy has a fixed block decomposition of every $x \in [N]$ into *last-digit groups*

$$x \;=\; A_{p-1}(x)\, A_{p-2}(x) \,\cdots\, A_1(x)\, A_0(x),$$

with the following properties:

- For each $i \in \{0, 1, \ldots, p-1\}$, the level-$i$ block $A_i(x)$ consists of exactly k^i digits.
- For each $i \in \{1, \ldots, p-1\}$, the block $A_i(x)$ encodes information about level-i ancestors of x and is used to distinguish x from all necessary ancestors.
- The level-0 block $A_0(x)$ encodes unique information regarding the number x itself.

# of digits	k^{p-1}	k^{p-2}	$\cdots$	k^i	$\cdots$	k^2	k	1
number x	A_{p-1}	A_{p-2}	$\cdots$	A_i	$\cdots$	A_2	A_1	A_0

Original Algorithm and Bound

The original algorithm simply concatenates all ancestor groups of the inputs at the same level and selects one unused unique digit as the last digit to construct the output. The last digit of the output is different from all its p-level ancestors, guaranteeing novelty.

$$P_1 \quad \boxed{0123}\,\boxed{45}\,\boxed{6} + \boxed{\mathrm{EDCB}}\,\boxed{\mathrm{A9}}\,\boxed{8} \Longrightarrow \boxed{\mathrm{45A9}}\,\boxed{68}\,\boxed{7}$$

$$P_2 \quad \boxed{\mathrm{45A9}}\,\boxed{68}\,\boxed{7} + \boxed{3210}\,\boxed{\mathrm{BC}}\,\boxed{\mathrm{D}} \Longrightarrow \boxed{\mathrm{68BC}}\,\boxed{\mathrm{7D}}\,\boxed{5}$$

$$P_3 \quad \boxed{\mathrm{68BC}}\,\boxed{\mathrm{7D}}\,\boxed{5} + \boxed{6543}\,\boxed{21}\,\boxed{0} \Longrightarrow \boxed{\mathrm{7D21}}\,\boxed{50}\,\boxed{9}$$

For the $(3, 2)$ novelty game: $\mathfrak{B}_0(3, 2) \leq 15^7 \approx 171$ million.

Original general bound [3]

$$\mathfrak{B}_0(p, k) \leq f(k, p+1)^{f(k,p)},$$

where

$$f(k, p) = 1 + k + k^2 + \cdots + k^{p-1} = \frac{k^p - 1}{k - 1}.$$

Main Theorems

Lemma (Minimal Strategy Lemma)

The set of all outputs must be the same as the set of all inputs for a winning strategy on a minimum number of vertices. In other words, $R([N]) = [N]$.

Lemma (Range Reduction Lemma)

Let $X = [N]$ *have a winning strategy* S *whose range is* $Y = \{y_1, \dots, y_M\}$ *with* $M < N$. *Define* $X_2 = [M]$. *Then there is a bijection*

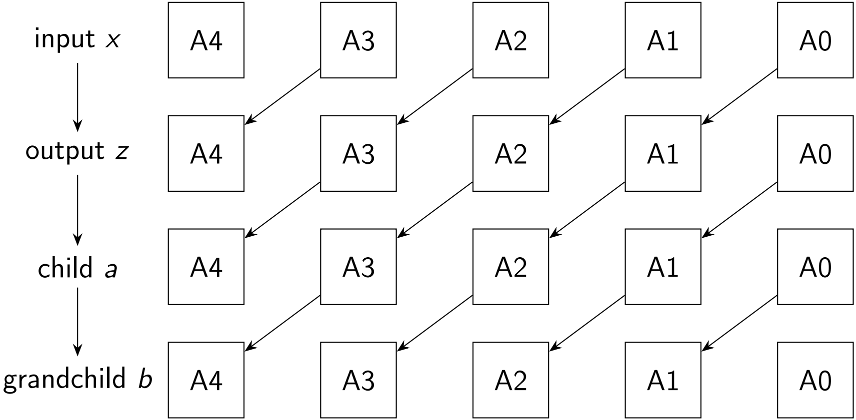
$$f : Y \to X_2, \quad f(y_i) = i,$$

and a corresponding winning strategy S_2 *on* X_2.

Implication: if we reduce the number of valid outputs for winning strategies, then the bound will be reduced.

