The Center of a Tambara-Yamagami-Like Category

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Motivation

- Fusion categories can be thought of as a generalization of finite groups.
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- Mathematically, the twist and braiding structures in modular categories form S and T matrices. This encodes a projective representation of the modular group $SL(2, \mathbb{Z})$ by sending

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- Applications in topological quantum field theory and representation theory of quantum groups.
- How to construct examples of modular tensor categories?

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- Equipping each object X in a fusion category $\mathcal{C}$ with a family of half-braiding isomorphisms $\{\gamma_Y : Y \otimes X \xrightarrow{\sim} X \otimes Y\}_{Y \in \mathcal{C}}$ constructs the Drinfeld center.

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Let $\Bbbk$ be an algebraically closed field and let $\mathcal{C}$ be a spherical fusion category. Then $\mathcal{Z}(\mathcal{C})$, the Drinfeld Center of $\mathcal{C}$, is a modular category.

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- The Tambara-Yamagami category is one of the few fusion categories with all data explicitly computed.
- A paper by Gelaki, Naidu, and Nikshych provides a technique to compute the center of a graded fusion category. As an example, they computed the center of the Tambara-Yamagami category.

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- A paper by Gelaki, Naidu, and Nikshych provides a technique to compute the center of a graded fusion category. As an example, they computed the center of the Tambara-Yamagami category.
- This project is about using this technique to compute the center of a Tambara-Yamagami-like category defined in a paper by Galindo, Lentner, and Möller.

Introduction to Categories

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    - Sets and functions between sets.
    - Groups and group homomorphisms.
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Definition (Categories)

A **category** consists of a collection of **objects** and **morphisms**, in which we can **compose** morphisms, there exists an **identity morphism** for each object, and the composition of morphisms is **associative**.

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[loop above, "\mathrm{id}_X"] \arrow[dr, "g\circ f"'] & Y \arrow[d, "g"] \arrow[loop above, "\mathrm{id}_Y"] \\
& Z \arrow[loop below, "\mathrm{id}_Z"]
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Example of Categories

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The category FdVect (over $\Bbbk$).

- Objects: finite-dimensional vector spaces.
- Morphisms: linear maps between vector spaces.

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What desirable properties of FdVect do we have?

Fusion Categories: definition via an example

A **fusion category** is a category $\mathcal{C}$ equipped with following structures:

- **Direct sum:** $\oplus$ between vector spaces.

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Satisfying the following properties:

- **Abelianess:** for a linear map T, $\ker(T) \in \mathrm{FdVect}$, $\mathrm{im}(T) \in \mathrm{FdVect}$.

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- **Finite:** V is finite dimensional.
- $\Bbbk$**-linearity:** $\mathrm{Hom}(V, W) \in \mathrm{FdVect}$.

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- **Finite:** V is finite dimensional.
- **$\Bbbk$-linearity:** $\mathrm{Hom}(V, W) \in \mathrm{FdVect}$.
- **Duality:** exists the dual vector space V^*, for $\phi \in V^*$, $v \in V$, $\phi(v) \in \Bbbk$.

Schur's Lemma

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If $\mathcal{C}$ is abelian, $\Bbbk$-linear and finite, when $\Bbbk$ is algebraically closed, $\mathrm{Hom}_{\mathcal{C}}(X, Y) \cong \Bbbk$ *if* $X \cong Y$ *and* $\mathrm{Hom}_{\mathcal{C}}(X, Y) = 0$ *otherwise.*

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Example

In FdRep(G), any G-linear map between two irreducible representations is either 0 or a multiple of the identity.

Building a Fusion Category

Example

Let G be a finite group. Let $\mathcal{C}$ be a fusion category over $\mathbb{C}$ with

- Simple objects labeled by δ_x, where $x \in G$.
- Tensor product given by $\delta_x \otimes \delta_y = \delta_{xy}$.

