ORBITAL INTEGRALS OVER LINEAR GROUPS AS LOCAL DENSITIES

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Abstract. Orbital integrals are central to the representation theory of reductive groups, with applications to the trace formula, isogeny classes of elliptic curves, and the (relative) Langlands program. Yet explicit computations are difficult beyond GL_2. Building on methods of Gekeler and Achter–Gordon, we extend finite counting techniques for orbital integrals to $\mathrm{GL}_n(\mathbb{Q}_p)$ and $\mathrm{SL}_n(\mathbb{Q}_p)$. For GL_n, we relate orbital integrals to limits of density ratios over finite quotients of $\mathbb{Z}_p$, yielding explicit formulas with respect to the geometric measure. We further treat arbitrary bi-$\mathrm{GL}_n(\mathbb{Z}_p)$-invariant spherical test functions that detect distinct double cosets. For SL_n, we introduce new conjugacy criteria and a modified ratio that accounts for orbit splitting. In the case of SL_2, we show that these limits recover orbital integrals for both geometric and canonical measures. In all settings, we prove that the corresponding ratios converge to the desired orbital integrals.

Contents

1. Introduction

Background and motivation. Orbital integrals are key objects of study in the representation theory of reductive groups as well as the theory of automorphic forms [Kot05]. Specifically, let GL_n be the algebraic group of invertible matrices and let $\mathbb{Q}_p$ denote the field of p-adic numbers (where p is an odd prime). We are interested in the computation of integrals of certain test functions over conjugation orbits of (regular semisimple) matrices in $\mathrm{GL}_n(\mathbb{Q}_p)$. An important example is the orbital integral of the characteristic function $\mathbb{1}_{\mathrm{GL}_n(\mathbb{Z}_p)}$. Given a linear transformation $\gamma\colon \mathbb{Q}_p^n \to \mathbb{Q}_p^n$, this integral measures the "volume" of the set of bases of $\mathbb{Q}_p^n$ with respect to which the matrix of γ has integer coefficients.

These orbital integrals play an important role in the representation theory of $\mathrm{GL}_n(\mathbb{Q}_p)$ [Kot05]. They show up, historically, in trace formulas like the Arthur-Selberg trace formula [Lan01], where orbital integrals encode contributions from conjugacy classes on the geometric side of the trace formula. They can also be used to calculate the cardinality of an ordinary isogeny class of elliptic curves over $\mathbb{F}_p$, as a consequence of Langlands and Kottwitz's description of points on a Shimura variety over a finite field [AG17].

Orbital integrals also play an important role in recent active areas such as the Beyond Endoscopy conjectures [Lan04] and the relative Langlands program as a whole [Zha17, GGP12, BZSV24], where they are used to isolate contributions from specific Langlands-functorial transfers. Furthermore, many problems of representation theory and harmonic analysis on p-adic groups can be reduced to the explicit evaluation of a family of orbital integrals. They control character values, the Plancherel formula [Sil96], and spectral decompositions on adelic quotients, so computational techniques for orbital integrals have wide-ranging consequences in modern number theory.

Statement of the problem. Despite their important role, explicit computations of orbital integrals are notoriously difficult. Even more so, they have been shown to be difficult to compute; for example, [Hal94] establishes relations between orbital integrals and point counts on hyperelliptic curves which are known to be hard to compute and not expressible as polynomials. Most existing research has been concentrated in simple cases such as integrals over GL_2.

In the case of GL_2, Achter and Gordon [AG17] related some specific orbital integrals over GL_2 to local densities introduced by Gekeler in [Gek03]. Specifically, they establish a powerful method for computing the orbital integral of a regular semisimple element $\gamma \in \mathrm{GL}_2(\mathbb{Q}_p)$ against the characteristic function of the maximal compact subgroup:

$$\int_{\mathrm{Orb}(\gamma)} \mathbb{1}_{\mathrm{GL}_2(\mathbb{Z}_p)}(g^{-1}\gamma g)\, d\mu.$$

This method was further extended in [AAGG23] to work with the more general group GSp_{2g}. However, the Achter–Gordon approach, both for GL_2 and GSp_{2g}, was restricted to a specific orbit and test function.

The present paper aims to significantly broaden the applicability of this method by extending the theory to the linear algebraic groups GL_n and SL_n. This creates a more widely applicable framework for computing orbital integrals via their connection to finite counting problems.

Overview of results. The main results for GL_n include an explicit formula relating the limit of the local ratios to the orbital integral under a measure known as the geometric measure. For SL_2, a new local ratio is introduced in order to deal with a phenomenon where stable orbits split into multiple rational orbits in $\mathrm{SL}_2(\mathbb{Q}_p)$. Then, the limit of these ratios is explicitly connected to orbital integrals under both the geometric and canonical measures.

In Section 2, we collect necessary background knowledge about topics such as orbits, measures, the Steinberg map, and local ratios.

In Section 3, we extend Achter and Gordon's method from GL_2 to GL_n, computing the explicit factor relating the orbital integral to the limit of the local ratios.

In Section 4, we further extend their result to arbitrary bi-$\mathrm{GL}_n(\mathbb{Z}_p)$-invariant test functions rather than just the characteristic function of $\mathrm{GL}_n(\mathbb{Z}_p)$, thereby detecting arbitrary double cosets. Specifically, for some $\lambda_1 \geq \cdots \geq \lambda_n$, let

$$S \coloneqq \mathrm{GL}_n(\mathbb{Z}_p) \operatorname{diag}(p^{\lambda_1}, p^{\lambda_2}, \ldots, p^{\lambda_n}) \mathrm{GL}_n(\mathbb{Z}_p).$$

We demonstrate the following:

Theorem 4.4. *Let $\gamma \in S$ be regular semisimple, and let ϕ_λ denote the characteristic function of S. Then,*

$$\nu(\gamma) = p^{-(n-1)(\lambda_1+\cdots+\lambda_n)} \cdot \frac{p^{n^2-1}}{\#\,\mathrm{SL}_n(\mathbb{F}_p)} O^{\mathrm{geom}}_{\mathrm{GL}_n}(\gamma, \phi_\lambda).$$

In Section 5, we explore conjugacy of matrices in $\mathrm{SL}_n(\mathbb{Q}_p)$ and establish a criterion that we will later add as an extra local condition on the densities to isolate the SL_n-conjugacy class within a GL_n-conjugacy class.

Theorem 5.15. *Let $\gamma \in \mathrm{SL}_n(\mathbb{Q}_p)$ be regular semisimple. An element g in the stable orbit of γ is in the same SL_n-orbit as γ if and only if*

$$\det\left(v \mid \gamma v \mid \cdots \mid \gamma^{n-1} v\right) \equiv \det\left(v \mid gv \mid \cdots \mid g^{n-1} v\right) \pmod{N_{F[\gamma]/F}(F[\gamma]^\times)},$$

where v is any common cyclic vector for γ and g.

In particular, in the case of SL_2, conjugacy requires the matching of the norm class of the top-right elements of the matrices in addition to GL_n-conjugacy. We also give a more direct and elementary proof of this in Appendix B.

In Section 6, we use the results from Section 5 to define a local ratio for detecting conjugacy in $\mathrm{SL}_n(\mathbb{Q}_p)$. We explore how the definition can be modified to be more explicit in the cases of $n = 2$ and $n = 3$.

In Section 7, we relate the limit of the SL_2 ratios to the values of orbital integrals over SL_2 under the canonical measure (and the geometric measure). We establish the following relation:

Theorem 7.8. *The relation between the limit of the ratios and the orbital integral under the canonical measure is given by*

$$\nu^{\mathrm{SL}_2}(\gamma) = p^{-\delta} \cdot (1 - \chi p^{-1})^{-1} \cdot O(\gamma),$$

where δ and χ are defined as in Section 7.

This also allows us to provide an explicit formula for the limits of the SL_2 ratios.

Finally, Section 8 shows that the limits involved in local ratios all eventually stabilize. Furthermore, we can predict when the sequence of ratios will stabilize, and we establish an exact formula for orbital integrals as a finite point count instead of the limit of point counts:

Theorem 8.15. *Let $\gamma \in \mathrm{GL}_n(\mathbb{Z}_p)$ be a regular semisimple element. Then for all $k \geq 2\delta + n$ we have*

$$\frac{\#S_k(\mathfrak{c}(\gamma))}{p^{kn(n-1)}} = \nu(\gamma) = \frac{p^{n^2-1}}{\#\,\mathrm{SL}_n(\mathbb{F}_p)} \cdot O^{\mathrm{geom}}_{\mathrm{GL}_n}(\gamma, \mathbb{1}_{\mathrm{GL}_n(\mathbb{Z}_p)}).$$

In particular,

$$O^{\mathrm{geom}}_{\mathrm{GL}_n}(\gamma, \mathbb{1}_{\mathrm{GL}_n(\mathbb{Z}_p)}) = \frac{\prod_{i=2}^{n}(1-p^{-i})}{p^{(2\delta+n)n(n-1)-1}} \#S_k(\mathfrak{c}(\gamma)).$$

where $\delta = \delta(\gamma)$.

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2. Background

2.1. **Preliminaries.** This section details definitions and preliminary results that will be used throughout the paper. In this paper, we fix an odd prime p. We denote the field of p-adic numbers by $\mathbb{Q}_p$ and the ring of p-adic integers by $\mathbb{Z}_p$.

Remark 2.1. The arguments in this paper extend to any nonarchimedean local field K of characteristic 0 with ring of integers $\mathcal{O}_K$ and maximal ideal $\mathfrak{p}$. We work over $\mathbb{Q}_p$ for notational simplicity, using the fact that $\mathbb{Z}_p/p^k\mathbb{Z}_p \cong \mathbb{Z}/p^k\mathbb{Z}$. For general K, the quotients $\mathcal{O}_K/\mathfrak{p}^k$ are finite local rings of size q^k, where $q = |\mathcal{O}_K/\mathfrak{p}|$, and need not be isomorphic to $\mathbb{Z}/q^k\mathbb{Z}$.

2.1.1. *Rational orbits and stable orbits.* A central consideration in this paper, when dealing with SL_n, is the phenomenon of conjugation orbits (conjugacy classes) splitting when passing from the algebraic closure down to the base field.

Let G be a reductive group[1] over a field F, and let $\gamma \in G(F)$.

The *rational orbit* (also *conjugation orbit* or just *orbit*) of γ is the set of elements in $G(F)$ conjugate to γ by an element of $G(F)$:

$$\mathrm{Orb}(\gamma) \coloneqq \{g^{-1}\gamma g \colon g \in G(F)\}.$$

This set can be identified with the quotient space $G_\gamma(F)\backslash G(F)$, where

$$G_\gamma \coloneqq \{g \in G \colon g\gamma = \gamma g\}$$

is the centralizer of γ. Elements in $\mathrm{Orb}(\gamma)$ are said to be *rationally conjugate* to γ.

The *stable orbit* of γ is the set of elements in $G(F)$ conjugate to γ by an element of $G(\overline{F})$:

$$\mathrm{Orb}^{\mathrm{st}}(\gamma) \coloneqq \{g^{-1}\gamma g \colon g \in G(\overline{F})\} \cap G(F).$$

Here $\overline{F}$ denotes the algebraic closure of F. Elements in $\mathrm{Orb}^{\mathrm{st}}(\gamma)$ are said to be *stably conjugate* to γ.

[1] A reductive group is a linear algebraic group whose unipotent radical is trivial, i.e., it has no nontrivial connected normal unipotent subgroup.

The definitions imply that $\mathrm{Orb}(\gamma) \subseteq \mathrm{Orb}^{\mathrm{st}}(\gamma)$. The potential difference between these two sets is a subtle but important phenomenon.

