Generalizations of Jarnik's theorem via total density of certain subspaces

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MIT PRIMES Conference

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Outline

- Introduction and Motivating Examples
- Results
- Total Density

Dirchlet's Theorem

Density of Rationals

For all $\alpha \in \mathbb{R}$ and $\epsilon > 0$, there exists $q \in \mathbb{N}$ and $p \in \mathbb{Z}$ such that

$$\left| \alpha - \frac{p}{q} \right| < \epsilon$$

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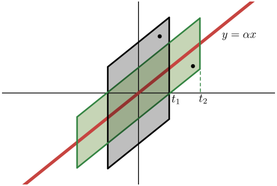
Dirichlet's Theorem

For all real numbers $\alpha \in \mathbb{R}$ and $t \geq 1$, there exists $p, q \in \mathbb{Z}$ such that

$$1 \leq q \leq t \text{ and } |q\alpha - p| < \frac{1}{t}$$



Higher Dimensions

Dirichlet's Theorem for $M_{m,n}(\mathbb{R})$

For all matrices $A \in M_{m,n}(\mathbb{R})$ $t \geq 1$, there exists $\mathbf{q} \in \mathbb{Z}^n \setminus \{0\}$ and $\mathbf{p} \in \mathbb{Z}^m$ such that

$$\|\mathbf{q}\| \leq t \text{ and } \|A\mathbf{q} - \mathbf{p}\| < \frac{1}{t^{\frac{n}{m}}}$$

where $\|\cdot\|$ is the supremum norm.

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- Approximation of linear forms: m=1. Given $(x_1, \ldots, x_n) \in \mathbb{R}^n$, find $(q_1, \ldots, q_n) \in \mathbb{Z}^n \setminus \{0\}$ and $p \in \mathbb{Z}$ minimizing $|x_1 q_1 + \cdots + x_n q_n - p|$.
- Simultaneous approximation: n=1. Given $(x_1, \ldots, x_n) \in \mathbb{R}^n$, find $q \in \mathbb{Z} \setminus \{0\}$ $(p_1, \ldots, p_n) \in \mathbb{Z}^m$ minimizing $\max(|x_1 q - p_1|, \ldots, |x_n q - p_n|)$.

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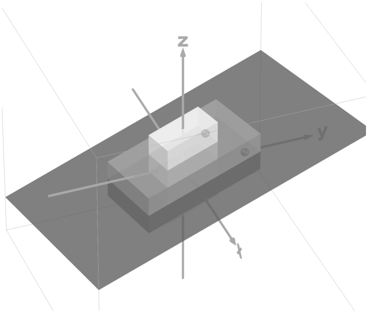

Singularity and Uniform Approximability

Definition: Given a matrix $A \in M_{m,n}(\mathbb{R})$, we say that A is singular if for all $\epsilon > 0$, there exists a $t_0 \in \mathbb{R}^+$ such that for all $t \geq t_0$, there exists $\mathbf{q} \in \mathbb{Z}^n \setminus \{0\}$ and $\mathbf{p} \in \mathbb{Z}^m$ such that

$$\|\mathbf{q}\| \leq t \text{ and } \|A\mathbf{q} - \mathbf{p}\| \leq \frac{\epsilon}{t^{\frac{n}{m}}}$$

Definition: Given a non-increasing function $f : \mathbb{R}^+ \to \mathbb{R}^+$ and matrix $A \in M_{m,n}(\mathbb{R})$, we say that A is uniformly f-approximable (or just f-uniform) if for all large enough $t > 0$, there exists $\mathbf{q} \in \mathbb{Z}^n \setminus \{0\}$ and $\mathbf{p} \in \mathbb{Z}^m$ such that

$$\|\mathbf{q}\| \leq t \text{ and } \|A\mathbf{q} - \mathbf{p}\| \leq f(t)$$

Definition: Given a matrix $A \in M_{m,n}(\mathbb{R})$, we say A is totally irrational if $A\mathbf{q} - \mathbf{p} \neq 0$ for all $\mathbf{q} \in \mathbb{Z}^n \setminus \{0\}$ and $\mathbf{p} \in \mathbb{Z}^n$.

