Honey, I Shrunk the Convex QCQP

Dimension Reduction for Smooth Convex Optimization via Color Refinement

Bryan Zhu
Mentor: Ziang Chen

Bellevue Senior High School

MIT PRIMES Conference
October 18, 2025

Motivation

- Convex optimization is generally more tractable than nonconvex optimization.
- However, standard algorithms for convex optimization can still exhibit rapidly increasing runtimes.
- Dimension reduction can reduce memory and runtime costs, thus improving tractability of solvers.

Color refinement

- An algorithm that, given a graph G, seeks to find a coloring function $\eta : V(G) \to \mathbb{N}_0$ that satisfies stability; i.e. if the associated coloring $\mathcal{C} = \{\eta^{-1}(c) \mid c \in \eta(V(G))\}$, then for all $C_1, C_2 \in \mathcal{C}$ and $v_1, v_2 \in C_1$,

$$|N(v_1) \cap C_2| = |N(v_2) \cap C_2|$$

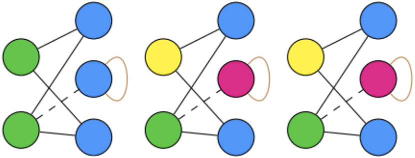
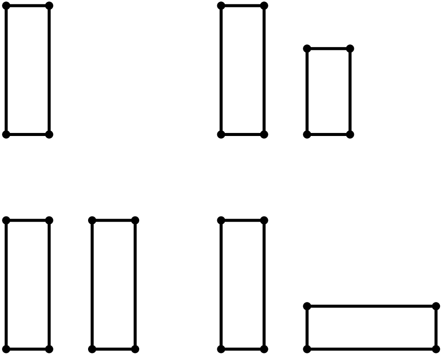


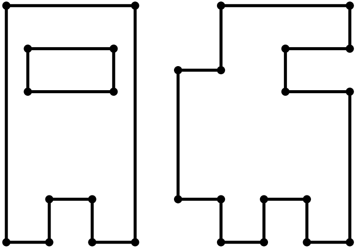
Figure: Chen et al., 2024

- Useful for isomorphism testing: if, in the coarsest stable partition $\mathcal{C}$ of $G_1 \cup G_2$, there exists $C \in \mathcal{C}$ such that $|V(G_1) \cap C| \neq |V(G_2) \cap C|$, then G_1 and G_2 are not isomorphic (Berkholz et al., 2015)

Color refinement

Color refinement is not perfect. For example, it will mark these two graphs below as isomorphic.





Linear programming and quadratic programming

Definition

A linearly constrained quadratic program (LCQP) takes the form

$$\begin{aligned} \text{minimize } & \frac{1}{2}x^\top Qx + c^\top x \\ \text{subject to } & Ax \leq b, \\ & l \leq x \leq u \end{aligned}$$

for $Q \neq 0$. If $Q = 0$, we call this a linear program (LP).

Past results

Definition

An equitable partition $(\mathcal{P}, \mathcal{Q})$ of a matrix A satisfies the condition that for any $S \in \mathcal{P}$ and $T \in \mathcal{Q}$,

$$\sum_{j \in T} A_{ij}$$

is constant across all $i \in S$, and

$$\sum_{i \in S} A_{ij}$$

is constant across all $j \in T$.

A variant of color refinement can find equitable partitions (Grohe et al., 2014).

Past results

Theorem (Grohe et al., 2014) [LPs]

Suppose colorings $\mathcal{P}$ and $\mathcal{Q}$ satisfy the following conditions:

- For all $T \in \mathcal{Q}$, (c_j, l_j, u_j) is equal for all $j \in T$.
- $(\mathcal{P}, \mathcal{Q})$ is equitable on A.
- For all $S \in \mathcal{P}$, b_i is equal for all $i \in S$.

Then an equivalent reduced LP can projected from the original.

Theorem (Mladenov et al., 2017) [Convex LCQPs]

Suppose colorings $\mathcal{P}$ and $\mathcal{Q}$ satisfy all of the above conditions, and also $(\mathcal{Q}, \mathcal{Q})$ is equitable on Q. An equivalent reduced LCQP can be projected from the original.

We will discuss how projection works later.

Example (part 1)

Example

Consider the convex LCQP given by

$$Q = \left[\begin{array}{ccc|c} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ \hline 0 & 0 & 0 & 2 \end{array}\right], \qquad c = \left[\begin{array}{c} 2 \\ 2 \\ 2 \\ \hline 3 \end{array}\right],$$

$$A = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 2 & 3 & 1 & 5 \\ 3 & 1 & 2 & 5 \end{array}\right], \qquad b = \begin{bmatrix} 100 \\ 100 \\ 100 \end{bmatrix}.$$

The coarsest reduction coloring is shown above. Specifically, $\mathcal{P} = \{\{1,2,3\}\}$ and $\mathcal{Q} = \{\{1,2,3\},\{4\}\}$.

