Bounds on the Distinguishing (Chromatic) Number of Posets

Aiden Jeong
Mentor: Yunseo Choi

Branham High School

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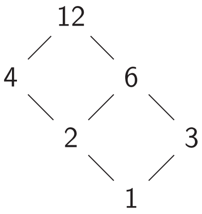

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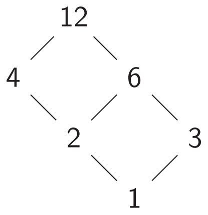

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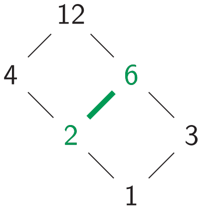

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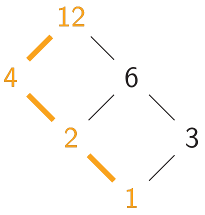

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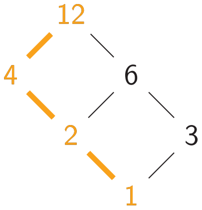
- $h(L_{12}) = 3$



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Poset Automorphisms

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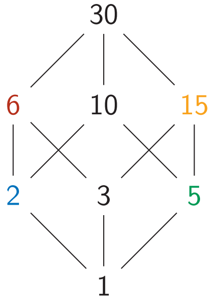

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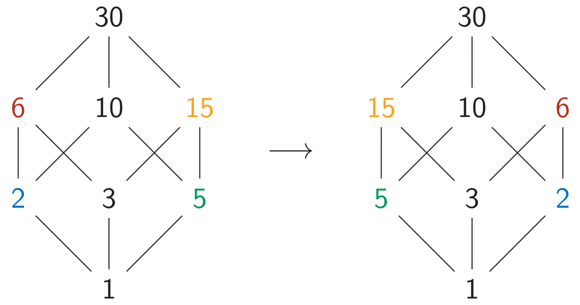

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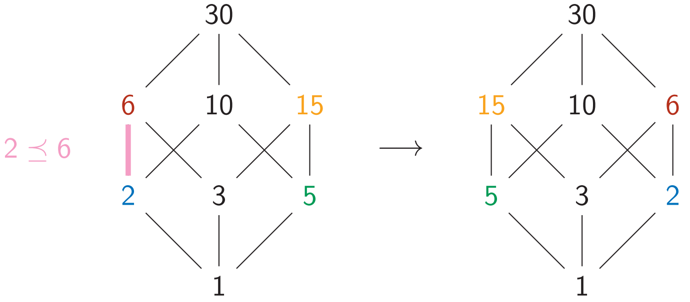

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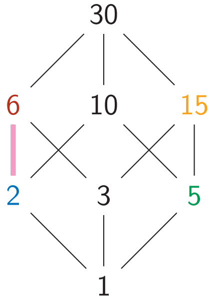
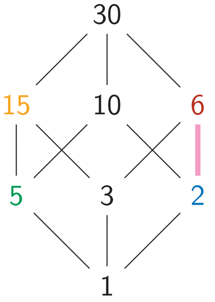


$\longrightarrow$



$\varphi(2) \preceq \varphi(6)$

The Distinguishing Number of a Poset

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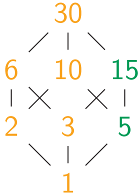

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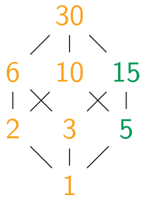

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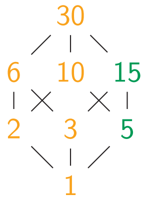

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The Distinguishing Number $D(P)$

- The minimum number of colors in a distinguishing coloring of P

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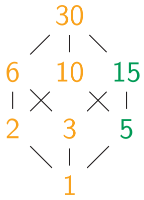

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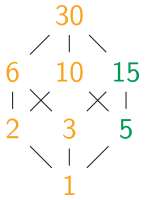

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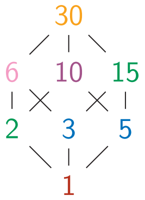

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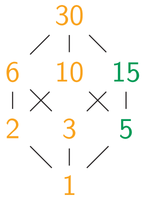

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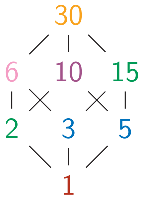