Jarnik's Theorem

Theorem (Jarnik (1959), Khintchine (1926))

Let $m, n \in \mathbb{N}$ where $n > 1$. Then for any non-increasing function $f : \mathbb{R}_+ \to \mathbb{R}_+$, the set of totally irrational f-uniform $m \times n$ matrices is uncountable and dense in $M_{m,n}(\mathbb{R})$.

In other words, the set of A such that $\|A\mathbf{q} - \mathbf{p}\| \leq f(t)$ has infinitely many solutions is uncountable and dense for all non-increasing functions f.

Main Theorem

Definition: Let $Q \subseteq \mathbb{R}^n \setminus \{0\}$ and $P \subseteq \mathbb{R}^m$. Given a function $f : \mathbb{R}^+ \to \mathbb{R}^+$ and matrix $A \in M_{m,n}(\mathbb{R})$, we say that A is f-uniform with respect to Q, P if for all large enough $t > 0$, there exists $\mathbf{q} \in Q$ and $\mathbf{p} \in P$ such that

$$\|q\| \leq t \text{ and } \|A\mathbf{q} - \mathbf{p}\| \leq f(t)$$

Let the set of matrices which are f-uniform with respect to Q, P be denoted by $\mathrm{UA}_{Q,P}(f)$. Also let $\mathrm{UA}^*_{Q,P}(f)$ be the set of matrices which are non-trivially f-uniform with respect to Q, P.

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Theorem (Kleinbock, Moshchevitin, Warren, Weiss, 2024)

Let $Q \subseteq \mathbb{R}^n \setminus \{0\}$ and $P \subseteq \mathbb{R}^m$ such that [important property] and let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be any non increasing function. In addition, let $\mathcal{S}$ be a countable collection of proper affine subspaces of $M_{m,n}(\mathbb{R})$. Then the set $\mathrm{UA}_{Q,P}(f) \setminus \bigcup_{S \in \mathcal{S}} S$ is uncountable and dense.

Generalizations

- Let $\mathbb{P}$ be the set of primes. Consider matrices $A \in M_{m,n}(\mathbb{R})$ such that for all large enough $t > 0$, there exists $\mathbf{q} \in \{2^k : k \in \mathbb{N}\}^n \setminus \{0\}$ and $\mathbf{p} \in \mathbb{P}^m$ such that

$$\|\mathbf{q}\| \leq t \text{ and } \|A\mathbf{q} - \mathbf{p}\| \leq f(t)$$

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- Fix integers $1 \leq a \leq n$ and $0 \leq b \leq m$ such that $a + b > m + 1$. Consider matrices $A \in M_{m,n}(\mathbb{R})$ such that for all large enough $t > 0$, there exists $\mathbf{q} \in \mathbb{Z}^a \times \{0\}^{n-a}$ and $\mathbf{p} \in \mathbb{Z}^b \times \{0\}^{m-b}$ such that

$$\|\mathbf{q}\| \leq t \text{ and } \|A\mathbf{q} - \mathbf{p}\| \leq f(t)$$

Generalizations

- Inhomogeneous approximation: Fix $\mathbf{b} \in \mathbb{R}^m$. Consider matrices $A \in M_{m,n}(\mathbb{R})$ such that for all large enough $t > 0$, there exists $\mathbf{q} \in \mathbb{Z}^n \setminus \{0\}$ and $\mathbf{p} \in \mathbb{Z}^m$ such that

  $$\|\mathbf{q}\| \leq t \text{ and } \|A\mathbf{q} - \mathbf{p} + \mathbf{b}\| \leq f(t)$$

  Note that this is equivalent to still considering $\|A\mathbf{q} - \mathbf{p}\|$ but letting $\mathbf{p} \in \mathbb{Z}^m - \mathbf{b}$ instead.

Results

Theorem (H., Neckrasov)

Let $n \geq 2$ and $m \geq 1$. Suppose $Q = \mathbf{a}\mathbb{Z} + \mathbf{b}\mathbb{Z}$ for linearly independent $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^n$ and $P = \mathbb{Z}^m + \mathbf{c}$ for some $\mathbf{c} \in \mathbb{Z}^m$. Then for all non-increasing functions $f : \mathbb{R}^+ \to \mathbb{R}^+$, the set $\mathrm{UA}^*_{P,Q}(f)$ is uncountable and dense.