Example (part 2)

Example

The reduced problem is given by

$$Q' = \begin{bmatrix} 6 & 0 \\ 0 & 2 \end{bmatrix}, \qquad c' = \begin{bmatrix} 6 \\ 3 \end{bmatrix},$$
$$A' = \begin{bmatrix} 6 & 5 \end{bmatrix}, \qquad b' = \begin{bmatrix} 100 \end{bmatrix}.$$

The optimum for the reduced problem is $x = \begin{bmatrix} -1 \\ -1.5 \end{bmatrix}$, and the optimum for the original problem is $x = \begin{bmatrix} -1 \\ -1 \\ -1 \\ -1.5 \end{bmatrix}$.

Generalizations

Definition

A polynomial optimization problem of degree d with n variables and m constraints takes the form

$$\begin{aligned} \text{minimize} \quad & \sum_{k=1}^{d} \frac{1}{k} \sum_{1 \le j_1, \ldots, j_k \le n} A^{(k)}_{j_1 \ldots j_k} \prod_{r=1}^{k} x_{j_r} \\ \text{subject to} \quad & \sum_{k=1}^{d} \frac{1}{k} \sum_{1 \le j_1, \ldots, j_k \le n} \left(P_i^{(k)}\right)_{j_1 \ldots j_k} \prod_{r=1}^{k} x_{j_r} \le b_i, \\ & l \le x \le u. \end{aligned}$$

We assume $A^{(k)}$ and $P_i^{(k)}$ are symmetric rank k coefficient tensors. We also define a rank $k+1$ tensor $P^{(k)}$ given by concatenating $P_i^{(k)}$.

Generalizations

Definition

A partition $(\mathcal{Q}_1, \ldots, \mathcal{Q}_k)$ of a rank k tensor $A^{(k)}$ is equitable if, for any $1 \leq r \leq k$, colors $T_1 \in \mathcal{Q}_1, \ldots, T_k \in \mathcal{Q}_k$, and $j_r^{(1)}, j_r^{(2)} \in T_r$,

$$\sum_{\substack{j_s \in T_s \\ s \neq r}} A^{(k)}_{j_1 \cdots j_{r-1} j_r^{(1)} j_{r+1} \cdots j_k} = \sum_{\substack{j_s \in T_s \\ s \neq r}} A^{(k)}_{j_1 \cdots j_{r-1} j_r^{(2)} j_{r+1} \cdots j_k}.$$

Definition

For a convex polynomial optimization problem, $(\mathcal{P}, \mathcal{Q})$ is a reduction coloring if it satisfies the following:

- For all $1 \leq k \leq d$, $(\mathcal{Q}, \ldots, \mathcal{Q})$ is equitable on $A^{(k)}$.
- For all $1 \leq k \leq d$, $(\mathcal{P}, \mathcal{Q}, \ldots, \mathcal{Q})$ is equitable on $P^{(k)}$.
- For each $S \in \mathcal{P}$, b_i is equal for every $i \in S$.
- For each $T \in \mathcal{Q}$, (l_j, u_j) is equal for every $j \in T$.

Generalizations

Definition

Given a reduction coloring $(\mathcal{P}, \mathcal{Q})$ of a convex polynomial optimization problem, we define an equivalent reduced problem with $|\mathcal{P}|$ constraints and $|\mathcal{Q}|$ variables:

- For each $A^{(k)}$ and $P^{(k)}$, the reduced tensors are given by summing color-based subtensors and keeping only one constraint per color.
- We average $\mathcal{Q}$ color blocks in l and u.
- We average $\mathcal{P}$ color blocks in b.

Generalizations

Theorem (Z. and Chen, 2025)

Consider any convex polynomial optimization problem. Let $(\mathcal{P}, \mathcal{Q})$ be a reduction coloring. If x is an optimum for the original problem, then x', given by averaging $\mathcal{Q}$ color blocks in x, is an optimum for the reduced problem. If x' is an optimum for the reduced problem, then x, given by $x_j = x'_T$ for all $j \in T$ and $T \in \mathcal{Q}$, is an optimum for the original problem.