This distinction arises from the action of the Galois group $\mathrm{Gal}(\overline{F}/F)$. Consider the orbit of γ over the algebraic closure,

$$\overline{\mathrm{Orb}}(\gamma) \coloneqq \{g^{-1}\gamma g \colon g \in G(\overline{F})\}.$$

The Galois group acts on this larger pointed set by acting on the matrix entries of the elements. We have the following:

- An element belongs to the stable orbit if it is part of $\overline{\mathrm{Orb}}(\gamma)$ and its entries are individually fixed by the Galois action (which is simply the condition that its entries lie in F).
- An element belongs to the rational orbit only if it can be written as $g^{-1}\gamma g$ where the matrix g itself has entries in F.

The key point is that an element $h = g^{-1}\gamma g$ can have all its entries in F even if the matrix g used to form it does not. This discrepancy can be measured using Galois cohomology. Example B.2 displays two matrices stably conjugate but not rationally conjugate.

In this paper, the algebraic group G will either be GL_n or SL_n. While the distinction between orbit types is crucial for SL_n, the situation simplifies for GL_n.

Proposition 2.2. *Rational orbits and stable orbits in $\mathrm{GL}_n(F)$ are equal.*

Proof. We follow the sketch of the proof in [Kot05, p.406]. This is a consequence of Galois descent and Hilbert's Theorem 90, applied to the following exact sequence of pointed sets with Galois action:

$$1 \to G_\gamma(\overline{F}) \to G(\overline{F}) \to (G_\gamma\backslash G)(\overline{F}) \to 1.$$

The key observation is that maximal tori of GL_n are quasi split, hence cohomologically trivial. Indeed, maximal tori correspond to centralizers of regular semisimple elements. Given such an element γ, its centralizer can be identified with $F[\gamma]^\times$, or in other words, its centralizer is $\mathrm{Res}_{F[\gamma]/F}\,\mathbb{G}_m$, which is a quasi split torus and therefore cohomologically trivial by Shapiro's lemma and Hilbert 90. We get the following sequence

$$1 \to G_\gamma(F) \to G(F) \xrightarrow{\varphi} (G_\gamma\backslash G)(F) \to H^1(F, G_\gamma(\overline{F})) = 1$$

and we may conclude that

$$\mathrm{Orb}^{\mathrm{st}}(\gamma) = (G_\gamma\backslash G)(F) = \mathrm{Im}(\varphi) = \mathrm{Orb}(\gamma),$$

thus showing that rational orbits are equal to stable orbits in $\mathrm{GL}_n(F)$. $\square$

This greatly simplifies considerations in the case of GL_n. This proposition does not hold true in general for SL_n, however, as is shown in Appendix B.

2.1.2. *Cartan decomposition.* To describe the test functions we integrate, we recall the Cartan decomposition. Using Gaussian elimination over $\mathbb{Z}_p$, we can write

$$\mathrm{GL}_n(\mathbb{Q}_p) = \bigsqcup_{\lambda_1 \geq \cdots \geq \lambda_n} \mathrm{GL}_n(\mathbb{Z}_p)\,\mathrm{diag}\big(p^{\lambda_1}, \ldots, p^{\lambda_n}\big)\ \mathrm{GL}_n(\mathbb{Z}_p),$$

where each $\lambda_i \in \mathbb{Z}$. This is the *Cartan decomposition.*

Given $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$, we let p^λ denote the matrix

$$\mathrm{diag}(p^{\lambda_1}, \ldots, p^{\lambda_n}) \in \mathrm{GL}_n(\mathbb{Q}_p).$$

When λ satisfies $\lambda_1 \geq \cdots \geq \lambda_n$, it is called a *dominant coweight*. Let $\mathbb{Z}^{n,+}$ denote the set of dominant coweights. Furthermore, let $K_n \coloneqq \mathrm{GL}_n(\mathbb{Z}_p)$. We can rewrite the Cartan decomposition as

$$\mathrm{GL}_n(\mathbb{Q}_p) = \bigsqcup_{\lambda \in \mathbb{Z}^{n,+}} K_n p^\lambda K_n. \tag{2.1}$$

In this paper, we will study orbital integrals of bi-K_n-invariant functions with compact support (i.e., functions $f : \mathrm{GL}_n(\mathbb{Q}_p) \to \mathbb{C}$ such that $f(k_1 g k_2) = f(g)$ for all $g \in \mathrm{GL}_n(\mathbb{Q}_p)$ and $k_1, k_2 \in K_n$). By the Cartan decomposition, functions of the form $\mathbb{1}_{K_n p^\lambda K_n}$ span the space of such functions, and thus, it suffices to compute integrals of characteristic functions of each double coset.

2.1.3. *Regular and semisimple elements.* In this section, we define the type of elements we will study. Assume that F is a perfect field.

Given $X \in M_n(F)$, we let G_X denote its centralizer in GL_n.

Definition 2.3. Let $X \in M_n(F)$. We say that X is *semisimple* if its orbit is a closed subspace of GL_n.

Equivalently, the matrix X is semisimple when it is diagonalizable when viewed as an element of $M_n(\overline{F})$.

Example 2.4. If $x \in F^\times$ is not a square, then the matrix $\begin{pmatrix} 0 & x \\ 1 & 0 \end{pmatrix}$ is semisimple but not diagonalizable over F. Its centralizer is the set of matrices $\begin{pmatrix} a & bx \\ b & a \end{pmatrix}$ which is isomorphic to the group $\mathrm{Res}_{F(\sqrt{x})/F}\, \mathbb{G}_m(F) = F(\sqrt{x})^\times$. Its orbit is the set of matrices of trace 0 and determinant $-x$, which is a closed condition. This corresponds to matrices $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ such that $a^2 + bc = x$, which gives us a geometric interpretation of the orbit as a conic in F^3.

For any $X \in M_n(F)$, we may define its regularity as

$$r_X = \frac{\dim G_X - \mathrm{rank}(\mathrm{GL}_n)}{2} = \frac{\dim G_X - n}{2}.$$

Definition 2.5. We say that X is *regular* if $r_X = 0$.

Equivalently, regular elements are elements with centralizers of minimal dimension.

In particular, $X \in M_n(\mathbb{Q}_p)$ is *regular semisimple* if its characteristic polynomial has n distinct roots over $\overline{\mathbb{Q}_p}$.

Example 2.6. When $n = 2$, we have $r_X \in \{0, 1\}$ and therefore all nonscalar matrices are regular.

For example for any $x \in \mathbb{Q}_p$, the matrix $\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$ is semisimple but not regular, the matrix $\begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix}$ is regular but not semisimple. They have the same characteristic polynomials but they are not conjugate.

Example 2.7. In GL_3, the following matrices are all representatives of the unipotent conjugacy classes of regularity $0, 1, 3$ respectively.

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The centralizers are respectively matrices of the form

$$\begin{pmatrix} x & y & z \\ 0 & x & y \\ 0 & 0 & x \end{pmatrix}, \begin{pmatrix} x & y & z \\ 0 & s & t \\ 0 & 0 & x \end{pmatrix}, \begin{pmatrix} \star & \star & \star \\ \star & \star & \star \\ \star & \star & \star \end{pmatrix},$$

which have dimension $3, 5, 9$ respectively.

Note that those are all GL_3 unipotent orbits, classified by their Jordan blocks, of type $(3), (2, 1), (1, 1, 1)$ respectively. Therefore, there is no unipotent matrix of regularity 2.

Remark 2.8. More generally, if G is a connected reductive group acting on a homogeneous space X, we say that x is semisimple if the stabilizer is closed, and regular if that stabilizer has minimal dimension.

In particular, the notions of semisimple and regular are valid for $X \in \mathrm{Lie}(G)$. In this exposition we use the fact that GL_n embeds in its Lie algebra to define the terms simultaneously on the group and its Lie algebra.

Next, we recall the Jordan decomposition.

Proposition 2.9 (Jordan decomposition)**.** *Let $\gamma \in \mathrm{GL}_n(F)$. We may write $\gamma = \gamma_s\gamma_u = \gamma_u\gamma_s$ where γ_s is semisimple and γ_u is unipotent.*

Similarly, if $X \in M_n(F)$, we can decompose $X = X_s + X_n$ where X_s is semisimple and X_n is nilpotent, and $[X_s, X_n] = 0$.

It is clear that if $X \in M_n(F)$ we have $\exp(X)_s = \exp(X_s)$ and $\exp(X)_u = \exp(X_n)$.

The reason why we tend to restrict our attention to regular semisimple matrices is because criteria for conjugacy are nicer in the case of regular semisimple matrices, as shown by the following lemma.

Lemma 2.10. *Let $\gamma \in \mathrm{GL}_n(F)$ be regular semisimple. For all $M \in \mathrm{GL}_n(F)$, we have that $M \in \mathrm{Orb}(\gamma)$ if and only if M has the same characteristic polynomial as γ.*

Proof. The forward direction is true as characteristic polynomials are invariant under conjugation (this is well-known).

For the backward direction, we will begin by showing that $M \in \mathrm{Orb}^{\mathrm{st}}(\gamma)$ and conclude using Proposition 2.2.

If two matrices X and Y are semisimple and have the same characteristic polynomial χ, then they are both similar (in $\mathrm{GL}_n(\overline{F})$) to the diagonal matrix whose diagonal entries equal the multiset of the roots of χ. Thus, they are stably conjugate (i.e. conjugate by a matrix in $\mathrm{GL}_n(\overline{F})$).

Therefore, it suffices to show that if M has the same characteristic polynomial as γ, then it is semisimple. Since M has n distinct eigenvalues, each eigenvalue has geometric multiplicity 1, meaning that M is diagonalizable in $\mathrm{GL}_n(\overline{F})$. We conclude that M is semisimple. $\square$

2.1.4. *The Steinberg quotient.* The *Steinberg map* $\mathfrak{c}\colon M_n \to \mathbb{A}^n$ maps an element $X \in M_n$ to the coefficients of its characteristic polynomial:

$$\mathfrak{c}(X) = (\mathrm{tr}(X), \dots, \det(X)).$$

The codomain of $\mathfrak{c}|_{\mathrm{GL}_n}$ is called the *Steinberg quotient* of GL_n, which we will denote by

$$\mathbb{A}_{\mathrm{GL}_n} = \mathbb{A}^{n-1} \times \mathbb{G}_m.$$

Remark 2.11. The map $\mathfrak{c}$ is sometimes called the "Chevalley map," hence the notation.

As seen in Lemma 2.10, the Steinberg quotient characterizes the space of conjugacy classes for regular semisimple elements in $\mathrm{GL}_n(F)$, as these elements are conjugate if and only if their characteristic polynomials are equal. However, it does not fully characterize the space of conjugacy classes in $\mathrm{SL}_n(F)$. In addition, an element's image in the Steinberg quotient of $\mathrm{GL}_n(F)$ does not determine the element's inclusion in a given double coset. Thus, the Steinberg quotient will be extended in Section 4 in order to encapsulate the extra information needed to differentiate between conjugacy classes as well as double cosets.

Let us list a few straightforward properties of the Steinberg quotient.

Lemma 2.12. *Let $X \in M_n(F)$. We have $\mathfrak{c}(X) = \mathfrak{c}(X_s)$ and $\mathfrak{c}(X_n) = (0, \dots, 0)$. If $\gamma \in \mathrm{GL}_n(F)$, then $\mathfrak{c}(\gamma_u) = \left((-1)^k \binom{n}{k}\right)_{1 \le k \le n}$, where γ_u is the unipotent part of γ in the Jordan decomposition.*

Proof. The characteristic polynomial of any nilpotent matrix is λ^n whereas the characteristic polynomial of a unipotent matrix is

$$(\lambda - 1)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k \lambda^{n-k},$$

thus concluding the proof. □

Lemma 2.13. *For all $\mathbf{x} \in F^n$, the set $\mathfrak{c}^{-1}(\mathbf{x})$ contains a unique (open) regular orbit as well as a unique (closed) semisimple $\mathrm{GL}_n(F)$-orbit.*

2.1.5. *Orbital integrals and measures.* A major focus of the paper is computing orbital integrals over $\mathrm{GL}_n(\mathbb{Q}_p)$. A central task is to define a suitable measure on $\mathrm{Orb}(\gamma)$. When the group $G = \mathrm{GL}_n(\mathbb{Q}_p)$ acts on itself by conjugation, the orbit $\mathrm{Orb}(\gamma)$ can be identified with the $\mathbb{Q}_p$-points of the homogeneous space $G_\gamma \backslash G$, where G_γ is the centralizer of γ.