Distinguishing and Chromatic

The Distinguishing Chromatic Number $\chi_D(P)$

- The least number of colors in a distinguishing chromatic coloring of P

Summary of Results

Notation

- Let P be a poset, L a lattice, and M a distributive lattice

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Quantity	Previous Upper Bound	Our Upper Bound
$D(P)$	$1+\lfloor \log_2 \lvert\mathrm{Aut}(P)\rvert \rfloor$ (Choi, 2022)	$1+\left\lceil \frac{\log_2 \lvert\mathrm{Aut}(P)\rvert}{f(P)} \right\rceil$
$D(L)$	-	$\lvert Q_L\rvert - h(L) + 2$
$\chi_D(L)$	-	$h(L) + \chi_D(Q_L)$
$D(M)$	2 (Collins & Trenk, 2021)	-
$\chi_D(M)$	$\lvert Q_M\rvert + \chi_D(Q_M) - 1$ (Collins & Trenk, 2021)	$\lvert Q_M\rvert + k$

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Lattices L

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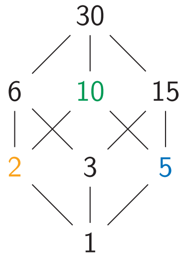

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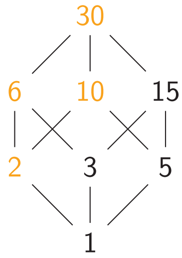

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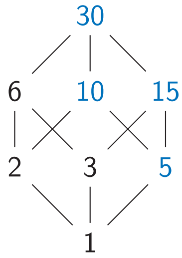

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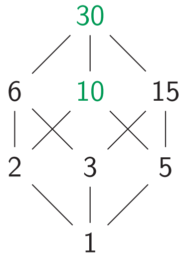

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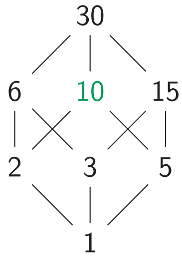

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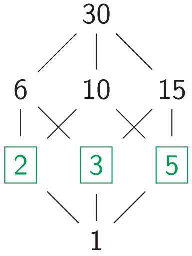
Example: Join-Irreducibles of L_{30}

- The join-irreducibles of L_{30} are 2, 3, and 5



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Theorem (J., 2025)

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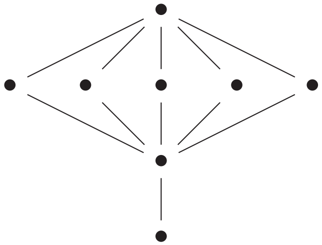

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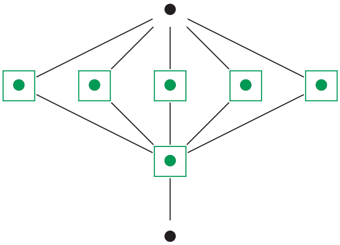

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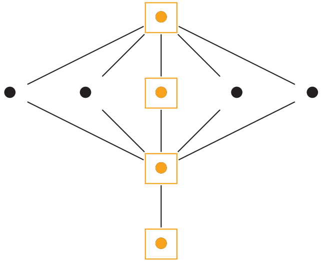

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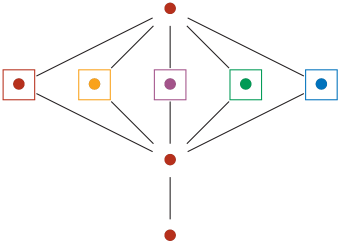

- $|Q_L| = 6$
- $h(L) = 3$
- $D(L) = 5$

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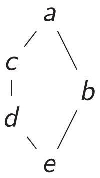

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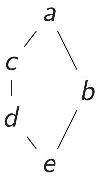

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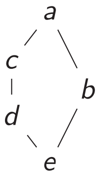

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- $(c \wedge b) \vee (c \wedge d) = e \vee d = d$

Results for Distributive Lattices I

Theorem (Collins & Trenk, 2021)

For a distributive lattice M such that $\chi_D(Q_M) \geq 3$, we have

$$\chi_D(M) \leq |Q_M| + \chi_D(Q_M) - 1.$$

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If M is a distributive lattice, we have $\chi_D(M) \leq |Q_M| + k$, where

$$k = \begin{cases} 1 & \text{if } |\max(Q_M)| = 1, \\ 2 & \text{if } |\max(Q_M)| \neq 1, 4, \\ 3 & \text{if } |\max(Q_M)| = 4. \end{cases}$$

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- Stronger except when $\chi_D(Q_M) = 2$ and $|\max(Q_M)| = 4$ or $\chi_D(Q_M) = 3$ and $|\max(Q_M)| = 4$

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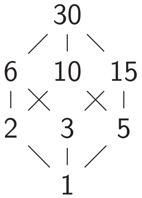

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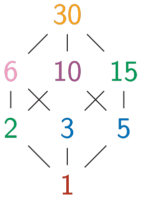

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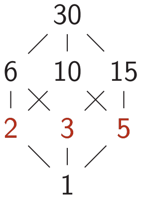
- $\chi_D(M) = 5$
- $|\max(Q_M)| = 3$



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