# Classifying Yetter-Drinfeld Modules over the Pansera Algebras $H_{2n^2}$

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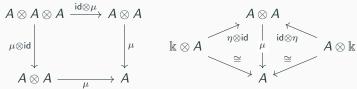
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**Crash Course in Hopf Algebras** 

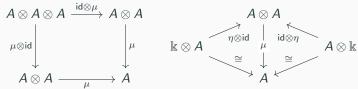
# **Algebras and Coalgebras**

- An algebra is a vector space A with a ring structure:
  - Has multiplication  $\mu: A \otimes A \to A$ , unit  $\eta: \mathbb{k} \to A$  satisfying:

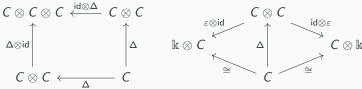


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- A coalgebra is "dual" to an algebra, reversing the maps:
  - Has comultiplication  $\Delta: A \to A \otimes A$ , counit  $\varepsilon: A \to \mathbb{k}$  satisfying:

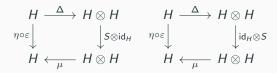


ullet Think of  $\Delta$  as a splitting map,  $\varepsilon$  as a "projection" map.

• When we have a v.s H with an algebra structure  $\mu, \eta$  and a coalgebra structure  $\Delta, \varepsilon$ , we want some compatibility between them.

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- An antipode map  $S: H \rightarrow H$  is a map satisfying:



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- Kind of an inverse map!
- When a bialgebra *H* has an antipode *S*, we say *H* is a **Hopf algebra**.

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# **Examples of Hopf Algebras**

- Let *G* be a finite group.
- Group Algebra: Take  $\mathbb{k}G = \{\sum k_i g_i \mid k_i \in \mathbb{k}, g_i \in G\}.$ 
  - Multiplication by  $\mu(g_1, g_2) = g_1g_2$ , extended linearly.
  - Comultiplication by  $\Delta(g) = g \otimes g$ , extended linearly.
  - Unit by  $\eta(1) = e$ , counit by  $\varepsilon(g) = 1$ , both extended linearly.
  - Antipode by  $S(g) = g^{-1}$ .

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- Dual of a Group Algebra: Take:  $(\Bbbk G)^* = \{f : \Bbbk G \to \Bbbk \mid \text{ is linear}\}.$ 
  - This has a basis by  $f_g : \Bbbk G$  defined by  $f_g(h) = \delta_{gh}$ .
  - Multiplication by  $\mu(f_{g_1}, f_{g_2})(h) = f_{g_1}(h)f_{g_2}(h)$ .
  - Comultiplication by  $\Delta(f_g)(h) = \sum_{g_1,g_2 \in G} f_{g_1} \otimes f_{g_2}$ .
  - Unit by  $1 \to \text{id}$ , counit by  $f_g \to \delta_{ge}$ , both extended linearly.
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- Key idea: Dual of a Group Algebra has a commutative multiplication

# Pansera Algebras $H_{2n^2}$

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  - Small enough to work with easily.
  - Still exhibits lots of interesting behavior.
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- Generated by x, y, z, q a primitive n root of unity with:

$$x^{n} = y^{n} = 1, xy = yx, zx = yz, zy = xz, z^{2} = \frac{1}{n} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} q^{-ij} x^{i} y^{j}.$$

• Comultiplication is given by:

$$\Delta(x) = x \otimes x, \Delta(y) = y \otimes y, \Delta(z) = \frac{1}{n} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} q^{-ij}(x^i z) \otimes (y^j z).$$

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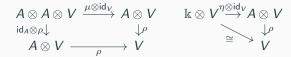
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•  $\{x^i y^j\}$  forms a copy of  $\mathbb{k}[\mathbb{Z}_n \times \mathbb{Z}_n]$ ,  $\{x^i y^j z\}$  is a "twisted" copy.