Since both G and G_γ are locally compact topological groups, they admit Haar measures [Gle10], allowing us to endow the orbit $\mathrm{Orb}(\gamma)$ with a G-invariant quotient measure. The properties of this measure depend crucially on the normalization of the Haar measures on G and G_γ. Two important normalizations are used in the literature:

- The *canonical measure*, denoted $\mu_\gamma^{\mathrm{can}}$ (or μ^{can}), is the quotient measure on $G_\gamma \backslash G$ obtained by equipping both G and G_γ with Haar measures normalized to give the standard maximal compact subgroups measure 1[2]. This

[2]This generally depends on a choice of integral model over $\mathbb{Z}_p$. In our case, when G and G_γ split over an unramified extension, we normalize the measures so that the maximal compact subgroups $G(\mathbb{Z}_p)$ and $G_\gamma(\mathbb{Z}_p)$ have volume 1. The only exception is when G_γ splits only over a *ramified*

normalization is standard in the theory of automorphic forms and is used in [Kot05].

- The *geometric measure*, denoted $\mu_\gamma^{\mathrm{geom}}$ (or μ^{geom}), is defined via the Steinberg map $\mathfrak{c}\colon \mathrm{GL}_n \to \mathbb{A}_{\mathrm{GL}_n}$. For regular semisimple γ, the orbit $\mathrm{Orb}(\gamma)$ is the fiber $\mathfrak{c}^{-1}(\mathfrak{c}(\gamma))$. Fix Haar measures dg on $G = \mathrm{GL}_n(\mathbb{Q}_p)$ and da on $\mathbb{A}_{\mathrm{GL}_n}(\mathbb{Q}_p)$. Then there is a unique measure μ_a^{geom} on each fiber $\mathfrak{c}^{-1}(a)$ such that the disintegration formula
$$\int_G f(g)\,dg \;=\; \int_{\mathbb{A}_{\mathrm{GL}_n}(\mathbb{Q}_p)} \left(\int_{\mathfrak{c}^{-1}(a)} f(x)\, d\mu_a^{\mathrm{geom}}(x) \right) da$$
holds for all compatible test functions f. We write $\mu_\gamma^{\mathrm{geom}} := \mu_{\mathfrak{c}(\gamma)}^{\mathrm{geom}}$. This normalization is convenient for the local ratios considered below.

Remark 2.14. There is an equivalent, concrete way to describe μ^{geom}. Fix Haar measures dg on $G = \mathrm{GL}_n(\mathbb{Q}_p)$ and da on $\mathbb{A}_{\mathrm{GL}_n}(\mathbb{Q}_p)$. For $a \in \mathbb{A}_{\mathrm{GL}_n}(\mathbb{Q}_p)$ and any small open neighborhood B of a, set
$$R(B) := \frac{\mathrm{vol}_G(\mathfrak{c}^{-1}(B))}{\mathrm{vol}_{\mathbb{A}_{\mathrm{GL}_n}}(B)}.$$
As B shrinks to $\{a\}$ (e.g., along p-adic balls), the ratios $R(B)$ converge to the μ_a^{geom}–measure of the fiber $\mathfrak{c}^{-1}(a)$. In fact, this method can be extended to support the following. Let $E \subset \mathrm{GL}_n(\mathbb{Q}_p)$ be an open set. Then,
$$\lim_{B \to \{a\}} \frac{\mathrm{vol}_G(\mathfrak{c}^{-1}(B) \cap E)}{\mathrm{vol}_{\mathbb{A}_{\mathrm{GL}_n}}(B)} = \mu_a^{\mathrm{geom}}(\mathfrak{c}^{-1}(a) \cap E).$$
This fact will be used repeatedly in proofs regarding the convergence of local ratios.

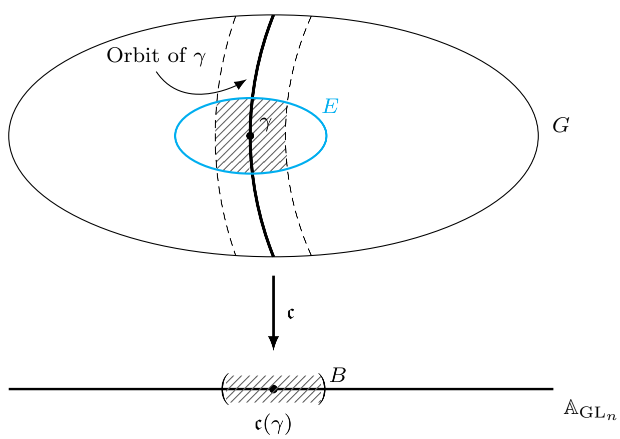

FIGURE 1. Illustration of Remark 2.14

Equipped with these measures, we can now define the orbital integrals.

extension (equivalently, when γ has eigenvalues in a ramified field). In that case the canonical measure is defined so that the maximal compact of the Néron model has volume 1, which may differ slightly from $G_\gamma(\mathbb{Z}_p)$; see Remark 7.2 for the precise adjustment.

Definition 2.15. Let $\gamma \in \mathrm{GL}_n(\mathbb{Q}_p)$ be regular semisimple. Its centralizer, denoted $T = G_\gamma$, is a maximal torus in G. Let $\phi\colon \mathrm{GL}_n(\mathbb{Q}_p) \to \mathbb{C}$ be a compactly supported and locally constant function. The orbital integrals of ϕ with respect to the geometric and canonical measures are:

$$O^{\mathrm{geom}}_{\mathrm{GL}_n}(\gamma, \phi) \coloneqq \int_{T(\mathbb{Q}_p)\backslash \mathrm{GL}_n(\mathbb{Q}_p)} \phi(g^{-1}\gamma g)\, d\mu^{\mathrm{geom}}_\gamma,$$
$$O^{\mathrm{can}}_{\mathrm{GL}_n}(\gamma, \phi) \coloneqq \int_{T(\mathbb{Q}_p)\backslash \mathrm{GL}_n(\mathbb{Q}_p)} \phi(g^{-1}\gamma g)\, d\mu^{\mathrm{can}}_\gamma.$$

In this paper, we will be dealing with specific test functions. When it is clear which test function (or measure) is being used, we may use a shorthand notation such as $O(\gamma)$.

2.1.6. *Gekeler's ratios.* The ratios used in this paper are motivated by a powerful heuristic introduced by Gekeler in his work studying isogeny classes of elliptic curves over finite fields [Gek03]. A central object associated with an elliptic curve is its Frobenius endomorphism. The Frobenius endomorphism can be represented as a conjugacy class of matrices in $\mathrm{GL}_2(\mathbb{Z}_\ell)$ (for $\ell \neq p$), whose characteristic polynomial is $X^2 - tX + p = 0$, where t is the trace of Frobenius.

Gekeler's idea was to assume that the distribution of these Frobenius conjugacy classes was uniform among all possible matrix conjugacy classes. This assumption implies that the proportion of elliptic curves with a given Frobenius trace t should be proportional to the number of all matrices that have trace t and determinant p. This transforms a difficult arithmetic question into a more tractable problem of counting matrices with a prescribed characteristic polynomial.

Definition 2.16. Let $\gamma \in \mathrm{GL}_n(\mathbb{Q}_p) \cap M_n(\mathbb{Z}_p)$. We define

$$S_k(\gamma) \coloneqq \{g \in \mathrm{GL}_n(\mathbb{Z}_p/p^k)\colon \mathfrak{c}(\gamma) \equiv \mathfrak{c}(g) \bmod p^k\},$$

and the local ratios

$$\nu_k(\gamma) = \frac{\# S_k}{\# \mathrm{GL}_n(\mathbb{Z}_p/p^k)/\#\mathbb{A}_{\mathrm{GL}_n}(\mathbb{Z}_p/p^k)}.$$

Also, define $\nu(\gamma)$ to be the limit of these ratios, i.e.,

$$\nu(\gamma) \coloneqq \lim_{k\to\infty} \nu_k(\gamma).$$

Note that when the context is clear and γ is fixed, we will occasionally omit γ and only write S_k and ν_k.

Since we may be dealing with many Gekeler-style ratios, when context is unclear, the above ratio may be called $\nu_k^{\mathrm{GL}_n}(\gamma)$ for more specificity.

2.1.7. *Hensel's lemma.* In order to show convergence of finite-level symbols, we will make use of strong versions of Hensel's lemma in the multivariate case.

Proposition 2.17 ([Con])**.** *For $\mathbf{a} = (a_1, a_2, \ldots, a_d)$ in $\mathbb{Q}_p^d$, define*

$$\|\mathbf{a}\|_p = \max_i |a_i|_p.$$

If $f(X_1, \ldots, X_d) \in \mathbb{Z}_p[X_1, \ldots, X_d]$ and some $\mathbf{a} \in \mathbb{Z}_p^d$ satisfies

$$|f(\mathbf{a})|_p < \|(\nabla f)(\mathbf{a})\|_p^2$$

then there is an $\boldsymbol{\alpha} \in \mathbb{Z}_p^d$ such that $f(\boldsymbol{\alpha}) = 0$ and $\|\boldsymbol{\alpha} - \mathbf{a}\|_p < \|(\nabla f)(\mathbf{a})\|_p$.

Theorem 2.18 (Multivariate Hensel's lemma [Bou98, Corollary 2-3, p. 225])**.** *Let $f_1, \ldots, f_r \in \mathbb{Z}_p[X_1, \ldots, X_n]$ with $r \leq n$. Define the Jacobian of f as the $r \times n$ matrix $J_f(\mathbf{x}) = \left(\frac{\partial f_i}{\partial X_j}(\mathbf{x})\right)$. where $\mathbf{x} \in \mathbb{Z}_p^n$. Assume that there are $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{Z}_p^r$ and $I = \{j_1, \ldots, j_r\}$ so that the corresponding $r \times r$ minor*

$$\min(\mathrm{val} f_i(\mathbf{a}))_{1 \leq i \leq r} > 2\mathrm{val}\left(\frac{\partial f_i}{\partial X_{j_k}}(\mathbf{a})\right)_{1 \leq i,k \leq r} = 2e$$

for all i. In other words, there is e so that one of the minors of $J_f(\mathbf{a})$ is nonzero modulo p^e and $f(\mathbf{a}) \equiv 0 \pmod{p^{2e}}$.