Example (part 1)

Example

Consider a convex quadratically constrained quadratic program (QCQP), which is constrained by $\frac{1}{2}x^\top P_i x \leq b_i$ for all $1 \leq i \leq m$, where

$$Q = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \qquad c = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix},$$

$$P_1 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad P_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix},$$

$$P_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \qquad b = \begin{bmatrix} 100 \\ 100 \\ 100 \end{bmatrix}.$$

The coarsest reduction coloring is the unit coloring.

Example (part 2)

Example

The reduced problem is given by

$$Q' = \begin{bmatrix} 6 \end{bmatrix}, \qquad c' = \begin{bmatrix} 6 \end{bmatrix},$$
$$P' = \begin{bmatrix} 4 \end{bmatrix}, \qquad b' = \begin{bmatrix} 100 \end{bmatrix}.$$

The optimum for the reduced problem is $x = \begin{bmatrix} -1 \end{bmatrix}$, and the optimum for the original problem is $x = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$.

Note on non-polynomial cases

Definition

For a general smooth convex optimization problem, a coloring $(\mathcal{P}, \mathcal{Q})$ is a reduction coloring if it satisfies the following conditions for all $\hat{x}$ that satisfy $\hat{x}_{j_1} = \hat{x}_{j_2}$ if j_1 and j_2 share a color in $\mathcal{Q}$:

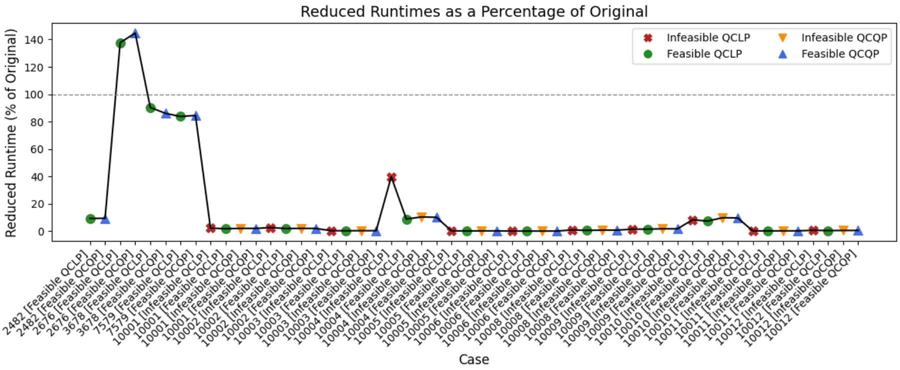
- If $T \in \mathcal{Q}$ and $j_1, j_2 \in T$, then $\frac{\partial F}{\partial x_{j_1}}\Big|_{x=\hat{x}} = \frac{\partial F}{\partial x_{j_2}}\Big|_{x=\hat{x}}$.
- If $S \in \mathcal{P}$, $T \in \mathcal{Q}$, and $j_1, j_2 \in T$, then

$$\frac{\partial}{\partial x_{j_1}} \left(\sum_{i \in S} G_i \right) \Bigg|_{x=\hat{x}} = \frac{\partial}{\partial x_{j_2}} \left(\sum_{i \in S} G_i \right) \Bigg|_{x=\hat{x}}.$$

- For all i that share some color in $\mathcal{P}$, $G_i(\hat{x})$ is equal.
- For each $S \in \mathcal{P}$, b_i is equal for every $i \in S$.
- For each $T \in \mathcal{Q}$, (l_j, u_j) is equal for every $j \in T$.

Experimental results

- We use QPLIB, a library of quadratic programs (Furini et al., 2018).
- Average runtime percentage of original is about 15.0%.



Acknowledgments

I would like to thank my mentor, Dr. Ziang Chen, for his extensive guidance and help in this project. I would also like to thank MIT PRIMES and its organizers for making this work possible.

References

- C. Berkholz, P. Bonsma, and M. Grohe. Tight lower and upper bounds for the complexity of canonical colour refinement. arXiv preprint arXiv:1509.08251, 2015.
- Z. Chen, X. Chen, J. Liu, X. Wang, and W. Yin. Expressive power of graph neural networks for (mixed-integer) quadratic programs. arXiv preprint arXiv:2406.05938, 2024.
- F. Furini et al. QPLIB: A library of quadratic programming instances. Mathematical Programming Computation, 2018, DOI:10.1007/s12532-018-0147-4.
- M. Grohe et al. Dimension reduction via colour refinement. Algorithms - ESA 2014, vol. 8737, pages 505-516, 2014, DOI:10.1007/978-3-662-44777-2_42.
- M. Mladenov, L. Kleinhans, and K. Kersting. Lifted inference for convex quadratic programs. Proceedings of the AAAI Conference on Artificial Intelligence, vol. 31, no. 1, 2017.

Thank you! Any questions?