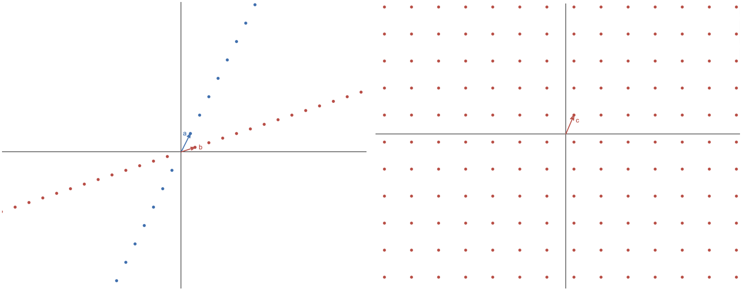


Figure: $Q = \mathbf{a}\mathbb{Z}^+ + \mathbf{b}\mathbb{Z}^+$ and $P = \mathbb{Z}^m + \mathbf{c}$

Theorem (H., Neckrasov)

Let $n \geq 2$ and $m \geq 1$, and fix $1 \leq k \leq \min(m+1, n)$. Suppose Q is the union of two k-dimensional lattices contained in two distinct k-dimensional subspaces of $\mathbb{R}^n$ and P is some $m-k+1$-dimensional lattice. Then for all non-increasing functions $f : \mathbb{R}^+ \to \mathbb{R}^+$, the set $\mathrm{UA}_{P,Q}(f)$ is uncountable and dense.

Remark: In the case where $k = 1$, we get the previous theorem.

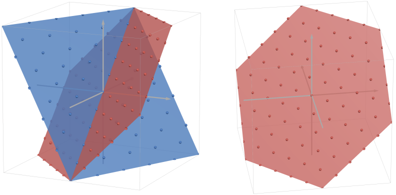
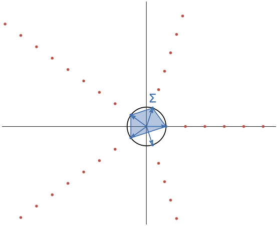


Figure: Q and P for $m = n = 3$, $k = 2$, and $m - k + 1 = 2$

Results

Theorem (H., Neckrasov)

Let $\Sigma \subseteq S^{n-1}$ be a set such that for all $\theta \in \Sigma$, $0 \in \mathrm{conv}(\Sigma \setminus \{\theta\})$ when embedded in $\mathbb{R}^n$, Π be some half space in $\mathbb{R}^m$. Also let $Q = \mathbb{Z}^+\Sigma$ and $P = \Pi \cap \mathbb{Z}^m$. Then for all non-increasing functions $f : \mathbb{R}^+ \to \mathbb{R}^+$, the set $\mathrm{UA}_{Q,P}(f)$ is uncountable and dense.



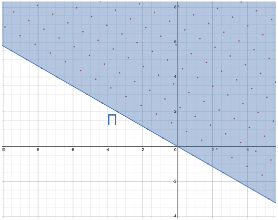


Figure: $Q = \mathbb{Z}^+\Sigma$ and $P = \Pi \cap \mathbb{Z}^m$ for $m = n = 2$

Zero-sets

Definition: Let $\mathbf{q} \in (\mathbb{R}^n \setminus \{0\})$ $\mathbf{p} \in \mathbb{R}^n$, $Q \subseteq \mathbb{R}^n \setminus \{0\}$, and $P \subseteq \mathbb{R}^m$.

- Define
$$L_{\mathbf{q},\mathbf{p}} = \{A \in M_{m,n}(\mathbb{R}) : A\mathbf{q} - \mathbf{p} = 0\}$$
Note that this is an affine subspace of codimension m in $M_{m,n}(\mathbb{R})$.
- Define $\mathcal{L}_{Q,P} = \{L_{\mathbf{q},\mathbf{p}} : \mathbf{q} \in Q, \mathbf{p} \in P\}$.