- *If $e = 0$ then there is a unique $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{Z}_p^n$ so that $f_i(\mathbf{x}) = 0$ for all i and $x_i = a_i$ if $i \notin I$, and $x_i \equiv a_i \pmod p$ if $i \in I$.*
- *(Implicit functions theorem) In general, there are power series $\phi_{i_1}, \ldots, \phi_{i_r}$ with no constant terms so that for all $\mathbf{t} = (t_i)_{i \in \{1,\ldots,n\} \setminus I} \in (\mathbb{Z}_p^\times)^{n-r}$ we have*

$$f(\mathbf{a} + p^e \varphi(\mathbf{t})) = 0, \text{ where } \varphi(\mathbf{t}) = \begin{cases} \phi_i(\mathbf{t}) & i \in I \\ p^e t_i & i \notin I \end{cases}.$$

3. Relating orbital integrals to local ratios in GL_n

In this section, we expand upon the work in [AG17] by extending the space of matrices from GL_2 to GL_n. Let γ be a regular semisimple element of $\mathrm{GL}_n(\mathbb{Z}_p)$. To relate the local ratios to the orbital integrals, we must first consider the subset of $G(\mathbb{Z}_p)$ as defined below:

$$V_k(\gamma) \coloneqq \{g \in \mathrm{GL}_n(\mathbb{Z}_p) \colon \mathfrak{c}(g) \equiv \mathfrak{c}(\gamma) \bmod p^k\}.$$

Furthermore, let

$$V(\gamma) \coloneqq \bigcap_{k \geq 1} V_k(\gamma). \tag{3.1}$$

Similarly define $U_k(\mathfrak{c}(\gamma))$ be the neighborhood of $\mathfrak{c}(\gamma)$ in $\mathbb{A}_{\mathrm{GL}_n}(\mathbb{Z}_p)$ defined by

$$U_k(\mathfrak{c}(\gamma)) \coloneqq \{x \in \mathbb{A}_{\mathrm{GL}_n}(\mathbb{Z}_p) \colon x \equiv \mathfrak{c}(\gamma) \bmod p^k\}.$$

We define the auxiliary ratio

$$u_k(\gamma) \coloneqq \frac{\mathrm{vol}_{\mu^{\mathrm{can}}_{\mathrm{GL}_n}}(V_k(\gamma))}{\mathrm{vol}(U_k(\mathfrak{c}(\gamma)))} = \frac{\mathrm{vol}_{\mu^{\mathrm{can}}_{\mathrm{GL}_n}}(V_k(\gamma))}{p^{-nk}}.$$

Now, we take the limit of these ratios and relate it to the limit of the local ratios and the orbital integrals.

Proposition 3.1. *We have*

$$\lim_{k \to \infty} u_k(\gamma) = O^{\mathrm{geom}}_{\mathrm{GL}_n}(\gamma, \mathbb{1}_{\mathrm{GL}_n(\mathbb{Z}_p)}).$$

Proof. Notice that $V(\gamma)$, acting as a limit for $V_k(\gamma)$, becomes the intersection of $\mathrm{Orb}(\gamma)$ and $\mathrm{GL}_n(\mathbb{Z}_p)$ as equality of characteristic polynomials implies conjugacy in $\mathrm{GL}_n(\mathbb{Q}_p)$.

Then $O^{\mathrm{geom}}_G(\gamma, \mathbb{1}_{\mathrm{GL}_n(\mathbb{Z}_p)})$ is just the volume of $V(\gamma)$ as a subset of $\mathrm{Orb}(\gamma)$ with respect to the geometric measure. Let $a_0 = \mathfrak{c}(\gamma) \in \mathbb{A}_{\mathrm{GL}_n}(\mathbb{Z}_p)$. Notice that we can write $V_k(\gamma) = \mathfrak{c}^{-1}(U_k(a_0)) \cap \mathrm{GL}_n(\mathbb{Z}_p)$. Thus,

$$\begin{aligned}\lim_{k\to\infty} u_k(\gamma) &= \lim_{k\to\infty} \frac{\mathrm{vol}_{|\mu_G|}\left(\mathfrak{c}^{-1}(U_k(a_0)) \cap \mathrm{GL}_n(\mathbb{Z}_p)\right)}{\mathrm{vol}_{|\mu_A|}(U_k(a_0))} \\ &= \mathrm{vol}_{|\mu_\gamma^{\mathrm{geom}}|}(V_k(\gamma))\end{aligned}$$

by the definition of the geometric measure (see Remark 2.14). □

Now we relate this auxiliary ratio to our local ratios.

Proposition 3.2. *We have*

$$\lim_{k\to\infty} u_k(\gamma) = \frac{\#\,\mathrm{SL}_n(\mathbb{F}_p)}{p^{n^2-1}} \cdot \nu(\gamma) = \prod_{i=2}^{n} (1-p^{-i}) \cdot \nu(\gamma).$$

Proof. Let $\pi_k = \pi_k^{\mathrm{GL}_n} \colon \mathrm{GL}_n(\mathbb{Z}_p) \to \mathrm{GL}_n(\mathbb{Z}_p/p^k)$ be the reduction map mod p^k. Note that $V_k = \pi_k^{-1}(S_k)$.

Notice that for each root of $\mathfrak{c}(g) \equiv \mathfrak{c}(\gamma)$ in $\mathrm{GL}_n(\mathbb{Z}_p/p^k)$, we have a fiber of volume $(p^{-k})^{n^2}$ in V_k. Thus, we only need to count the number of fibers in V_k, which is $\#S_k$, meaning we have:

$$\mathrm{vol}_{\mu^{\mathrm{can}}_{\mathrm{GL}_n}}(V_k) = \frac{\#S_k}{p^{kn^2}}.$$

By the definition of $u_k(\gamma)$, we have

$$\begin{aligned}u_k(\gamma) &= \frac{\mathrm{vol}_{\mu^{\mathrm{can}}_{\mathrm{GL}_n}}(V_k(\gamma))}{p^{-nk}} = \frac{\#S_k \cdot p^{-kn^2}}{p^{-kn}} \\ &= \frac{\#S_k}{p^{(n^2-n)k}} = \frac{\#S_k \cdot \#\,\mathrm{SL}_n(\mathbb{F}_p)}{\#\,\mathrm{SL}_n(\mathbb{F}_p) \cdot p^{n^2(k-1)} p^{-kn+1} p^{n^2-1}} \\ &= \frac{\#\,\mathrm{SL}_n(\mathbb{F}_p)}{p^{n^2-1}} \cdot \nu_k(\gamma).\end{aligned}$$

Because $\lim_{k\to\infty} \nu_k(\gamma) = \nu(\gamma)$, we have that

$$\lim_{k\to\infty} u_k(\gamma) = \frac{\#\,\mathrm{SL}_n(\mathbb{F}_p)}{p^{n^2-1}} \cdot \nu(\gamma) = \prod_{i=2}^{n} (1-p^{-i}) \cdot \nu(\gamma).$$

This concludes the proof. □

Combining Proposition 3.1 and Proposition 3.2, we get the following:

Theorem 3.3. *The local ratios relate to the orbital integrals via*

$$\nu(\gamma) = \frac{p^{n^2-1}}{\#\,\mathrm{SL}_n(\mathbb{F}_p)} \cdot O^{\mathrm{geom}}_{\mathrm{GL}_n}(\gamma, \mathbb{1}_{\mathrm{GL}_n(\mathbb{Z}_p)}),$$

where $\#\,\mathrm{SL}_n(\mathbb{F}_p) = p^{n^2-1} \prod_{i=2}^{n} (1-p^{-i})$.

Remark 3.4. In general, we expect stable orbital integrals with the geometric measure over more general split groups G (like GL_n) with simply connected semisimple derived subgroup to differ from the local ratios by a factor of $\frac{p^{\dim(G^{\mathrm{der}})}}{\#G^{\mathrm{der}}(\mathbb{F}_p)}$.

Note that what "local ratio" would mean in that case is ambiguous. Indeed, for such groups, the maximal tori are no longer cohomologically trivial, so stable orbits may be strictly larger than rational orbits, and simple equality of characteristic polynomials will not suffice. Even for stable orbital integrals, we do not want to count matrices in $M_n(\mathbb{Z}/p^k\mathbb{Z})$ since they do not all lift to elements of $G(\mathbb{Q}_p)$. A candidate could be the tangent space of G at γ.

4. Integrating general spherical functions

Recall the Cartan decomposition for $\mathrm{GL}_n(\mathbb{Q}_p)$ in (2.1). For $M \in M_n(\mathbb{Z}_p)$ such that $\det(M) \equiv 0 \pmod p$, the matrix M doesn't lie in $\mathrm{GL}_n(\mathbb{Z}_p)$ as the determinant isn't invertible in $\mathbb{Z}_p$. This implies that the intersection of the $\mathrm{Orb}(M)$ with $\mathrm{GL}_n(\mathbb{Z}_p)$ is empty. Thus, the double coset here $K_n p^\lambda K_n$ acts as a replacement for $\mathrm{GL}_n(\mathbb{Z}_p)$ so that the volume isn't 0.

In this case, the characteristic polynomial of an element isn't sufficient to determine which double coset the element is in. To remedy this, we must extend the definition of the Steinberg quotient to record the dominant coweights. First, we define the function:

Definition 4.1. Let
$$\mathrm{inv}\colon \mathrm{GL}_n(\mathbb{Q}_p) \to \mathbb{Z}^{n,+}$$
be the Cartan invariant map, mapping $\gamma \in \mathrm{GL}_n(\mathbb{Q}_p)$ to the unique $\lambda \in \mathbb{Z}^{n,+}$ such that $\gamma \in K_n p^\lambda K_n$. Note, we compute $\mathrm{inv}(\gamma)$ by Gaussian elimination with row and column operations reducing γ to diagonal form.

Now, we can extend our Steinberg map:

Definition 4.2. Define the *extended Steinberg map*,
$$\mathfrak{c}_{\mathrm{inv}}\colon \mathrm{GL}_n \to \mathbb{A}^{n-1} \times \mathbb{G}_m \times \mathbb{Z}^{n,+},$$
by
$$\gamma \mapsto (\mathrm{tr}(\gamma), \dots, \det(\gamma), \mathrm{inv}(\gamma))$$
We also let
$$\mathbb{A}_{\mathrm{inv}} := \mathbb{A}^{n-1} \times \mathbb{G}_m \times \mathbb{Z}^{n,+}$$
be the *extended Steinberg quotient*.

Since local ratios involve computations modulo p^k, we require a reduction map for our extended Steinberg map, like before:
$$\pi_k^{\mathbb{A}_{\mathrm{inv}}}\colon \mathbb{A}^n(\mathbb{Z}_p) \times \mathbb{Z}^{n,+} \to \mathbb{A}^n(\mathbb{Z}_p/p^k) \times (\mathbb{Z}/k \cup \{\infty\})^{n,+}.$$
As before, the map reduces the coefficients of the characteristic polynomial modulo p^k and the elements of λ such that if $0 \le \lambda_i < k$, it gets sent to itself, and if $\lambda_i \ge k$, it gets sent to ∞. With this, we can define our local ratio.

Definition 4.3. Let γ be a regular semisimple element in $\mathrm{GL}_n(\mathbb{Q}_p)$. Define
$$S_k^{\mathrm{inv}} := \left\{ g \in M_n(\mathbb{Z}_p/p^k)\colon \pi_k^{\mathbb{A}_{\mathrm{inv}}}(\mathfrak{c}_{\mathrm{inv}}(g)) = \pi_k^{\mathbb{A}_{\mathrm{inv}}}(\mathfrak{c}_{\mathrm{inv}}(\gamma)) \right\}$$
The local ratio is thus
$$\nu_k(\gamma) := \frac{\# S_k^{\mathrm{inv}}}{\# \mathrm{GL}_n(\mathbb{Z}_p/p^k)/\mathbb{A}_{\mathrm{GL}_n}(\mathbb{Z}_p/p^k)}, \qquad \lim_{k\to\infty} (\nu_k(\gamma)) = \nu(\gamma).$$

Theorem 4.4. *Let γ be a regular semisimple element in $\mathrm{GL}_n(\mathbb{Q}_p)$ such that $\gamma \in K_n p^\lambda K_n$, where $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{Z}^{n,+}$. Let ϕ_λ denote the characteristic function of $K_n p^\lambda K_n$. Then,*
$$\nu(\gamma) = p^{-(n-1)(\lambda_1+\cdots+\lambda_n)} \cdot \frac{p^{n^2-1}}{\# \mathrm{SL}_n(\mathbb{F}_p)} O_{\mathrm{GL}_n}^{\mathrm{geom}}(\gamma, \phi_\lambda).$$