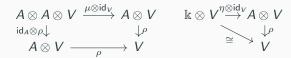
Modules, Comodules,

**Yetter-Drinfeld Modules** 

• An A-module is a v.s V with a action  $\rho: A \otimes V \to V$  satisfying:

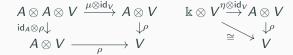


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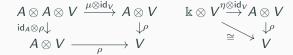
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$$\begin{array}{cccc} V & \xrightarrow{\delta} & C \otimes V & & V \xrightarrow{\delta} & C \otimes V \\ \downarrow \delta \downarrow & & \downarrow \Delta \otimes \mathrm{id}_V & & \swarrow & \downarrow \varepsilon \otimes \mathrm{id}_V \\ C \otimes V & \xrightarrow{\mathrm{id}_C \otimes \delta} & C \otimes C \otimes V & & & \Bbbk \otimes V \end{array}$$

• We can think of a coaction as a factoring map, sending  $v \to \sum_i c_i \otimes v_i$ 

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#### Yetter-Drinfeld Modules

- What happens when we have V both a module and a comodule?
- When these satisfy a certain compatibility condition, we say we have a *Yetter-Drinfeld module*.

#### Yetter-Drinfeld Modules

- ullet What happens when we have V both a module and a comodule?
- When these satisfy a certain compatibility condition, we say we have a Yetter-Drinfeld module.
- Why do we care about these?
  - Encodes "combinatorial" information about the Hopf algebra.
  - The Andruskiewitsch–Schneider program provides a method for classifying finite-dimensional Hopf algebras over  $H_{2n^2}$ .
  - Analogue of classifying groups with a given quotient group.
  - First ingredient in above method is the Yetter-Drinfeld Modules.

**Problem and Results** 

### Research Question + Motivation

Question: What are the simple Hopf algebras over  $H_{2n^2}$ ?

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- Hopf algebras are important tools.
  - In quantum mechanics, operators  $\hat{x}$ ,  $\hat{p}$  do not commute.
  - Hopf algebras encode this non-commutative multiplication of functions.
  - Concrete applications in quantum computing, TQFTs.

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- Hopf algebras are important tools.
  - In quantum mechanics, operators  $\hat{x}$ ,  $\hat{p}$  do not commute.
  - Hopf algebras encode this non-commutative multiplication of functions
  - Concrete applications in quantum computing, TQFTs.
- Classification of Hopf algebras is a fundamental question.
  - Unfortunately, little is known.
  - Classifying Hopf algebras over  $H_{2n^2}$  already advances classification.
  - Insights gained could be used in more general classification programs.

#### **Our Results**

#### Theorem (2025, Prasad, Pollastri, Plavnik, Spencer)

The following is a complete list of simple comodules over  $H_{2n^2}$ :

- 1. There are  $n^2$  one-dimensional simple comodules.
- 2. There is one n-dimensional simple comodule.

In particular, the category of comodules over  $H_{2n^2}$  can be realized as a Tambara-Yamagami category.

#### Theorem (2025, Prasad, Pollastri, Plavnik, Spencer)

The following is a complete list of simple Yetter-Drinfeld modules over  $H_{2n^2}$ :

- 1. There are  $2n^2$  1-dimensional simple Yetter-Drinfeld modules.
- 2. There are  $\frac{n^2(n^2-1)}{2}$  2-dimensional simple Yetter-Drinfeld modules.
- 3. There are  $2n^2$  n-dimensional simple Yetter-Drinfeld modules.

In particular, the category of Yetter-Drinfeld modules over  $H_{2n^2}$  can be realized as the center of a Tambara-Yamagami category.

#### **Future Work**

#### Future work includes:

- Computing the fusion rules of  $H_{2n^2}\mathcal{YD}$
- Applying this to determine quasitriangular structures over  $H_{2n^2}$
- Use this to find the Nichols algebras over  $H_{2n^2}$
- Ultimately classify Hopf algebras over  $H_{2n^2}$ .

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  - Many thanks to the MIT-PRIMES Program for organizing this mentorship.
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  - Many thanks to my parents and friends for supporting me throughout the research process.

### Questions

# THANK YOU FOR LISTENING! Questions?

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