Measures on Oligomorphic Groups

Thanh Can
Mentor: Dr. Thomas Rüd

Jackson-Reed High School

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Oligomorphic Groups

Definition (Oligomorphic group)

An oligomorphic group is a group G acting on a set Ω such that G has finitely many orbits on Ω^n for all $n \geq 0$.

Example

Let G be a group and Ω be a finite set. For $n \geq 0$, the action of G on Ω^n induces at most $|\Omega|^n$ orbits, where $|\Omega|^n$ is finite. Thus, G is an oligomorphic group with its respective action on Ω.

Non-example

Consider the action of $(\mathbb{R}, +)$ on $\mathbb{R}$ given by $r \cdot x = r + x$ for reals r, x. The action has one orbit on $\mathbb{R}$, yet uncountably many orbits on $\mathbb{R}^2$.

Oligomorphic Groups

Example (S_∞)

Let S_∞ be the infinite symmetric group acting on $\Omega = \{1, 2, \dots\}$. Each orbit on Ω^n corresponds to a partition of $\{1, \dots, n\}$ where each subset contains indices whose corresponding elements are equal. Below is an example for $n = 5$, where equal entries have the same color:

$$(\omega_1, \ \omega_2, \ \omega_3, \ \omega_4, \ \omega_5) \xrightarrow{\sigma \in S_\infty} (\omega_1', \ \omega_2', \ \omega_3', \ \omega_4', \ \omega_5')$$

$\Updownarrow$ orbit only depends on equality pattern

$$\{\{1, 4\}, \{2\}, \{3, 5\}\}$$

Oligomorphic Groups

Example $(\mathrm{Aut}(\mathbb{Q}, <))$

Let $\mathrm{Aut}(\mathbb{Q}, <)$ denote the group of order-preserving bijections $f : \mathbb{Q} \to \mathbb{Q}$ acting on $\Omega = \mathbb{Q}$. Below is an example of an orbit for $n = 5$:

$$\begin{array}{c} < \quad > \\ (\omega_1, \omega_2, \omega_3, \omega_4, \omega_5) \\ < \end{array} \xrightarrow{f \in \mathrm{Aut}(\mathbb{Q}, <)} \begin{array}{c} < \quad > \\ (\omega'_1, \omega'_2, \omega'_3, \omega'_4, \omega'_5) \\ < \end{array}$$

orbit depends on equality and ordering pattern

$$\Updownarrow$$

$$\{1, 4\} < \{3, 5\} < \{2\}$$

Combinatorial Interpretation

- Fix an oligomorphic group G and a set Ω.
- For each $n \geq 0$, consider each orbit $R \subseteq \Omega^n$ as an n-ary relation on Ω i.e. R takes input elements $x_1, \ldots, x_n \in \Omega$ and outputs true if $(x_1, \ldots, x_n) \in R$ and false otherwise.
- The finite subsets of Ω equipped with the relations induced by the orbits form a class of finite relational structures.

Example ($\mathrm{Aut}(\mathbb{Q}, <)$)

The action of $\mathrm{Aut}(\mathbb{Q}, <)$ induces three 2-ary relations on $\mathbb{Q}$: $R_>$, $R_<$, and $R_=$. If $a > b$, then $R_>(a, b)$ is true while $R_<(a, b)$ and $R_=(a, b)$ are false.

Remark

The 2-ary relations on $\mathbb{Q}$ determine all n-ary relations for all $n \geq 2$.

“Gluing” Finite Relational Structures

Example (Finite sets with a total order)

$$
\begin{tikzcd}
& \begin{cases} a' < b' < c < c' \\ a' < b' < c' < c \\ a' < b' < c = c' \end{cases} & \\
a' < b' < c \arrow[ur, hook] & & a' < b' < c' \arrow[ul, hook'] \\
& a < b \arrow[ul, hook'] \arrow[ur, hook] &
\end{tikzcd}
$$

Amalgamations

Definition (Amalgamation)

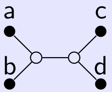
Let $i : Z \to X$ and $j : Z \to Y$ be embeddings. An amalgamation of X and Y over Z is a triple (Z', i', j') such that $i' : Y \to Z'$ and $j' : X \to Z'$ are jointly surjective embeddings into Z' and the following diagram commutes:

$$\begin{tikzcd}
 & Z' & \\
X \arrow[ur, hook, "j'"] & & Y \arrow[ul, hook', "i'"'] \\
 & Z \arrow[ul, hook', "i"] \arrow[ur, hook, "j"'] &
\end{tikzcd}$$

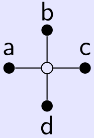
Treelike Structures

Definition (Tree structure)

A tree structure is a set of leaves of a tree T (denoted $L(T)$) along with a quartic relation R such that $R(a, b; c, d)$ is true iff a, b, c, d are distinct leaves and the simple path from a to b does not intersect the simple path from c to d.