Proof. Let $\pi_k^M\colon M_n(\mathbb{Z}_p) \to M_n(\mathbb{Z}_p/p^k)$ be the reduction map mod p^k, so we can define

$$V_k := (\pi_k^M)^{-1}(S_k^{\mathrm{inv}})$$

We define the set

$$U_k := \{g \in \mathbb{A}_{\mathrm{inv}}(\mathbb{Z}_p)\colon \pi_k^{\mathbb{A}_{\mathrm{inv}}}(g) = \pi_k^{\mathbb{A}_{\mathrm{inv}}}(\gamma)\}$$

as the $(p^{-k})^n$ neighborhood around the image of γ under the Steinberg map. For $k \geq \mathrm{val}(\det(\gamma))$, we have that $V_k \subset K_n p^\lambda K_n$, and thus,

$$V_k = \mathfrak{c}_{\mathrm{inv}}^{-1}(\gamma) \cap K_n p^\lambda K_n$$

We can therefore express our orbital integral as

$$\begin{aligned} O^{\mathrm{geom}}_{\mathrm{GL}_n}(\gamma, \phi_\lambda) &= \lim_{k\to\infty} \frac{\mathrm{vol}_{\mu^{\mathrm{can}}_{\mathrm{GL}_n}}(\mathfrak{c}_{\mathrm{inv}}^{-1}(U_k) \cap K_n p^\lambda K_n)}{\mathrm{vol}_{\mu^{\mathrm{can}}_{A}}(U_k)} \\ &= \lim_{k\to\infty} \frac{|\det(\gamma)|^{-n}\,\mathrm{vol}_{\mu^{\mathrm{can}}_{\mathrm{Mat}_n}}(\mathfrak{c}_{\mathrm{inv}}^{-1}(U_k) \cap K_n p^\lambda K_n)}{|\det(\gamma)|^{-1}\,\mathrm{vol}_{\mu^{\mathrm{can}}_{\mathbb{A}^n}}(U_k)} \\ &= p^{(n-1)(\lambda_1+\cdots+\lambda_n)} \cdot \lim_{k\to\infty} \frac{p^{-kn^2}\#S_k^{\mathrm{inv}}}{p^{-nk}} \\ &= p^{(n-1)(\lambda_1+\cdots+\lambda_n)} \frac{\#\,\mathrm{SL}_n(\mathbb{F}_p)}{p^{n^2-1}}\nu(\gamma). \end{aligned}$$

Rearranging, we get

$$\nu(\gamma) = p^{-(n-1)(\lambda_1+\cdots+\lambda_n)} \cdot \frac{p^{n^2-1}}{\#\,\mathrm{SL}_n(\mathbb{F}_p)} O^{\mathrm{geom}}_{\mathrm{GL}_n}(\gamma, \phi_\lambda).$$

□

Remark 4.5. Note that for $n = 2$, to have $\mathrm{inv}_k(\gamma_0) = \mathrm{inv}_k(\gamma)$ is equivalent to $|\det(\gamma_0)| = |\det(\gamma)|$ and $\min(\mathrm{val}(\gamma_0)) = \min(\mathrm{val}(\gamma))$, where $\min(\mathrm{val}(\gamma))$ denotes the minimum valuation of the entries of γ, because the minimum valuation is preserved under conjugation by $\mathrm{GL}_n(\mathbb{Z})$. Therefore, for local ratios in $n = 2$, we can replace inv_k with $\min(\mathrm{val})$ for a more efficient computation.

5. A characterization of SL_n-conjugacy

This section was inspired by the explicit results in Appendix B regarding SL_2 conjugacy. The goal of this section is to make sense of the criterion mentioned in Theorem B.7 as a generalization of the orientation of a basis, and extend it to higher-dimensional examples.

Fix a dimension $n \geq 2$. Let F be a field and let $V = F^{\oplus n}$. Let $G = \mathrm{GL}(V)$ and $S = \mathrm{SL}(V)$. For $\gamma \in G$, we let $\mathrm{Orb}_G(\gamma)$ (resp. $\mathrm{Orb}_S(\gamma)$) denote the G-orbit (resp. S-orbit) of γ.

Given $g \in G$ and $v \in V$, define the n-tuple $\Lambda_g(v)$ as

$$\Lambda_g(v) = (v, gv, \ldots, g^{n-1}v).$$

Note that G acts on n-tuples in V^n via

$$g \cdot (v_1, \ldots, v_n) := (gv_1, \ldots, gv_n).$$

Lemma 5.1. *Let $\gamma \in G$. For all $g \in G$ we have*

$$\Lambda_\gamma(gv) = g\Lambda_{g^{-1}\gamma g}(v).$$

Proof. The proof is by direct application of the definitions. □

Define the set $\mathcal{C} \subset G \times V$ of *cyclic pairs*

$$\mathcal{C} = \{(g, v) \in G \times V \ : \ \mathrm{Span}_F \Lambda_g(v) = V\}.$$

Definition 5.2. Given an n-tuple of vectors $(v_1, \dots, v_n) \in V^n$, we define its (signed) volume to be

$$\mathrm{vol}((v_1, \dots, v_n)) = \det\left(\begin{array}{c|c|c|c} v_1 & v_2 & \cdots & v_n \end{array} \right).$$

Corollary 5.3. *If g commutes with γ, then $\Lambda_\gamma(gv) = g\Lambda_\gamma(v)$. In that case, we have*

$$\mathrm{vol}(\Lambda_\gamma(gv)) = \det(g)\,\mathrm{vol}(\Lambda_\gamma(v)).$$

Given $\gamma \in G$, let G_γ denote its centralizer in G.

Corollary 5.4. *Let $g, h \in G$ such that $g^{-1}\gamma g = h^{-1}\gamma h$. Then,*

$$\mathrm{vol}(\Lambda_\gamma(gv)) \equiv \mathrm{vol}(\Lambda_\gamma(hv)) \pmod{\det(G_\gamma)}.$$

Proof. Let $c = gh^{-1}$. From $g^{-1}\gamma g = h^{-1}\gamma h$, we get $c\gamma = \gamma c$, so $c \in G_\gamma$.
Then,

$$\mathrm{vol}(\Lambda_\gamma(gv)) = \mathrm{vol}(\Lambda_\gamma(chv)) = \det(c)\,\mathrm{vol}(\Lambda_\gamma(hv)),$$

which yields the desired result. □

Recall that given a regular semisimple element $\gamma \in G$, the F-algebra $F[\gamma]$ has no torsion (by regularity) and is therefore an n-dimensional F-vector space. We may identify V with $F[\gamma]$ so that the action of γ is the same, seen as an element of $\mathrm{GL}(V)$ or $F[\gamma]^\times$. Without the semisimplicity assumption, we would get an identification of the semisimple part of γ in G with $\gamma \in F[\gamma]$.

Under this identification, we have $G_\gamma \cong F[\gamma]^\times$ and $\det(x) = N_{F[\gamma]/F}(x)$ for all $x \in F[\gamma]^\times$. We can therefore establish the following.

Proposition 5.5. *Assume $\gamma \in G$ is regular semisimple. We have*

$$\det(G_\gamma) = N_{F[\gamma]/F}(F[\gamma]^\times).$$

Definition 5.6. Let $\gamma \in G$ be a regular semisimple element. Define the map

$$\mathcal{V}_\gamma \colon V \to F/N_{F[\gamma]/F}(F[\gamma]^\times),\ v \mapsto \mathrm{vol}(\Lambda_\gamma(v)) \pmod{N_{F[\gamma]/F}(F[\gamma]^\times)}.$$

Corollary 5.3 is telling us that $\mathcal{V}_\gamma(G_\gamma v) = \{\mathcal{V}_\gamma(v)\}$ and therefore $\mathcal{V}_\gamma$ factors through the finite quotient V/G_γ.

Definition 5.7. Let π_G and π_V denote the usual projection maps from $\mathcal{C} \subset G \times V$ to G and V, respectively.

Proposition 5.8. *Let $\gamma \in G$ be a regular semisimple element. Under our identification, we have*

$$\pi_V(\pi_G^{-1}(\gamma)) = F[\gamma]^\times.$$

Proof. Since γ is regular semisimple, its characteristic polynomial is also its minimal polynomial. Therefore, the set $\Lambda_\gamma(1) = (1, \gamma, \dots, \gamma^{n-1})$ must have rank n. By Corollary 5.3 we get that all elements in $F[\gamma]^\times = G_\gamma \cdot 1$ belong to $\pi_V(\pi_G^{-1}(\gamma))$.

Conversely, if $x \notin F[\gamma]^\times$ then $\dim(F[\gamma]x) < n$ and therefore $\Lambda_\gamma(x)$ cannot have full-rank. □

Corollary 5.9. *Let $\gamma \in G_\gamma$ be a regular semisimple element. Then the map $\mathcal{V}_\gamma$ is constant on $\pi_V(\pi_G^{-1}(\gamma))$.*

Remark 5.10. Let $G^{ell} \subset G$ the set of regular semisimple elements γ so that $F[\gamma]$ is a field. Then Proposition 5.8 tells us that $G^{ell} \times V \backslash \{0\} \subset \mathcal{C}$.

Example 5.11. Assume $n = 2$ and $\gamma \in G$ is regular semisimple. We have that $F[\gamma] \cong F \oplus F$ when γ is diagonalizable, and $F[\gamma]$ is a field otherwise. Consequently, we can write

$$V/G_\gamma = \begin{cases} \{\{0\}, F^\times \times \{0\}, \{0\} \times F^\times, F[\gamma]^\times\} & \text{if } \gamma \text{ is diagonalizable} \\ \{\{0\}, F[\gamma]^\times\} & \text{else} \end{cases}.$$

In the second case, the zero vector generates a rank 0 lattice, whereas any vector in $F[\gamma]^\times$ gives rise to a basis of V. In the first case, vectors in $F^\times \times \{0\}$ and $\{0\} \times F^\times$ are eigenvectors for γ and therefore their images under powers of γ generate 1-dimensional eigenspaces.

Definition 5.12. Let $\gamma \in G$ be a regular semisimple element. Define the map $\omega_\gamma \colon \operatorname{Orb}_G^{\mathrm{st}}(\gamma) \to F^\times / N_{F[\gamma]/F}(F[\gamma]^\times)$ by

$$\omega_\gamma(\gamma') = \frac{\operatorname{vol}(\Lambda_{\gamma'}(v))}{\operatorname{vol}(\Lambda_\gamma(v))} \pmod{N_{F[\gamma]/F}(F[\gamma]^\times)},$$

where $v \in \pi_V(\pi_G^{-1}(\gamma)) \cap \pi_V(\pi_G^{-1}(\gamma'))$.

Proposition 5.13. *The map ω_γ is well-defined.*

Proof. Firstly, let us observe that the set $\pi_V(\pi_G^{-1}(\gamma)) \cap \pi_V(\pi_G^{-1}(\gamma'))$ is nonempty because $F[\gamma]^\times$ and $F[\gamma']^\times$ are both Zariski open sets and therefore must intersect.

Since γ, γ' are conjugate then $\det(G_\gamma) = \det(G_{\gamma'}) = N_{F[\gamma]/F}(F[\gamma]^\times)$ and therefore $\omega_\gamma(\gamma')$ does not depend on the choice of v. $\square$

Example 5.14. Let $p = 3$. Consider the two matrices

$$\gamma = \begin{pmatrix} 0 & 0 & 0 & -2 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \ \gamma' = \begin{pmatrix} 0 & 0 & 0 & -2/3 \\ 0 & 3 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Both matrices are semisimple and have characteristic polynomial $\lambda^4 + \lambda^2 + 2$. Pick $v = 1 \oplus 0 \oplus 0 \oplus 0$ which is a cyclic vector for both matrices, and

$$\operatorname{vol} \Lambda_\gamma(v) = 1, \ \operatorname{vol} \Lambda_{\gamma'}(v) = 27.$$

We get that $\omega_\gamma(\gamma') = 27$.

Theorem 5.15. *We have $\operatorname{Orb}_S(\gamma) = \omega_\gamma^{-1}(1)$. In other words, given $\gamma \in \mathrm{SL}_n(\mathbb{Q}_p)$ a regular semisimple element and g in the stable orbit of γ, g and γ are in the same SL_n-orbit if and only if*

$$\det\left(v \mid \gamma v \mid \cdots \mid \gamma^{n-1} v\right) \equiv \det\left(v \mid gv \mid \cdots \mid g^{n-1} v\right) \pmod{N_{F[\gamma]/F}(F[\gamma]^\times)},$$

where v is any common cyclic vector for γ and g.