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Question

What choices do we have for the associativity constraint

$$\alpha_{\delta_x,\delta_y,\delta_z} : (\delta_x \otimes \delta_y) \otimes \delta_z \to \delta_x \otimes (\delta_y \otimes \delta_z)$$

The Pentagon Axiom

$$
\begin{tikzcd}
 & ((W \otimes X) \otimes Y) \otimes Z \arrow[dl, "\alpha_{W,X,Y} \otimes \mathrm{id}_Z"'] \arrow[dr, "\alpha_{W \otimes X, Y, Z}"] & \\
(W \otimes (X \otimes Y)) \otimes Z \arrow[d, "\alpha_{W, X \otimes Y, Z}"'] & & (W \otimes X) \otimes (Y \otimes Z) \arrow[d, "\alpha_{W, X, Y \otimes Z}"] \\
W \otimes ((X \otimes Y) \otimes Z) \arrow[rr, "\mathrm{id}_W \otimes \alpha_{X,Y,Z}"] & & W \otimes (X \otimes (Y \otimes Z))
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$$\begin{tikzcd}
 & ((\delta_w \otimes \delta_x) \otimes \delta_y) \otimes \delta_z \arrow[dl, "\omega(w,x,y)"'] \arrow[dr, "\omega(wx,y,z)"] & \\
(\delta_w \otimes (\delta_x \otimes \delta_y)) \otimes \delta_z \arrow[d, "\omega(w,xy,z)"] & & (\delta_w \otimes \delta_x) \otimes (\delta_y \otimes \delta_z) \arrow[d, "\omega(w,x,yz)"] \\
\delta_w \otimes ((\delta_x \otimes \delta_y) \otimes \delta_z) \arrow[rr, "\omega(x,y,z)"] & & \delta_w \otimes (\delta_x \otimes (\delta_y \otimes \delta_z))
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- Simple objects labeled by δ_x, where $x \in G$.
- Tensor product given by $\delta_x \otimes \delta_y = \delta_{xy}$.
- The associativity constraint $\alpha_{\delta_x,\delta_y,\delta_z} : (\delta_x \otimes \delta_y) \otimes \delta_z \to \delta_x \otimes (\delta_y \otimes \delta_z)$ is parameterized by $\omega(x, y, z) \in \mathbb{C}$, where ω satisfies the **3-cocycle condition**

$$\omega(w, x, y)\omega(w, xy, z)\omega(x, y, z) = \omega(wx, y, z)\omega(w, x, yz)$$

for all $w, x, y, z \in G$.

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for all $w, x, y, z \in G$.

We can think of each δ_x as a "copy" of $\mathbb{C}$ labeled by x. The objects can then be seen as graded vector spaces, and morphisms are linear maps respecting the grading. (Thus the name of this category $\Bbbk\text{-FdVect}_G^\omega$.)

Tambara-Yamagami Categories

Now add one non-invertible object M, with the additional fusion rules given by

$$\mathbb{C}_a \otimes \mathrm{M} = \mathrm{M} = \mathrm{M} \otimes \mathbb{C}_a, \quad \mathrm{M} \otimes \mathrm{M} = \bigoplus_{a \in G} \mathbb{C}_a.$$

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Question

What are all the coherent associativity constraints we can put given this fusion rule?

Tambara-Yamagami Categories

Theorem (Tambara and Yamagami, 1998)

Let G be a finite (abelian) group, χ be a nondegenerate symmetric bilinear form, and τ be a square root of $|G|^{-1}$. The Tambara-Yamagami category $\mathrm{TY}(G, \chi, \tau)$ *parameterizes all fusion categories with this fusion rule. The associativity constraints are given by:*

$$\alpha_{\mathbb{C}_a,\mathbb{C}_b,\mathbb{C}_c} = \mathrm{id}_{\mathbb{C}_{a+b+c}}, \quad \alpha_{\mathbb{C}_a,\mathbb{C}_b,\mathrm{M}} = \mathrm{id}_{\mathrm{M}}, \quad \alpha_{\mathrm{M},\mathbb{C}_a,\mathbb{C}_b} = \mathrm{id}_{\mathrm{M}},$$

