Classifying Yetter-Drinfeld Modules over the Pansera Algebras H_{2n^2}

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Table of Contents

Crash Course in Hopf Algebras

Algebras and Coalgebras

- An algebra is a vector space A with a ring structure:
  - Has *multiplication* $\mu : A \otimes A \to A$, *unit* $\eta : \Bbbk \to A$ satisfying:

$$\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "\mathrm{id}\otimes\mu"] \arrow[d, "\mu\otimes\mathrm{id}"'] & A \otimes A \arrow[d, "\mu"] \\
A \otimes A \arrow[r, "\mu"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
 & A \otimes A \arrow[d, "\mu"] & \\
\Bbbk \otimes A \arrow[ur, "\eta\otimes\mathrm{id}"] \arrow[r, "\cong"'] & A & A \otimes \Bbbk \arrow[ul, "\mathrm{id}\otimes\eta"'] \arrow[l, "\cong"]
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- A coalgebra is ”dual” to an algebra, reversing the maps:
    - Has *comultiplication* $\Delta : A \to A \otimes A$, *counit* $\varepsilon : A \to \Bbbk$ satisfying:

$$\begin{tikzcd}
C \otimes C \otimes C & C \otimes C \arrow[l, "\mathrm{id}\otimes\Delta"'] \\
C \otimes C \arrow[u, "\Delta\otimes\mathrm{id}"] & C \arrow[l, "\Delta"] \arrow[u, "\Delta"']
\end{tikzcd}
\qquad
\begin{tikzcd}
 & C \otimes C \arrow[dl, "\varepsilon\otimes\mathrm{id}"'] \arrow[dr, "\mathrm{id}\otimes\varepsilon"] & \\
\Bbbk \otimes C & & C \otimes \Bbbk \\
 & C \arrow[uu, "\Delta"] \arrow[ul, "\cong"] \arrow[ur, "\cong"'] &
\end{tikzcd}$$

    - Think of Δ as a splitting map, ε as a ”projection” map.

Hopf Algebras

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- When the maps Δ, ε are algebra maps, we say H is a **bialgebra**.
- An *antipode* map $S : H \to H$ is a map satisfying:

$$\begin{tikzcd} H \arrow[r, "\Delta"] \arrow[d, "\eta\circ\varepsilon"'] & H\otimes H \arrow[d, "S\otimes \mathrm{id}_H"] \\ H & H\otimes H \arrow[l, "\mu"] \end{tikzcd} \qquad \begin{tikzcd} H \arrow[r, "\Delta"] \arrow[d, "\eta\circ\varepsilon"'] & H\otimes H \arrow[d, "\mathrm{id}_H\otimes S"] \\ H & H\otimes H \arrow[l, "\mu"] \end{tikzcd}$$

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  - Kind of an inverse map!
- When a bialgebra H has an antipode S, we say H is a **Hopf algebra**.

Examples of Hopf Algebras

- Let G be a finite group.
- Group Algebra: Take $\Bbbk G = \{\sum k_i g_i \mid k_i \in \Bbbk, g_i \in G\}$.
  - Multiplication by $\mu(g_1, g_2) = g_1 g_2$, extended linearly.
  - Comultiplication by $\Delta(g) = g \otimes g$, extended linearly.
  - Unit by $\eta(1) = e$, counit by $\varepsilon(g) = 1$, both extended linearly.
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- Dual of a Group Algebra: Take: $(\mathbb{k}G)^* = \{f : \mathbb{k}G \to \mathbb{k} \mid \text{ is linear}\}$.
  - This has a basis by $f_g : \mathbb{k}G$ defined by $f_g(h) = \delta_{gh}$.
  - Multiplication by $\mu(f_{g_1}, f_{g_2})(h) = f_{g_1}(h) f_{g_2}(h)$.
  - Comultiplication by $\Delta(f_g)(h) = \sum_{\substack{g_1, g_2 \in G \\ g_1 g_2 = g}} f_{g_1} \otimes f_{g_2}$.
  - Unit by $1 \to \mathrm{id}$, counit by $f_g \to \delta_{ge}$, both extended linearly.
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  - Antipode by $f_g \to f_{g^{-1}}$.
- Key idea: Dual of a Group Algebra has a commutative multiplication

Pansera Algebras H_{2n^2}

- The Pansera algebras H_{2n^2} are a very interesting Hopf algebra.
  - Small enough to work with easily.
  - Still exhibits lots of interesting behavior.
  - Good "toy" example to understand general phenomena.

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- Generated by x, y, z, q a primitive n root of unity with:

$$x^n = y^n = 1, xy = yx, zx = yz, zy = xz, z^2 = \frac{1}{n}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} q^{-ij}x^i y^j.$$

- Comultiplication is given by:

$$\Delta(x) = x \otimes x, \Delta(y) = y \otimes y, \Delta(z) = \frac{1}{n}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} q^{-ij}(x^i z) \otimes (y^j z).$$

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- $\{x^i y^j\}$ forms a copy of $\mathbb{k}[\mathbb{Z}_n \times \mathbb{Z}_n]$, $\{x^i y^j z\}$ is a "twisted" copy.