Proof. Let $\gamma' \in \operatorname{Orb}_G(\gamma)$ and let $g \in G$ so that $\gamma' = g^{-1}\gamma g$. By Proposition 5.1 we know that

$$\Lambda_\gamma(gv) = g \cdot \Lambda_{\gamma'}(v).$$

We get two facts out of this: firstly $gv \in \pi_V(\pi_G^{-1}(\gamma))$ since the set generated by gv is a basis, and secondly $\mathcal{V}_\gamma(gv) = \det(g)\mathcal{V}_{\gamma'}(v)$. However, Proposition 5.8 tells us that gv and v are both in $F[\gamma]^\times$ and therefore $\mathcal{V}_\gamma(gv) = \mathcal{V}_\gamma(v)$. We obtain that

$$\omega_\gamma(\gamma') = \frac{\det(g^{-1})\mathcal{V}_\gamma(gv)}{\mathcal{V}_\gamma(v)} = \det(g)^{-1} \pmod{N_{F[\gamma]/F}(F[\gamma]^\times)}.$$

Therefore, if $\gamma' \in \mathrm{Orb}_S(\gamma)$ then we may assume that $g \in S$ and therefore $\omega(\gamma') = 1$.

Conversely, assume that $\det(g) = N_{F[\gamma]/F}(x)$ for some $x \in F[\gamma]^\times = G_\gamma$. Define $h = x^{-1}g \in S$. We have $\det(h) = \det(x)^{-1}\det(g) = 1$ and $h^{-1}\gamma h = g^{-1}\underbrace{x\gamma x^{-1}}_{=\gamma}g = g^{-1}\gamma g = \gamma'$. Therefore, we conclude that $\gamma' \in \mathrm{Orb}_S(\gamma)$. □

Example 5.16. Continuing Example 5.14, we find that γ and γ' are conjugate in $\mathrm{SL}_4(\mathbb{Q}_3)$ if and only if 27 is a norm in $E = \mathbb{Q}_3[\lambda]/(\lambda^4 + \lambda^2 + 2)$. The polynomial $\lambda^4 + \lambda^2 + 2$ defines a degree 4 extension of $\mathbb{F}_3$ hence the extension E is the unique degree 4 unramified extension of $\mathbb{Q}_3$. Since $v_3(27) = 3$ is not divisible by 4, it is not a norm hence γ and γ' are $\mathrm{GL}_4(\mathbb{Q}_3)$ conjugate but not $\mathrm{SL}_4(\mathbb{Q}_3)$-conjugate.

Corollary 5.17. *Let $\gamma \in G$ be a regular semisimple element so that γ has an eigenvector $v \in V$. Then $\mathrm{Orb}_G(\gamma) = \mathrm{Orb}_S(\gamma)$.*

Proof. Let $\lambda \in F$ be an eigenvalue of γ corresponding to v. Then write the characteristic polynomial of γ as $p_\gamma(t) = (t-\lambda)f(t)$ for some polynomial $f(t)$. By the Chinese Remainder Theorem we have

$$F[\gamma] \cong F[t]/(p_\gamma(t)) \cong F[t]/(t-\lambda) \oplus F[t]/(f(t)) \cong F \oplus F[t]/(f(t)).$$

Under this decomposition, the norm of an element $(x, 1)$ where $x \in F$ is itself, hence $N_{F[\gamma]/F}(F[\gamma]^\times) = F^\times$ and therefore the map ω_γ is trivial. □

Remark 5.18. Note that if $\gamma \in G$ then $\mathrm{Orb}_S^{\mathrm{st}}(\gamma) = \mathrm{Orb}_G(\gamma)$. Let $\mathbf{S}_\gamma$ be the centralizer of γ in SL_n seen as an algebraic torus. We know that the number of S-orbits inside $\mathrm{Orb}_S^{\mathrm{st}}(\gamma)$ is equal to $|H^1(F, \mathbf{S}_\gamma(\overline{F}))|$. Write $F[\gamma] = F_1 \oplus \cdots \oplus F_r$ be a decomposition of $F[\gamma]$ as sum of fields. Consider the exact sequence on the $\overline{F}$-points of the sequence

$$1 \longrightarrow \mathbf{S}_\gamma \longrightarrow \underbrace{\prod_{i=1}^r \mathrm{Res}_{F_i/F}\,\mathbb{G}_m}_{=\mathbf{G}_\gamma} \xrightarrow{N_{F[\gamma]/F}} \mathbb{G}_m \longrightarrow 1,$$

where $\mathrm{Res}_{F_i/F}$ denotes the Weil restriction of scalars. The middle term is cohomologically trivial and therefore

$$\begin{aligned} H^1(F, \mathbf{S}_\gamma(\overline{F})) &\cong \mathrm{Coker}(H^0(F, \mathbf{G}_\gamma(\overline{F})) \longrightarrow H^0(F, \overline{F}^\times)) \\ &\cong F^\times / N_{F[\gamma]/F}(F[\gamma]^\times). \end{aligned}$$

Therefore we already know that the index of $N_{F[\gamma]/F}(F[\gamma]^\times)$ in $F^\times$ is equal to the number of S-orbits in the G-orbits. The upshot of Theorem 5.15 however, is that it gives us an explicit way to sort elements of $\mathrm{Orb}_G(\gamma)$ into S-orbits.

Corollary 5.19. *Assume that $\gamma \in G$ is regular semisimple and that $F[\gamma]$ is a field. Then $\gamma \in \mathrm{Orb}_S(\gamma)$ if and only if the two following conditions hold:*

(*i*) *The characteristic polynomials of γ and γ' are equal.*

(ii) $\dfrac{\det\left(v \mid \gamma v \mid \cdots \mid \gamma^{n-1}v\right)}{\det\left(v \mid \gamma' v \mid \cdots \mid (\gamma')^{n-1}v\right)} \in N_{F[\gamma]/F}(F[\gamma]^\times)$, *where v can be taken to be any nonzero vector in V.*

Proof. We have seen that the first condition ensures that $\gamma' \in \mathrm{Orb}_G(\gamma)$. The second condition is a reformulation of Theorem 5.15. The condition that $F[\gamma]$ is a field means that $F[\gamma]^\times = F[\gamma']^\times = V\backslash\{0\}$ hence $\mathcal{V}_\gamma$ and $\mathcal{V}_{\gamma'}$ are constant on $V\backslash\{0\}$ hence the result does not depend on the choice of v. □

Corollary 5.20. *Assume that $n = 2$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ be an elliptic regular semisimple element. Let $\gamma' = \begin{pmatrix} x & y \\ s & t \end{pmatrix} \in G$. The following are equivalent:*

(i) $\gamma' \in \mathrm{Orb}_S(\gamma)$;
(ii) $\det(\gamma) = \det(\gamma')$, $\mathrm{Tr}(\gamma) = \mathrm{Tr}(\gamma')$, *and* $c \equiv s \pmod{N_{F[\gamma]/F}(F[\gamma]^\times)}$;
(iii) $\det(\gamma) = \det(\gamma')$, $\mathrm{Tr}(\gamma) = \mathrm{Tr}(\gamma')$, *and* $b \equiv y \pmod{N_{F[\gamma]/F}(F[\gamma]^\times)}$.

Proof. This is a reformulation of Corollary 5.19. Indeed, if we take $v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ then

$$\frac{\det\left(\ v \mid \gamma v\ \right)}{\det\left(\ v \mid \gamma' v\ \right)} = \frac{\det\begin{pmatrix} 1 & a \\ 0 & c \end{pmatrix}}{\det\begin{pmatrix} 1 & x \\ 0 & s \end{pmatrix}} = \frac{c}{s},$$

which proves $(i) \Leftrightarrow (ii)$.

Similarly, taking $v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ shows $(i) \Leftrightarrow (iii)$. □

Example 5.21. Let $\gamma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\gamma' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. If $v = \langle a, b\rangle$ is any nonzero vector, then

$$\mathrm{vol}\left(\Lambda(v,\gamma)\right) = a^2 + b^2, \text{ and } \mathrm{vol}\left(\Lambda(v,\gamma')\right) = -(a^2+b^2).$$

Therefore, we get that γ and γ' are conjugate in $\mathrm{SL}_2(\mathbb{Q}_p)$ if and only if -1 is a norm of $\mathbb{Q}_p(\gamma)$.

Corollary 5.22. *Assume that $n = 3$ and $\gamma = \left(\gamma_{ij}\right)_{1\le i,j\le 3}$ be an elliptic regular semisimple element. Let $\gamma' = \left(\gamma'_{ij}\right)_{1\le i,j\le 3}$. If γ and γ' share the same characteristic polynomial, then they are S-conjugate if and only if*

$$\gamma_{21}^2\gamma_{32} + \gamma_{21}\gamma_{31}\gamma_{33} - \gamma_{21}\gamma_{22}\gamma_{31} - \gamma_{23}\gamma_{31}^2 \equiv {\gamma'_{21}}^2\gamma'_{32} + \gamma'_{21}\gamma'_{31}\gamma'_{33} - \gamma'_{21}\gamma'_{22}\gamma'_{31} - \gamma'_{23}{\gamma'_{31}}^2$$

mod $F^\times/N_{F[\gamma]/F}(F[\gamma]^\times)$.

Proof. This is following the same method as the previous corollary, taking $v = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. Note that if $F[\gamma]/F$ is not a field extension of degree 3, then $F[\gamma]$ must be isomorphic to $F^{\oplus 3}$ or $F \oplus E$, where E/F is a quadratic extension. In both cases, we have that $N_{F[\gamma]/F}(F[\gamma]^\times) = F^\times$ so the equation holds by default. □

6. A local ratio for SL_n rational orbits

In this section, we define a local ratio for matrices in SL_n which aims to detect rational conjugacy as in Theorem 5.15.

6.1. **Defining the ratio in general.** Recall that $\mathfrak{c}\colon \mathrm{SL}_n(\mathbb{Q}_p) \to \mathbb{A}_{\mathrm{SL}_n}$, the Steinberg map, takes a matrix to the n-tuple of coefficients of its characteristic polynomial.

Let $R_k = \mathbb{Z}_p/p^k\mathbb{Z}_p$ and let $v \in (R_k)^n$. Then, for each $h \in \mathrm{SL}_n(R_k)$, we call (h, v) a cyclic pair if $\{v, hv, \dots, h^{n-1}v\}$ spans R_k^n, or equivalently if

$$\mathrm{vol}(\Lambda_h(v)) = \det\left(\, v \, \middle| \, hv \, \middle| \, \cdots \, \middle| \, h^{n-1}v \, \right) \in R_k^\times.$$

We have $R_k[h] \subseteq M_n(R_k)$. Then, the determinant-of-multiplication defines a multiplicative norm map

$$N_{R_k[h]/R_k}\colon R_k[h]^\times \to R_k^\times, \; x \mapsto \det(m_x\colon R_k[h] \to R_k[h]).$$

If we fix any cyclic v for h, then we can define the invariant norm class

$$\mathcal{V}_{h,k} = [\mathrm{vol}(\Lambda_h(v))] \in R_k^\times / N_{R_k[h]/R_k}(R_k[h]^\times).$$

Since existence of a cyclic v is not guaranteed, we work over the cyclic locus, defined as

$$\mathrm{SL}_n(R_k)^{\mathrm{cyc}} := \{h \in \mathrm{SL}_n(R_k)\colon \exists v \text{ with } \mathrm{vol}(\Lambda_h(v)) \in R_k^\times\}.$$

For $\gamma \in \mathrm{SL}_n(\mathbb{Z}_p)$, let

$$\mathcal{F}_k(\gamma) = \{h \in \mathrm{SL}_n(R_k)\colon \mathfrak{c}(h) \equiv \mathfrak{c}(\gamma) \bmod p^k\}$$

and intersect

$$\mathcal{F}_k(\gamma)^{\mathrm{cyc}} = \mathcal{F}_k(\gamma) \cap \mathrm{SL}_n(R_k)^{\mathrm{cyc}}.$$

Definition 6.1. Let $\gamma \in \mathrm{SL}_n(\mathbb{Z}_p)$ be regular semisimple. We define the local ratio

$$\nu_k^{\mathrm{SL}_n}(\gamma) = \frac{\#\{h \in \mathcal{F}_k(\gamma)^{\mathrm{cyc}}\colon \mathcal{V}_{h,k} = \mathcal{V}_{\gamma,k}\}}{\#\mathcal{F}_k(\gamma)^{\mathrm{cyc}}}.$$

Furthermore, we define

$$\nu^{\mathrm{SL}_n}(\gamma) = \lim_{k\to\infty} \nu_k^{\mathrm{SL}_n}(\gamma)$$

whenever the limit exists.