Chain-Shifting Property

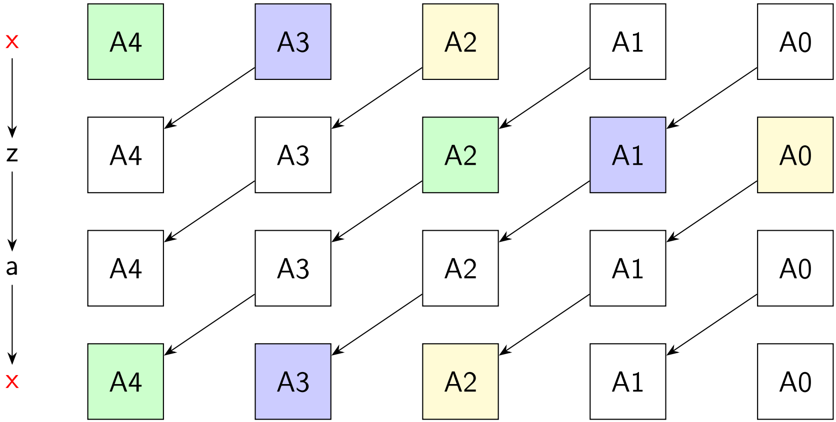
Ancestor groups of the number x are shifted left when converted by players from inputs to outputs.



Cycle Existence Condition

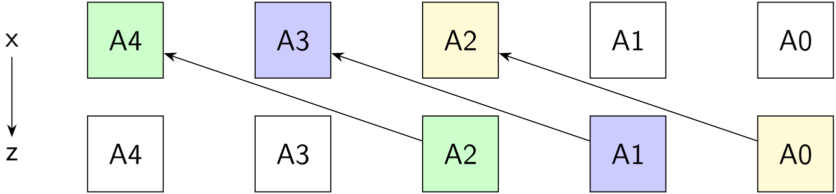
The input is x, the output is z, a is a child of z and parent of x.

Figure: The necessary condition for 3-cycle existence with 5 players



Cycle Avoidance Rules

We can avoid cycles by breaking the necessary condition.



Definition (Convertibility)

$A_i(x) \succ A_j(u)$ means that $A_i(x)$ eventually becomes part of $A_j(u)$ (the ancestor group of some descendant u) after $j - i > 0$ steps. Conversely, we write $A_i(x) \not\succ A_j(u)$ to indicate that such a conversion is not possible after $j - i > 0$ players.

For (p, k) novelty game, the rule to avoid cycles:

$$\begin{cases} A_0(z) \notin A_{p-1}(x) \\ A_0(z) \notin A_L(x), \text{where } \lfloor \frac{p}{2} \rfloor < L < p-1 \quad \text{if } \forall j > 0, A_j(z) \succ A_{j+L-1}(x) \end{cases}$$

Optimization 1: Input Avoidance

The last digit of the output number can be the same as one of the input numbers which does not appear in the ancestor digits of all inputs.

$$\boxed{0123}\ \boxed{45}\ \boxed{6} + \boxed{CBA9}\ \boxed{87}\ \boxed{6} \Longrightarrow \boxed{4587}\ \boxed{66}\ \boxed{6}$$

$$\boxed{0123}\ \boxed{45}\ \boxed{3} + \boxed{CBA9}\ \boxed{87}\ \boxed{6} \Longrightarrow \boxed{4587}\ \boxed{36}\ \boxed{6}$$

General Bound

$$\mathfrak{B}_1(p,k) \leq (f(k,p+1)-k)^{f(k,p)}$$

(3,2) Bound

$$\mathfrak{B}_1(3,2) \leq 13^7 \approx 63 \text{ million}$$

Optimization 2: Cycle Avoidance

The last digit of the output number is selected by following the cycle avoidance rules.

$$\boxed{0123}\ \boxed{45}\ \boxed{6} + \boxed{A987}\ \boxed{65}\ \boxed{4} \Longrightarrow \boxed{4565}\ \boxed{64}\ \boxed{4}$$

General Bound

$$\mathfrak{B}_2(p,k) \leq \mathfrak{D}(p,k)^{f(k,p)}, \text{ where}$$

$$\mathfrak{D}(p,k) = k^{p-s} f(k,s+1) - k^2 f(k,s) - \frac{kf(k,s)}{k-1} + \frac{sk^2}{k-1} + E + 1,$$

where $s = \lceil \frac{p}{2} \rceil - 1$, and E is a very small term.

(3,2) Bound

$$\mathfrak{B}_2(3,2) \leq 11^7 \approx 19 \text{ million}$$

Optimization 3: Reordering Ancestors

The last digit is the same as Optimization 2 of cycle avoidance, while the only difference is the digit order in each ancestor group $A_i(z)$ is fixed in non-decreasing order. As a result, some numbers like $4565644, 5546464$ become invalid outputs, and are replaced by the same valid output 4556464.

$$\boxed{0123}\,\boxed{45}\,\boxed{6} \;+\; \boxed{A987}\,\boxed{65}\,\boxed{4} \;\Longrightarrow\; \boxed{4556}\,\boxed{46}\,\boxed{4}$$

General Bound

$$\mathfrak{B}_3(p,k) \leq \prod_{i=0}^{p-1} \binom{\mathfrak{D}(p,k) + k^i - 1}{k^i}$$

(3,2) Bound

$$\mathfrak{B}_3(3,2) \leq \binom{11+4-1}{4} \cdot \binom{11+2-1}{2} \cdot 11 = 726726$$