$R(a, b; c, d)$ is true



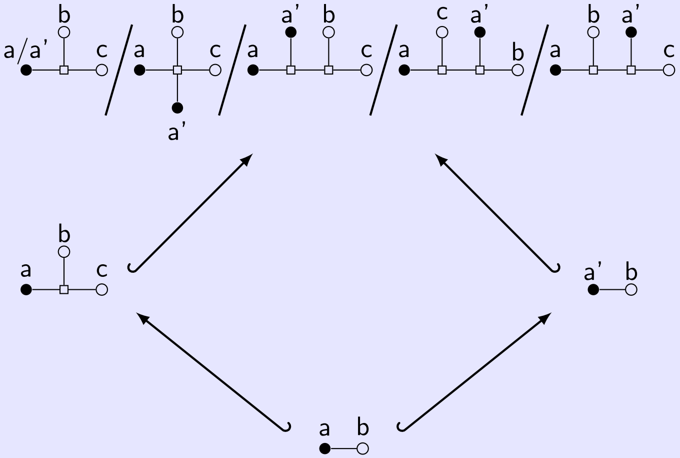
$R(a, b; c, d)$ is false

Definition (n-coloring)

Let T be a tree and $L(T)$ be the set of the leaves of T. We define an n-coloring of T to be a function $\sigma : L(T) \to \{1, \ldots, n\}$.

Amalgamations

Example (2-colored trees)



Measures

Definition (Measure)

A measure on a class of finite structures $\mathfrak{F}$ is a rule μ that assigns each embedding $i : Y \to X$ a quantity $\mu(i)$ valued in a commutative ring k such that the following conditions hold:

1. If i is an isomorphism then $\mu(i) = 1$.
2. If $i : X \to Y$ and $j : Y \to Z$ are embeddings, then $\mu(j \circ i) = \mu(j)\mu(i)$.
3. Let $i : Z \to X$ and $j : Z \to Y$ be embeddings, and $i'_\alpha : Y \to Z'_\alpha$ be all amalgamations of X and Y over Z for $\alpha = 1, \ldots, n$. Then $\mu(i) = \sum_{\alpha=1}^{n} \mu(i'_\alpha)$.

Remark

Measures are key for constructing new tensor categories out of classes of finite relational structures.

Measures

Example (Finite sets with a total order)

$$
\begin{tikzcd}
 & \begin{cases} a' < b' < c < c' \ (\alpha = 1) \\ a' < b' < c' < c \ (\alpha = 2) \\ a' < b' < c = c' \ (\alpha = 3) \end{cases} & \\
a' < b' < c \arrow[ur, hook, "j'_\alpha"] & & a' < b' < c' \arrow[ul, hook', "i'_\alpha"'] \\
 & a < b \arrow[ul, hook', "i"] \arrow[ur, hook, "j"'] &
\end{tikzcd}
$$

- $\mu(i) = \mu(i'_1) + \mu(i'_2) + \mu(i'_3)$
- $\mu(j) = \mu(j'_1) + \mu(j'_2) + \mu(j'_3)$

Results on Measures

Theorem (Harman–Snowden, 2022)

There is a unique family of $\mathbb{C}$-valued measures on the class of finite sets where if $i : Y \to X$ is an injection of sets with respective cardinalities m and n (where $m \leq n$), then

$$\mu_t(i) = (t - m)(t - m - 1) \cdots (t - n + 1)$$

given any $t \in \mathbb{C}$.

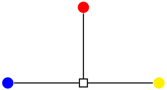
Remark

The measure on any embedding is determined by

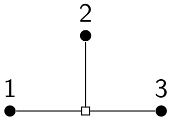
$$\mu_t(\emptyset \to \bullet) = t.$$

Cameron's Treelike Objects

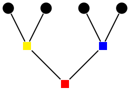
- Cameron proposed the following treelike objects, which become treelike structures by imposing the relation R:



n-colored trees



Trees with a total order



Node-colored rooted binary trees

- The goal of my project was to classify measures on these classes of treelike structures.

Results on Measures

Theorem (Harman–Nekrasov–Snowden, 2023)

There are two one-parameter $\mathbb{C}$-valued families of measures on the class of tree structures.

Theorem (C.)

There are no measures on the class of n-colored tree structures for all $n \geq 2$.

Theorem (C.)

There are no measures on the class of tree structures with a total order.

Results on Measures

Theorem (C.)

Given a directed rooted tree T with edges labeled by $\{1, \ldots, n\}$ and a distinguished vertex v (which could be the root), there is a corresponding $\mathbb{Z}\left[\frac{1}{2}\right]$-valued measure $\mu_{T,v}$ on the class of node-colored rooted binary tree structures with n colors.

Corollary (C.)

There are $(2n+2)^n$ measures on the class of node-colored rooted binary tree structures with n colors for all $n \geq 1$.

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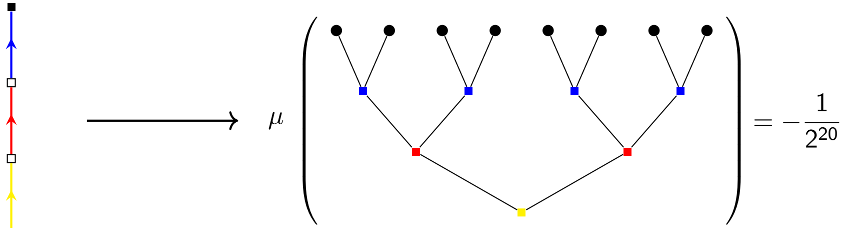
Tree $\Longleftrightarrow$ Measure



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