$$\alpha_{\mathbb{C}_a,\mathrm{M},\mathbb{C}_b} = \chi(a,b)\mathrm{id}_{\mathrm{M}} \quad \alpha_{\mathrm{M},\mathrm{M},\mathbb{C}_a} = \bigoplus_{b\in G} \mathrm{id}_{\mathbb{C}_b}, \quad \alpha_{\mathbb{C}_a,\mathrm{M},\mathrm{M}} = \bigoplus_{b\in G} \mathrm{id}_{\mathbb{C}_b},$$

$$\alpha_{\mathrm{M},\mathbb{C}_a,\mathrm{M}} = \bigoplus_{b\in G} \chi(a,b)\mathrm{id}_{\mathbb{C}_{a+b}}, \quad \alpha_{\mathrm{M},\mathrm{M},\mathrm{M}} = \bigoplus_{a,b\in G} \tau\chi(a,b)^{-1}\mathrm{id}_{\mathrm{M}}.$$

The Category $\mathrm{GLM}(G, \sigma, \omega, \delta, \epsilon)$

Add non-invertible objects $\mathsf{M}_{\bar{x}}$ parameterized by $\bar{x} \in G/2G$, with tensor products given by

$$\mathbb{C}_a \otimes \mathsf{M}_{\bar{x}} = \mathsf{M}_{\bar{x}+\bar{a}} = \mathsf{M}_{\bar{x}} \otimes \mathbb{C}_a, \quad \mathsf{M}_{\bar{x}} \otimes \mathsf{M}_{\bar{y}} = \bigoplus_{t \in \bar{x}+\bar{y}+\delta} \mathbb{C}_t.$$

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Theorem (Galindo, Lentner, and Möller, 2024)

Let (σ, ω) be an abelian 3-cocycle of a specific form, $\delta \in G/2G$, and ϵ be a sign choice. The following defines a coherent associativity structure:

$$\alpha_{\mathbb{C}_a,\mathbb{C}_b,\mathbb{C}_c} = \omega(a,b,c)\mathrm{id}_{\mathbb{C}_{a+b+c}}, \quad \alpha_{\mathbb{C}_a,\mathbb{C}_b,\mathsf{M}_{\bar{x}}} = \omega(a+b+\bar{x},a,b)\mathrm{id}_{\mathsf{M}_{\bar{x}+a+b}},$$

$$\alpha_{\mathsf{M}_{\bar{x}},\mathbb{C}_a,\mathbb{C}_b} = \omega(\bar{x}+\delta,a,b)\mathrm{id}_{\mathsf{M}_{\bar{x}+a+b}}, \quad \alpha_{\mathbb{C}_a,\mathsf{M}_{\bar{x}},\mathbb{C}_b} = \sigma(a,b)\mathrm{id}_{\mathsf{M}_{a+\bar{x}+b}}$$

$$\alpha_{\mathsf{M}_{\bar{x}},\mathsf{M}_{\bar{y}},\mathbb{C}_a} = \bigoplus_{t \in \bar{x}+\bar{y}+\delta} \omega(\bar{x},t,a)\mathrm{id}_{\mathbb{C}_{t+a}}, \quad \alpha_{\mathbb{C}_a,\mathsf{M}_{\bar{x}},\mathsf{M}_{\bar{y}}} = \bigoplus_{t \in \bar{x}+\bar{y}+\delta} \omega(a+\bar{x},a,t)\mathrm{id}_{\mathbb{C}_{a+t}},$$

$$\alpha_{\mathsf{M}_{\bar{x}},\mathbb{C}_a,\mathsf{M}_{\bar{y}}} = \bigoplus_{t \in \bar{x}+\bar{y}+\delta+a} \sigma(a,t)\mathrm{id}_{\mathbb{C}_t},$$

$$\alpha_{\mathsf{M}_{\bar{x}},\mathsf{M}_{\bar{y}},\mathsf{M}_{\bar{z}}} = \bigoplus_{t \in \bar{x}+\bar{y}+\delta, r \in \bar{y}+\bar{z}+\delta} \epsilon|2G|^{-\frac{1}{2}}\sigma(t,r)^{-1}\mathrm{id}_{\mathsf{M}_{\bar{x}+\bar{y}+\bar{z}+\delta}}.$$