This definition is general, but it is impractical to work with and compute because of the mystery surrounding $\mathcal{V}_{h,k}$. In the following subsections, we aim to provide simpler definitions of the ratio for low values of n (the dimension).

6.2. **A concrete definition in SL_2 using the Hilbert symbol.** First, we define a local version of the Hilbert symbol for $\mathbb{Z}_p/p^k$.

Definition 6.2. Let $a, b \in \mathbb{Z}_p/p^k$, and define $(a, b)_{p,k}$ to be 1 if there exists a solution (x, y, z) to $z^2 = ax^2 + by^2$, where $x, y, z \in \mathbb{Z}_p/p^k$ and at least one of x, y, z is a unit in $\mathbb{Z}_p/p^k$, and -1 otherwise. If $a, b \in \mathbb{Z}_p \setminus \{0\}$, then we define $(a, b)_{p,k} := (\overline{a}, \overline{b})_{p,k}$, where $\overline{a}$ and $\overline{b}$ represent a and b reduced modulo p^k, respectively.

Lemma 6.3. *Let $a, b \in \mathbb{Z}_p \setminus \{0\}$. As $k \to \infty$, the symbol $(a, b)_{p,k}$ converges to the usual Hilbert symbol $(a, b)_p$.*

Proof. The Hilbert symbol $(a,b)_p$ is 1 if and only if the equation

$$F(x,y,z) = z^2 - ax^2 - by^2 = 0$$

has a nontrivial solution in $\mathbb{Q}_p$. By clearing denominators and common factors of p, this is equivalent to the existence of a primitive solution in $\mathbb{Z}_p$, i.e., a solution $(x,y,z) \in \mathbb{Z}_p^3$ where at least one of x,y,z is a unit.

Suppose $(a,b)_p = 1$. Then, there is a primitive solution $(x_0,y_0,z_0) \in \mathbb{Z}_p^3$ to $F(x,y,z) = 0$. Taking this solution mod p^k yields $(a,b)_{p,k} = 1$ for all $k \geq 1$, so the desired result is true in this case.

The other case is $(a,b)_p = -1$. This means that the equation $F(x,y,z) = 0$ has no primitive solution in $\mathbb{Z}_p$. We will show that $(a,b)_{p,k} = -1$ for all sufficiently large k.

We claim that some K exists such that $(a,b)_{p,K} = -1$. Suppose to the contrary that $(a,b)_{p,k} = 1$ for all k. This implies, for each k, the existence of an approximate solution $\mathbf{v}_k = (x_k,y_k,z_k) \in \mathbb{Z}_p^3$ that satisfies $|F(\mathbf{v}_k)|_p \leq p^{-k}$ and $\|\mathbf{v}_k\|_p = 1$.

The gradient is

$$\nabla F(\mathbf{v}_k) = (-2ax_k, -2by_k, 2z_k).$$

Let $M = \|\nabla F(\mathbf{v}_k)\|_p$. Specifically, since $\|\mathbf{v}_k\|_p = 1$, we have

$$M = |2|_p \cdot \max(|ax_k|_p, |by_k|_p, |z_k|_p) \geq |2|_p \cdot \min(|a|_p, |b|_p, 1).$$

The lifting condition is $|F(\mathbf{v}_k)|_p < M^2$, which becomes $p^{-k} < M^2$. Since M is bounded below by a positive constant that does not depend on k, we can certainly find a large enough k such that $p^{-k} < M^2$. For such a k, Proposition 2.17 guarantees the existence of a true solution $\boldsymbol{\alpha} \in \mathbb{Z}_p^3$ to $F(\boldsymbol{\alpha}) = 0$. This contradicts the statement $(a,b)_p = -1$.

Thus, there must exist some integer K such that

$$(a,b)_{p,K} = -1.$$

Furthermore, if $(a,b)_{p,k}$ for some $k > K$, then such a solution would reduce to a primitive solution mod p^K, contradicting the fact that $(a,b)_{p,K} = -1$. So, we have $(a,b)_{p,k} = -1$ for all $k \geq K$, and thus it stabilizes to -1. ☐

We define this Hilbert symbol in order to preserve the fact that everything in the local ratio is a finite count. However, in practice, there are efficient ways to compute the Hilbert symbol directly, so one does not need to rely on this finite-level Hilbert symbol. An example is given in Appendix A.

Proposition 6.4. *Let $\gamma \in \mathrm{SL}_2(\mathbb{Z}_p)$ be regular semisimple. Then,*

$$\nu_k^{\mathrm{SL}_2}(\gamma) \sim \frac{\# \left\{ h \in \mathrm{SL}_2(\mathbb{Z}_p/p^k) \colon \begin{array}{l} \mathfrak{c}(h) \equiv \mathfrak{c}(\gamma) \bmod p^k, \\ (h_{12}, D)_{p,k} = (\gamma_{12}, D)_{p,k} \end{array} \right\}}{\# \mathrm{SL}_2(\mathbb{Z}_p/p^k) / \# \mathbb{A}_{\mathrm{SL}_2}(\mathbb{Z}_p/p^k)}.$$

(Here, the notation $\sim$ denotes having the same limit as $k \to \infty$.)

Proof. Follows from Corollary 5.20. ☐

6.3. **An invariant for** SL_3**.** Let $\gamma = \left(\gamma_{ij}\right)_{1\le i,j\le 3}$ be an elliptic regular semisimple element in $\mathrm{SL}_3(\mathbb{Z}_p)$. Define

$$\kappa(\gamma) = \gamma_{21}^2\gamma_{32} + \gamma_{21}\gamma_{31}\gamma_{33} - \gamma_{21}\gamma_{22}\gamma_{31} - \gamma_{23}\gamma_{31}^2.$$

According to Corollary 5.22, if γ and γ' have the same characteristic polynomial and $\kappa(\gamma) \equiv \kappa(\gamma') \pmod{\mathbb{Q}_p^\times/N_{\mathbb{Q}_p[\gamma]/\mathbb{Q}_p}(\mathbb{Q}_p[\gamma]^\times)}$, then they are conjugate. With this, we state the SL_3 ratio:

Proposition 6.5. *Let $\gamma \in \mathrm{SL}_3(\mathbb{Z}_p)$ be regular semisimple. Then,*

$$\nu_k^{\mathrm{SL}_3}(\gamma) \sim \frac{\#\left\{ h \in \mathrm{SL}_3(\mathbb{Z}_p/p^k) \colon \begin{array}{l} \mathfrak{c}(h) \equiv \mathfrak{c}(\gamma) \bmod p^k, \\ \kappa(h) \equiv \kappa(\gamma) \bmod R_k^\times/N_{R_k[h]/R_k}(R_k[h]^\times) \end{array} \right\}}{\#\,\mathrm{SL}_3(R_k)/\#\mathbb{A}_{\mathrm{SL}_3}(R_k)}.$$

7. Explicit computations for SL_2 local ratios

Because computations are much simpler in SL_2, we are able to use experimental data (see Appendix A) to conjecture the following as an explicit formula of these SL_2 ratios, which we will prove in this section:

Proposition 7.1. *Let $\gamma \in \mathrm{SL}_2(\mathbb{Q}_p)$ be regular semisimple and have trace t. Let $D = t^2 - 4$ be the discriminant of the characteristic polynomial. Let $\delta = \lfloor \frac{\mathrm{val}(D)}{2} \rfloor$, i.e., $p^{2\delta}$ is the highest power of p^2 that divides D. Let $\chi = \left(\frac{D/p^{2\delta}}{p}\right)$ (this is the Legendre symbol). We have*

$$\nu^{\mathrm{SL}_2}(\gamma) = p^{-\delta} \cdot \begin{cases} \frac{p}{p-1} \cdot p^\delta, & \chi = 1, \\ \frac{1}{2} \cdot \frac{p^{\delta+1}-1}{p-1}, & \chi = 0, \\ \frac{p}{p+1} \cdot \frac{p^\delta - 1}{p-1} \text{ or } \frac{p}{p+1} \cdot \frac{p^{\delta+1}-1}{p-1}, & \chi = -1. \end{cases}$$

The two cases for $\chi = -1$ represent the two different conjugacy classes that γ may be in.

7.1. Relating the ratio with the orbital integral.

This section aims to prove Proposition 7.1 by relating the ratio with an orbital integral with known values.

7.1.1. *Formulating the problem in terms of existing results.* Fix choices of Haar measures on $\mathrm{SL}_2(\mathbb{Q}_p)$ and $T(\mathbb{Q}_p)$ (the centralizer); suppose they are dg and dg_γ, respectively. Let $d\dot{g} = \frac{dg}{dg_\gamma}$ be a measure on $T(\mathbb{Q}_p)\backslash \mathrm{SL}_2(\mathbb{Q}_p)$. For regular semisimple $\gamma \in \mathrm{SL}_2(\mathbb{Z}_p)$, define

$$O(\gamma) = \int_{T(\mathbb{Q}_p)\backslash \mathrm{SL}_2(\mathbb{Q}_p)} \mathbf{1}_{\mathrm{SL}_2(\mathbb{Z}_p)}(g^{-1}\gamma g)\, d\dot{g}.$$

The value of this orbital integral depends on the specific Haar measures chosen. In this section, we denote by $O(\gamma)$ the integral using the canonical measure, denoted μ^{can} (see Section 2.1.5).

Remark 7.2. In this case, the canonical measure is gotten by first letting dg be the Haar measure on $\mathrm{SL}_2(\mathbb{Q}_p)$ giving $\mathrm{SL}_2(\mathbb{Z}_p)$ volume 1. Then, the measure on the centralizer $T(\mathbb{Q}_p)$ depends on whether it is split or elliptic. In the split case, we choose the measure giving $T(\mathbb{Z}_p)$ volume 1. In the elliptic case, the connected component of the identity (denoted $T^\circ(\mathbb{Q}_p)$) is assigned a volume of 1.

Theorem 7.3 ([Rüd25])**.** *For regular semisimple $\gamma \in \mathrm{SL}_2(\mathbb{Z}_p)$, suppose γ has distinct eigenvalues $a, b \in \mathbb{Q}_p(\gamma)$ where $\mathbb{Q}_p(\gamma)$ is the smallest field containing the eigenvalues of γ. Let*

$$d_\gamma = \mathrm{val}\left(1 - \frac{a}{b}\right),$$

where val *is normalized with* $\mathrm{val}(p) = 1$*. Then,*

$$O(\gamma) = \begin{cases} p^{d_\gamma}, & \chi = 1 \quad \textit{(hyperbolic)}, \\ \frac{1}{2} \cdot \frac{p^{d_\gamma + \frac{1}{2}} - 1}{p-1}, & \chi = 0 \quad \textit{(ramified elliptic)}, \\ \frac{p^{d_\gamma} - 1}{p-1} \textit{ or } \frac{p^{d_\gamma + 1} - 1}{p-1}, & \chi = -1 \quad \textit{(unramified elliptic)}. \end{cases}$$

Remark 7.4. The measure used in [Rüd25] differs from μ^{can} in the ramified case. Specifically, the measure μ^{can} is half the measure on the orbit used in [Rüd25]. This factor arises because the measure used in [Rüd25] gives the entire centralizer volume 1, which consists of two connected components in the ramified case. We have appropriately scaled the cited value of $O(\gamma)$ to match our use of μ^{can}.