Total Density

Definition: Given $Q \subseteq \mathbb{R}^n \setminus \{0\}$ and $P \subseteq \mathbb{R}^m$, we say $\mathcal{L}_{Q,P}$ is totally dense if $\bigcup_{L \in \mathcal{L}_{Q,P}} L$ is dense in $M_{m,n}(\mathbb{R})$ and for every open $W \subseteq Y$ and $L \in \mathcal{L}_R$ where $W \cap L \neq \emptyset$,

$$\bigcup_{L' \in \mathcal{L}_{Q,P}, L' \cap L \cap W \neq \emptyset} L'$$

is not nowhere dense.

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Theorem (Kleinbock, Moshchevitin, Warren, Weiss, 2024)

Let $Q \subseteq \mathbb{R}^n \setminus \{0\}$ and $P \subseteq \mathbb{R}^m$ such that $\mathcal{L}_{Q,P}$ is totally dense and let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be any non increasing function. In addition, let $\mathcal{S}$ be a countable collection of proper affine subspaces of $M_{m,n}(\mathbb{R})$. Then the set $\mathrm{UA}_{Q,P}(f) \setminus \bigcup_{S \in \mathcal{S}} S$ is uncountable and dense.

$T_{m,n}$

Remark: The mapping $(\mathbf{q}, \mathbf{p}) \mapsto L_{\mathbf{q},\mathbf{p}}$ is not injective.

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Definition: Let $T_{m,n} = S^{n-1} \times \mathbb{R}^m$ and define the map $\mathfrak{pr} : (\mathbb{R}^n \setminus \{0\}) \times \mathbb{R}^n \to \mathbf{T}_{m,n}$, $(\mathbf{q}, \mathbf{p}) \mapsto \left(\frac{\mathbf{q}}{|\mathbf{q}|}, \frac{\mathbf{p}}{|\mathbf{q}|}\right)$.

Note that $A\mathbf{q} = \mathbf{p}$ if and only if $A\mathbf{q}/|\mathbf{q}| = \mathbf{p}/|\mathbf{q}|$, so $L_{\mathbf{q},\mathbf{p}} = L_{\mathfrak{pr}(\mathbf{q},\mathbf{p})}$.

Main Lemma

Theorem (H., Neckrasov)

Let $R \subseteq \mathbf{T}_{m,n}$. Fix $\mathbf{r_0} \in R$ and an open subset W of $M_{m,n}$ intersecting $L_{\mathbf{r_0}}$. Suppose there exists $\epsilon > 0$ such that the collection $\mathcal{L}_{R\setminus(B_\epsilon(\mathbf{r_0})\cup B_\epsilon(-\mathbf{r_0}))}$ is dense in W. Then, the set

$$\overline{\bigcup_{\mathbf{r}\in R:L_{\mathbf{r}}\cap L_{\mathbf{r_0}}\cap W\neq\emptyset} L_r}$$

has nonempty interior.

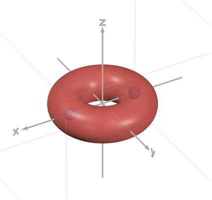

Figure: $T_{m,n} \setminus (B_\epsilon(\mathbf{r_0}) \cup B_\epsilon(-\mathbf{r_0}))$ for $m = n = 2$

Convex Hulls

Proposition

Let $\Sigma \subseteq S^{n-1}$ be a set such that $0 \in \text{conv}(\Sigma)$ when embedded in $\mathbb{R}^n$ and Π be some half space in $\mathbb{R}^m$. Also let $Q = \mathbb{Z}^+\Sigma$ and $P = \Pi \cap \mathbb{Z}^m$. Then $\mathcal{L}_{Q,P}$ is dense in $M_{m,n}(\mathbb{R})$.

Corollary

Let $\Sigma \subseteq S^{n-1}$ be a set such that for all $\theta \in \Sigma$, $0 \in \text{conv}(\Sigma \setminus \{\theta\})$ when embedded in $\mathbb{R}^n$ and Π be some half space in $\mathbb{R}^m$. Also let $Q = \mathbb{Z}^+\Sigma$ and $P = \Pi \cap \mathbb{Z}^m$. Then $\mathcal{L}_{Q,P}$ is totally dense in $M_{m,n}(\mathbb{R})$.

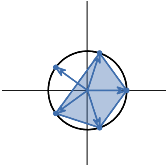

Figure: $0 \in \text{conv}(\Sigma \setminus \{\theta\})$

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THANK YOU!