Optimization 4: Merging Equivalent Ancestors

Each ancestor group is replaced by its equivalent ancestor group. $4565, 5546, 5646$ now become invalid ancestor group digits and are replaced by the same equivalent ancestor group 4456.

$$\boxed{0123}\ \boxed{45}\ \boxed{6} + \boxed{A987}\ \boxed{65}\ \boxed{4} \implies \boxed{4456}\ \boxed{46}\ \boxed{4}$$

$$\boxed{0123}\ \boxed{45}\ \boxed{4} + \boxed{789A}\ \boxed{46}\ \boxed{6} \implies \boxed{4456}\ \boxed{46}\ \boxed{4}$$

General Bound

$$\mathfrak{B}_4(p,k) \leq \prod_{i=0}^{p-1} \sum_{j=1}^{k^i} \binom{\mathfrak{D}(p,k)}{j}$$

(3,2) Bound

$$\mathfrak{B}_4 \leq \left[\binom{11}{1} + \binom{11}{2} + \binom{11}{3} + \binom{11}{4}\right] \cdot \left[\binom{11}{1} + \binom{11}{2}\right] \cdot 11 = 407286$$

Optimization 5: Merging Neighboring Ancestors

Each ancestor group is replaced by its "neighboring" ancestor group. $1413, 1314, 0304$ become invalid ancestor group digits and are replaced by the same neighboring ancestor group 0134.

$$\boxed{0123}\ \boxed{03}\ \boxed{6} + \boxed{789A}\ \boxed{14}\ \boxed{0} \implies \boxed{0134}\ \boxed{06}\ \boxed{4}$$

$$\boxed{0123}\ \boxed{14}\ \boxed{6} + \boxed{789A}\ \boxed{13}\ \boxed{6} \implies \boxed{0134}\ \boxed{06}\ \boxed{4}$$

General Bound

$$\mathfrak{B}_5(p,k) \leq \prod_{i=0}^{p-1} \binom{\mathfrak{D}(p,k)}{k^i}$$

(3,2) Bound

$$\mathfrak{B}_5(3,2) \leq \binom{11}{4}\binom{11}{2} \cdot 11 = 199650$$

Optimization 6: Pruning Unused Outputs

In the $(3, 2)$ novelty game, it is impossible to reach $A_0(z) = 9$ or $A_0(z) = A$ if $A_1(z)$ contains the hexadecimal digit 9 or A. Therefore, we can remove such unused outputs from the set of vertices to reduce the bound.

General Bound

$$\mathfrak{B}_6(p, k) \leq \prod_{i=0}^{p-1} \binom{\mathfrak{D}(p, k)}{k^i} - \binom{\mathfrak{D}(p, k) - 1}{k - 1} \cdot (k^{p-1} - k^2 + k) \cdot \prod_{i=2}^{p-1} \binom{\mathfrak{D}(p, k)}{k^i}$$

(3,2) Bound

$$\mathfrak{B}_5(3, 2) \leq 199650 - 6600 = 193050$$

Summary of Results

General Bound

$$\mathfrak{B}_5(p,k) \leq \prod_{i=0}^{p-1} \binom{\mathfrak{D}(p,k)}{k^i}$$

Table: Reduction of the $\mathfrak{B}(3,2)$ bound

$(3,2)$	opt0	opt1	opt2	opt3	opt4	opt5	opt6
bound	15^7	13^7	11^7	726726	407286	199650	193050
ratio%	100	36.7	11.4	0.425	0.238	0.117	0.113

$$f_5(p,k) = \frac{\mathfrak{B}_0(p,k)}{\mathfrak{B}_5(p,k)} \approx e^{2k^{\frac{p}{2}}} \prod_{i=0}^{p-1} (k^i)!$$

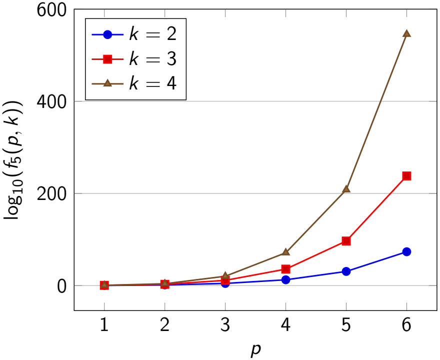


Figure: Log-scale plot of $f_5(p,k)$ for $k = 2$, $k = 3$, and $k = 4$.

Further Directions

Existing lower bound [3]:

$$\mathfrak{B}(p, k) \geq pk + 1$$

We have proved that for ancestor-bookkeeping strategies of the (p, k) novelty game:

$$\mathfrak{B}(p, k) \geq k^p + 1$$

Two conjectures on lower bounds:

- For ancestor-bookkeeping strategies with $p > 2$ players, $\mathfrak{B}(p, k) \geq \prod_{i=0}^{p-1} \binom{k^p+1}{k^i}$.
- $\mathfrak{B}(p, k) \geq \mathfrak{B}(2, k)^{k^{p-2}}$ for all winning strategies.

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