Generalizations of the Zesting Construction for Fusion Categories

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Agenda

1. Fusion categories using example of vector spaces
2. String diagrams
3. Braidings
4. Zesting construction
5. Generalizing zesting construction

Fusion Categories

Categorical Structure:

- Objects
  Finite dimensional vector spaces over field $\Bbbk$
- Maps between objects with associative composition and identity map of each object
  Linear maps between vector spaces, which we can compose

Abelian Structure:

- Direct sum of objects
  Direct sum of vector spaces. e.g. $\Bbbk^3 \oplus \Bbbk^5 = \Bbbk^8$

Monoidal Structure:

- Tensor product of objects, which distributes over direct sum
  Tensor product of vector spaces, which distributes over direct sum
- Associators $(V \otimes U) \otimes W \cong V \otimes (U \otimes W)$ and unit constraints $\mathbf{1} \otimes V \cong V$ and $V \otimes \mathbf{1} \cong V$ satisfying the triangle and pentagon axioms
  Canonical associators $a_{V,U,W} : (V \otimes U) \otimes W \cong V \otimes (U \otimes W)$ and unit constraints $\Bbbk \otimes V \cong V$ and $V \otimes \Bbbk \cong V$ $\Bbbk$

Fusion Categories

Rigidity

- Objects have left and right duals with evaluation and coevaluation morphisms
  The dual of vector space V is the space of linear maps $V \to \Bbbk$

Semisimplicity

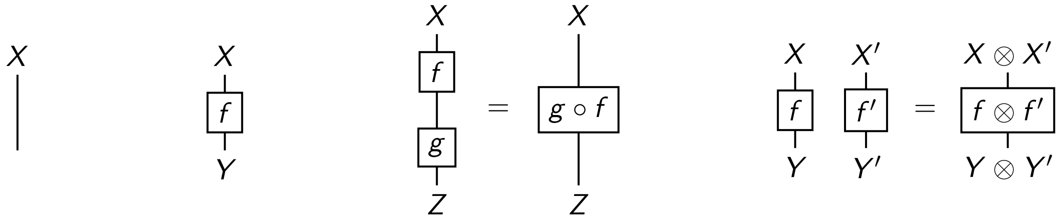
- An object X of an abelian category is **simple** if its only subobjects are $\mathbf{0}$ and X.
  The only simple vector space is $\Bbbk$
- Objects are **semisimple** if they factor as a direct sum of simple objects
  Vector spaces can be factored as a direct sum of $\Bbbk$

Definition

A **fusion category** is a semisimple rigid monoidal abelian category $\mathcal{C}$. [a].

[a] Also, it needs to be locally finite, k-linear, not equivalent to a direct sum of at least two nonzero multitensor categories, $\mathrm{End}_{\mathcal{C}}(\mathbf{1})$ is the base field k, and is "finite", which requires many more conditions...

String diagrams

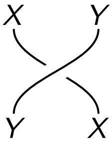


X $\qquad$ $f : X \to Y$ $\qquad$ $g \circ f : X \to Z$ $\qquad$ $f \otimes f' : X \otimes X' \to Y \otimes Y'$

Braidings

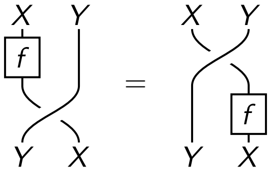
Braidings

- braidings $V \otimes U \cong U \otimes V$
  canonical braidings $V \otimes U \cong U \otimes V : v \otimes u \mapsto u \otimes v$

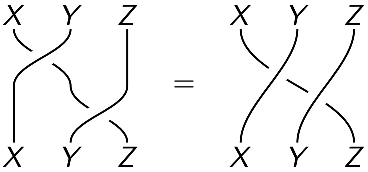


$c_{X,Y}$

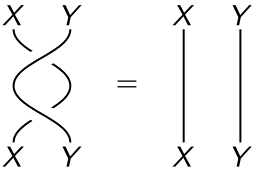
naturality



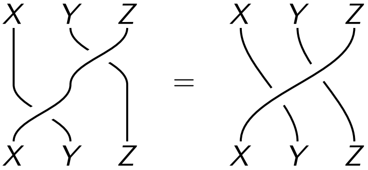
Braidings



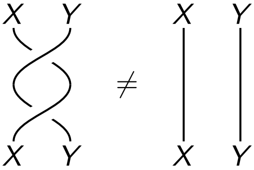
$(\mathrm{Id}_X \otimes c_{Y,Z}) \circ (c_{X,Y} \otimes \mathrm{Id}_Z)$ $\quad c_{X,Y\otimes Z}$



$c^{-1}_{X,Y} \circ c_{X,Y}$ $\quad \mathrm{Id}_{X,Y}$



$(c_{X,Y} \otimes \mathrm{Id}_Z) \circ (\mathrm{Id}_X \otimes c_{Y,Z})$ $\quad c_{X\otimes Y,Z}$



$c_{X,Y} \circ c_{X,Y}$ $\quad \mathrm{Id}_{X,Y}$

Graded Vector Spaces

A G-**graded vector space** is vector space V with decomposition

$$V = \bigoplus_{g \in G} V_g$$

A linear map f between two G-graded vector spaces V and W is the direct sum

$$f = \bigoplus_{g \in G} f_g$$

where $f_g : V_g \to W_g$ for each $g \in G$. Finally, a G-**graded tensor product** of two G-graded vector spaces is given by

$$(V \otimes_G W)_g = \bigoplus_{x, y \in G, xy = g} V_x \otimes W_y$$

for every $g \in G$.