An Overview of the Gelaki-Naidu-Nikshych Technique

Definition (Drinfeld Center, vague)

The Drinfeld center of a fusion category $\mathcal{C}$, $\mathcal{Z}(\mathcal{C})$, contains objects in the forms of (X, γ), where $X \in C$ and the half-braiding γ is a family of maps

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Definition (Graded Fusion Category)

Let G be a finite group. A fusion category $\mathcal{C}$ is G-graded if there is a decomposition

$$\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$$

satisfying $\otimes : \mathcal{C}_g \times \mathcal{C}_h \to \mathcal{C}_{gh}$.

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- Each object in the **equivariantization** of $\mathcal{C}$, $\mathcal{C}^G$, is a pair (X, u), where $X \in \mathcal{C}$ and $u = \{u_g : g.X \xrightarrow{\sim} X\}_{g \in G}$.

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- Each object in the **relative center** of $\mathcal{C}$, $\mathcal{Z}_{\mathcal{C}_e}(\mathcal{C})$, is a pair (X, γ), where $X \in \mathcal{C}$ and $\gamma = \{\gamma_Y : Y \otimes X \xrightarrow{\sim} X \otimes Y\}_{Y \in \mathcal{C}_e}$.

Theorem (Gelaki, Naidu, and Nikshych, 2001)

The relative center $\mathcal{Z}_{\mathcal{C}_e}(\mathcal{C})$ has a canonical braided G-crossed structure. Furthermore, there is an equivalence of braided fusion categories

$$\mathcal{Z}_{\mathcal{C}_e}(\mathcal{C})^G \xrightarrow{\sim} \mathcal{Z}(\mathcal{C}).$$

Our Results

Theorem

Let $\mathcal{C} = \mathrm{GLM}(G, \sigma, \omega, \delta, \epsilon)$*. The following is a complete list of* **simple objects** *of* $\mathcal{Z}_{\mathcal{D}}(\mathcal{C})^{\mathbb{Z}/2\mathbb{Z}} \cong \mathcal{Z}(\mathcal{C})$*.*

- $2|G||G_2|$ *objects of dimension 1 parameterized by an ordered tuple* (a, b, ν)*, where* $a + b \in G_2$ *and* $\nu \in \{\pm 1\}$*. The corresponding simple object is* $\mathbb{C}_{(a,b)}$*. The* $\mathbb{Z}/2\mathbb{Z}$*-equivariant structure is given by* $u_g = \nu\sqrt{\gamma_{g,g}(\mathbb{C}_{(a,b)})}\mathrm{id}$*.*
- $[G : 2G] \cdot |G|$ *objects of dimension 2 parameterized by an ordered tuple* $(\bar{x}, u, \Delta)$*, where* $\bar{x} \in G/2G$*,* $u \in G$*, and* $\Delta \in \{\pm 1\}$*. The corresponding simple object is* $\mathrm{M}_{(\bar{x},u)}$*. The* $\mathbb{Z}/2\mathbb{Z}$*-equivariant structure is given by* $u_g = \Delta\sqrt{\gamma_{g,g}(\mathrm{M}_{(\bar{x},u)})}\mathrm{id}$*.*
- $\frac{|G|(|G|-|G_2|)}{2}$ *objects of dimension* $\sqrt{|2G|}$ *parameterized by an unordered pair* (a, b)*, where* $a, b \in G$ *and* $a + b \neq G_2$*. The corresponding simple object is* $\mathbb{C}_{(a,b)} \oplus \mathbb{C}_{(-a-2b,b)}$*. The* $\mathbb{Z}/2\mathbb{Z}$*-equivariant structure is given by* $u_g = \gamma_{g,g}(\mathbb{C}_{(a,b)})\mathrm{id}_{\mathbb{C}_{(a,b)}} \oplus \mathrm{id}_{\mathbb{C}_{(-a-2b,b)}}$*.*

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