Modules, Comodules, Yetter-Drinfeld Modules

Modules and Comodules

- An A-module is a v.s V with a *action* $\rho : A \otimes V \to V$ satisfying:

$$
\begin{array}{ccc}
A \otimes A \otimes V & \xrightarrow{\mu \otimes \mathrm{id}_V} & A \otimes V \\
{\scriptstyle \mathrm{id}_A \otimes \rho}\downarrow & & \downarrow{\scriptstyle \rho} \\
A \otimes V & \xrightarrow[\rho]{} & V
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- A C-comodule is a v.s V with coaction $\delta : V \to C \otimes V$ satisfying:

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- We can think of a coaction as a factoring map, sending $v \to \sum_i c_i \otimes v_i$

Yetter-Drinfeld Modules

- What happens when we have V both a module and a comodule?
- When these satisfy a certain compatibility condition, we say we have a *Yetter-Drinfeld module*.

Yetter-Drinfeld Modules

- What happens when we have V both a module and a comodule?
- When these satisfy a certain compatibility condition, we say we have a *Yetter-Drinfeld module*.
- Why do we care about these?
  - Encodes "combinatorial" information about the Hopf algebra.
  - The Andruskiewitsch–Schneider program provides a method for classifying finite-dimensional Hopf algebras over H_{2n^2}.
  - Analogue of classifying groups with a given quotient group.
  - First ingredient in above method is the Yetter-Drinfeld Modules.

Problem and Results

Research Question + Motivation

$\boxed{\textbf{Question: What are the simple Hopf algebras over } H_{2n^2}\textbf{?}}$

Why ask this?

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Why ask this?

- Hopf algebras are important tools.
    - In quantum mechanics, operators $\hat{x}, \hat{p}$ do not commute.
    - Hopf algebras encode this non-commutative multiplication of functions.
    - Concrete applications in quantum computing, TQFTs.

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  - In quantum mechanics, operators $\hat{x}, \hat{p}$ do not commute.
  - Hopf algebras encode this non-commutative multiplication of functions.
  - Concrete applications in quantum computing, TQFTs.
- Classification of Hopf algebras is a fundamental question.
  - Unfortunately, little is known.
  - Classifying Hopf algebras over H_{2n^2} already advances classification.
  - Insights gained could be used in more general classification programs.

Our Results

Theorem (2025, Prasad, Pollastri, Plavnik, Spencer)

The following is a complete list of simple comodules over H_{2n^2}:

1. *There are n^2 one-dimensional simple comodules.*
2. *There is one n-dimensional simple comodule.*

In particular, the category of comodules over H_{2n^2} can be realized as a Tambara-Yamagami category.

Theorem (2025, Prasad, Pollastri, Plavnik, Spencer)

The following is a complete list of simple Yetter-Drinfeld modules over H_{2n^2}:

1. *There are $2n^2$ 1-dimensional simple Yetter-Drinfeld modules.*
2. *There are $\frac{n^2(n^2-1)}{2}$ 2-dimensional simple Yetter-Drinfeld modules.*
3. *There are $2n^2$ n-dimensional simple Yetter-Drinfeld modules.*

In particular, the category of Yetter-Drinfeld modules over H_{2n^2} can be realized as the center of a Tambara-Yamagami category.

Future Work

Future work includes:

- Computing the fusion rules of ${}^{H_{2n^2}}_{H_{2n^2}}\mathcal{YD}$
- Applying this to determine quasitriangular structures over H_{2n^2}
- Use this to find the Nichols algebras over H_{2n^2}
- $\boxed{\text{Ultimately classify Hopf algebras over } H_{2n^2}}$.

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THANK YOU FOR LISTENING!

Questions?

References i

N. Andruskiewitsch and H.-J. Schneider.
On the classification of finite-dimensional pointed hopf algebras.
Annals of Mathematics, 171(1):375–417, 2010.

N. Andruskiewitsch and C. Vay.
Finite dimensional hopf algebras over the dual group algebra of the symmetric group in three letters.
Communications in Algebra, 39(12):4507–4517, 2011.

I. Heckenberger and H.-J. Schneider.
Hopf Algebras and Root Systems, volume 247 of Mathematical Surveys and Monographs.
American Mathematical Society, Providence, RI, 2020.

References ii

C. Lomp.
Generalized kac-paljutkin algebras, 2025.

D. Pansera.
A class of semisimple hopf algebras acting on quantum polynomial algebras, 2017.

Y. Shi.
Finite-dimensional hopf algebras over the kac–paljutkin algebra h_8.
Revista de la Unión Matemática Argentina, 60(1):265–298, 2019.