Graded Fusion Category

Let G be a group. A fusion category $\mathcal{C}$ is G-**graded** if there is a decomposition

$$\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$$

of $\mathcal{C}$ into a direct sum of full abelian subcategories such that the tensor product maps $\mathcal{C}_g \times \mathcal{C}_h$ to $\mathcal{C}_{gh}$ for all $g, h \in G$.

Example

The category Vect_G of graded vector spaces is a graded fusion category:

$$\mathrm{Vect}_G = \bigoplus_{g \in G} \mathrm{Vect}_g$$

Associative Zesting

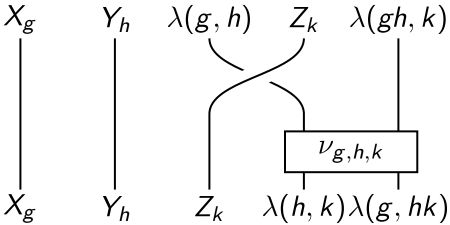
Let $\mathcal{C}$ be a G-graded braided fusion category. The data of an **associative zesting** over $\mathcal{C}$ consists of

- λ: a map $G^2 \to$ group of invertible objects of $\mathcal{C}_e$.
- ν: a collection of isomorphisms $\nu_{g,h,k} : \lambda(g,h) \otimes \lambda(gh,k) \to \lambda(h,k) \otimes \lambda(g,hk)$.

Additionally we require normalization $\lambda(e,g) = \lambda(g,e) = \mathbf{1}$ and $\nu_{g,e,h} = \mathrm{Id}_{\lambda(g,h)}$.

$$X_g \otimes Y_h \qquad \longrightarrow \qquad X_g \otimes Y_h \otimes \lambda(g,h)$$

$$a_{X_g,Y_h,Z_k} \qquad \longrightarrow$$



Associative Zesting

Example

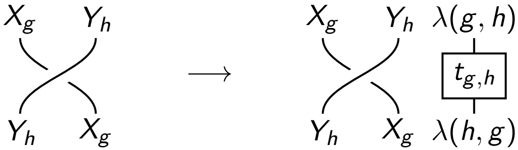
Applying zesting to Vect_G can yield Vect_G^{ω} for $\omega : G^3 \to \mathbb{k}$ satisfying

$$\omega(g,h,k)\omega(g,hk,l)\omega(h,k,l) = \omega(gh,k,l)\omega(g,h,kl).$$

The associator in this case is $a_{X_g,Y_h,Z_k} = \omega(g,h,k) \cdot \mathrm{Id}_{X_g \otimes Y_h \otimes Z_k}$.

Braided Zesting

A **Braided Zesting** (λ, ν, t) is an associative zesting (λ, ν) equipped with a collection of isomorphisms $t(g, h) : \lambda(g, h) \to \lambda(h, g)$ satisfying normalization condition $t(e, g) = t(g, e) = \mathrm{Id}_{\mathbf{1}}$.



Theorem ([DGP+22])

Two braided extensions of $\mathcal{C}_e$ by G have the same group homomorphism $G \to BrPic(\mathcal{C}_e)$ if and only if they are related by braided zesting.

Unnormalized Zesting

What if zesting data wasn't normalized (i.e. $\lambda(e, g) = \lambda(g, e) = \mathbf{1}$ and $\nu_{g,e,h} = \mathrm{Id}_{\lambda(g,h)}$)?

Theorem

Any unnormalized associative zesting (λ', ν') is "equivalent" to a normalized associative zesting (λ, ν). That is, $\mathcal{C}^{\lambda',\nu'}$ is equivalent to $\mathcal{C}^{\lambda,\nu}$ as $\mathcal{C}_e$ extensions by grading group G.

General Zesting

What if λ doesn't have to be in $\mathcal{C}_e$?

- $X_g \otimes Y_h \otimes \lambda(g,h)$ is not necessarily in $\mathcal{C}_{gh}$.
- We define a new group operation $\cdot$ such that $X_g \otimes Y_h \otimes \lambda(g,h) \in \mathcal{C}_{g\cdot h}$

Theorem

Two $\mathcal{C}_e$ extensions by G and G' with group morphisms $\phi_g : G \to BrPic(\mathcal{C}_e)$ and $\phi_h : G' \to BrPic(\mathcal{C}_e)$, respectively, are related by general braided zesting if and only if there is a bijection (as sets) $f : G \to G'$ such that $\phi_h \circ f = \phi_g$

Example

Starting from Vect_G, general zesting can modify its data into Vect_H for any group H with the same cardinality as G.

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References

Colleen Delaney, César Galindo, Julia Plavnik, Eric C. Rowell, and Qing Zhang.
Braided zesting and its applications.
Communications in Mathematical Physics, 386(1):1–55, 2021.

Colleen Delaney, César Galindo, Julia Plavnik, Eric Rowell, and Qing Zhang.
g-crossed braided zesting, 2022.

Pavel Etingof, Shlomo Gelaki, Dmitri Nikshych, and Victor Ostrik.
Tensor Categories, volume 205 of *Mathematical Surveys and Monographs*.
American Mathematical Society, Providence, RI, 2015.

Pavel Etingof, Dmitri Nikshych, and Victor Ostrik.
Fusion categories and homotopy theory.
Quantum Topology, 1(3):209–273, 2010.

Peter Selinger.
A survey of graphical languages for monoidal categories.
arXiv preprint arXiv:0908.3347, 2009.