We begin by explicitly relating the quantities d_γ and $\delta \coloneqq \lfloor \frac{\mathrm{val}(D)}{2} \rfloor$.

Lemma 7.5. *We have*

$$d_\gamma = \begin{cases} \delta, & \chi \in \{-1, 1\}, \\ \delta + \frac{1}{2}, & \chi = 0. \end{cases}$$

Proof. Suppose $\chi_\gamma(x) = x^2 - tx + 1$ is the characteristic polynomial of γ. Then we have

$$a, b = \frac{-t \pm \sqrt{D}}{2} \implies b - a = \pm\sqrt{D}.$$

Then,

$$\begin{aligned} d_\gamma &= \mathrm{val}\left(1 - \frac{a}{b}\right) = \mathrm{val}\left(\frac{b-a}{b}\right) \\ &= \mathrm{val}\left(\frac{\sqrt{D}}{b}\right) = \frac{1}{2}\,\mathrm{val}(D) - \mathrm{val}(b). \end{aligned}$$

Since $\chi_\gamma \in \mathbb{Z}_p[x]$ and it is monic, the roots a and b are integral over $\mathbb{Z}_p$. Hence, they both have nonnegative valuation. However, at the same time, we get $\mathrm{val}(a) + \mathrm{val}(b) = \mathrm{val}(ab) = \mathrm{val}(1) = 0$, so $\mathrm{val}(a) = \mathrm{val}(b) = 0$. Thus,

$$d_\gamma = \frac{1}{2}\,\mathrm{val}(D).$$

By the definition of δ, we have $\delta = \lfloor d_\gamma \rfloor$. Notice that $\chi \in \{-1, 1\}$ whenever $\mathrm{val}(D)$ is even and $\chi = 0$ whenever $\mathrm{val}(D)$ is odd. This yields the desired result. □

Using this result, we can rewrite the values for $O(\gamma)$ as follows:

$$O(\gamma) = \begin{cases} p^{\delta}, & \chi = 1, \\ \frac{1}{2} \cdot \frac{p^{\delta+1} - 1}{p-1}, & \chi = 0, \\ \frac{p^{\delta} - 1}{p-1} \text{ or } \frac{p^{\delta+1} - 1}{p-1}, & \chi = -1. \end{cases}$$

This reduces Proposition 7.1 to showing that

$$\nu^{\mathrm{SL}_2}(\gamma) = p^{-\delta} \cdot O(\gamma) \cdot \begin{cases} \frac{p}{p-1}, & \chi = 1, \\ 1, & \chi = 0, \\ \frac{p}{p+1}, & \chi = -1. \end{cases}$$

7.1.2. *Proving Proposition 7.1.* We first relate the limit of the ratios to the orbital integral with the geometric measure, and then we relate the geometric and canonical measures.

Lemma 7.6. *Let $O^{geom}(\gamma)$ be the rational orbital integral with the geometric measure. We have $\nu^{\mathrm{SL}_2}(\gamma) = O^{geom}(\gamma)$.*

Proof. We first introduce notation. Let $G = \mathrm{SL}_2(\mathbb{Q}_p)$, K denote its maximal compact subgroup $\mathrm{SL}_2(\mathbb{Z}_p)$, and $G_k = \mathrm{SL}_2(\mathbb{Z}_p/p^k)$.

Let $t = \mathrm{tr}(\gamma)$. Let $\pi_k \colon K \to G_k$ be the reduction map mod p^k.

We define the following subsets of K to reflect the two conditions required for rational conjugacy.

$$\begin{aligned} \mathcal{A}_k(\gamma) &= \{h \in K \colon \mathrm{tr}(h) \equiv t \bmod p^k\}, \\ \mathcal{B}_k(\gamma) &= \{h \in K \colon (a(h), D)_{p,k} = (a(\gamma), D)_{p,k}\}, \\ \mathcal{V}_k(\gamma) &= \mathcal{A}_k(\gamma) \cap \mathcal{B}_k(\gamma) = \left\{ h \in K \colon \begin{array}{c} \mathrm{tr}(h) \equiv t \bmod p^k, \\ (a(h), D)_{p,k} = (a(\gamma), D)_{p,k} \end{array} \right\}. \end{aligned}$$

Next, define

$$\mathcal{V}(\gamma) = \bigcap_{k \geq 1} \mathcal{V}_k(\gamma).$$

By Lemma 6.3, the set $\mathcal{V}(\gamma)$ consists precisely of all of the matrices in the rational orbit of γ who also have integer coefficients. In other words,

$$\mathcal{V}(\gamma) = \mathrm{Orb}(\gamma) \cap K.$$

For convenience, we define $S_k(\gamma)$ to be the set referenced in the numerator of the definition of $\nu_k(\gamma)$:

$$S_k(\gamma) = \pi(\mathcal{V}_k(\gamma)) = \left\{ h \in G_k \colon \begin{array}{c} \mathrm{tr}(h) \equiv t \bmod p^k, \\ (a(h), D)_{p,k} = (a(\gamma), D)_{p,k} \end{array} \right\} \subset G_k.$$

Because of this shorthand, we can write the following:

$$\nu_k^{\mathrm{SL}_2}(\gamma) = \frac{\# S_k(\gamma)}{\# G_k / p^k}.$$

Because $\mu_G(K) = 1$ and Haar measures are left/right invariant, the fibers of π_k each have measure $\frac{1}{\# G_k}$. Thus,

$$\mu_G(\mathcal{V}_k(\gamma)) = \frac{\# S_k(\gamma)}{\# G_k} \implies \nu_k^{\mathrm{SL}_2}(\gamma) = \frac{\mu_G(\mathcal{V}_k(\gamma))}{p^{-k}}.$$

Now, let $U_k(\gamma) = t + p^k\mathbb{Z}_p$ (recall that t is the trace of γ). Normalizing the Haar measure on $\mathbb{Q}_p$ to give $\mathbb{Z}_p$ volume 1, we have $\mu_{\mathbb{Z}_p}(U_k(\gamma)) = p^{-k}$. Then,

$$\begin{aligned}\lim_{k\to\infty} \nu_k^{\mathrm{SL}_2}(\gamma) &= \lim_{k\to\infty} \frac{\mu_G(\mathcal{V}_k(\gamma))}{p^{-k}} = \lim_{k\to\infty} \frac{\mu_G(\mathcal{V}_k(\gamma))}{\mu_{\mathbb{Z}_p}(U_k(\gamma))} \\ &= \lim_{k\to\infty} \frac{\mu_G(\mathfrak{c}^{-1}(U_k(\gamma)) \cap \mathcal{B}_k(\gamma))}{\mu_{\mathbb{Z}_p}(U_k(\gamma))} = \mu^{\mathrm{geom}}(\mathcal{V}(\gamma)).\end{aligned}$$

The last step is by the definition of the geometric measure (see Remark 2.14). Since $\mathcal{V}(\gamma)$ consists of all the elements in K that satisfy the trace and Hilbert symbol condition, it is the intersection of K and the rational orbit of γ. The measure of this is $O^{\mathrm{geom}}(\gamma)$. □

Next, in order to relate $O^{\mathrm{geom}}(\gamma)$ to $O(\gamma)$ which uses the canonical measure, we must relate μ^{geom} and μ^{can}.

Theorem 7.7. *We have*

$$\frac{\mu^{geom}}{\mu^{can}} = p^{-\delta} \cdot (1 - \chi p^{-1})^{-1}.$$

Proof. Let $|\Delta(\gamma)| = \sqrt{|D(\gamma)|_p}$. From [Gor22, Eq. 36], we get

$$\mu^{\mathrm{geom}} = \frac{|\Delta(\gamma)|}{\mathrm{vol}_{\omega_T}(T^\circ)} \mu^{\mathrm{can}}.$$

From [Gor22, Example 2.8], we have

$$|\omega_T| = \sqrt{\Delta_{E/\mathbb{Q}_p}} \cdot |\omega^{\mathrm{can}}|,$$

where $\Delta_{E/\mathbb{Q}_p}$ is the discriminant of $E/\mathbb{Q}_p$.

In general, according to [Gor22, Thm 2.6], we have

$$\mathrm{vol}_{\omega^{\mathrm{can}}}(T^\circ) = \frac{\#T(\mathbb{F}_p)}{p^{\dim T}}.$$

In our case, the torus T has dimension 1. Thus,

$$\mu^{\mathrm{geom}} = \frac{|\Delta(\gamma)| \cdot p}{\sqrt{\Delta_{E/\mathbb{Q}_p}} \cdot \#T(\mathbb{F}_p)} \mu^{\mathrm{can}}. \tag{7.1}$$

Now, we do the cases separately:

- Case $\chi = 1$ (T is split): In this case,

$$|\Delta(\gamma)| = p^{-\mathrm{val}(D)/2} = p^{-\delta}.$$

  Also, note that $\Delta_{E/\mathbb{Q}_p} = 1$. Since T is the split torus $\{\mathrm{diag}(a, a^{-1})\}$, taking $\mathbb{F}_p$-points gives $T(\mathbb{F}_p) \simeq \mathbb{F}_p^\times$, so $\#T(\mathbb{F}_p) = p - 1$. Putting this together with (7.1) yields the desired result.
- Case $\chi = 0$ (T is ramified elliptic): In this case, we have $|\Delta(\gamma)| = p^{-\delta-\frac{1}{2}}$. Also, we have $\Delta_{E/\mathbb{Q}_p} = p$. To count the number of elements in $T(\mathbb{F}_p)$, note that $T = \mathrm{Res}^1_{E/\mathbb{Q}_p} \mathbb{G}_m$. It can be shown that $\#T(\mathbb{F}_p) = p$ for a ramified extension. Putting this together with (7.1) yields the desired result.

- Case $\chi = -1$ (T is unramified elliptic): In this case, we get $|\Delta(\gamma)| = p^{-\delta}$. Also, we get $\Delta_{E/\mathbb{Q}_p} = 1$. To count the number of elements in $T(\mathbb{F}_p)$, note that $T = \mathrm{Res}^1_{E/\mathbb{Q}_p} \mathbb{G}_m$. We have the exact sequence of tori
$$1 \longrightarrow \mathrm{Res}^1_{E/\mathbb{Q}_p} \mathbb{G}_m \longrightarrow \mathrm{Res}_{E/\mathbb{Q}_p} \mathbb{G}_m \xrightarrow{N_{E/\mathbb{Q}_p}} \mathbb{G}_m \longrightarrow 1.$$
Since $E/\mathbb{Q}_p$ is unramified, taking $\mathbb{F}_p$-points yields
$$1 \longrightarrow T(\mathbb{F}_p) \longrightarrow \mathbb{F}_{p^2}^\times \xrightarrow{N_{\mathbb{F}_{p^2}/\mathbb{F}_p}} \mathbb{F}_p^\times \longrightarrow 1.$$
Thus, we may count $\#T(\mathbb{F}_p) = \frac{p^2-1}{p-1} = p+1$. Putting this together with (7.1) yields the desired result.

This concludes the proof. □

This yields the final result of this section, which also proves Proposition 7.1.

Theorem 7.8. *Let $\gamma \in \mathrm{SL}_2(\mathbb{Q}_p)$ be regular semisimple and have trace t. Let $D = t^2 - 4$ be the discriminant of the characteristic polynomial. Let $\delta = \lfloor \frac{\mathrm{val}(D)}{2} \rfloor$, i.e., $p^{2\delta}$ is the highest power of p^2 that divides D. Let $\chi = \left(\frac{D/p^{2\delta}}{p}\right)$ (this is the Legendre symbol). The relation between the limit of the ratios and the orbital integral under the canonical measure is given by*
$$\nu^{\mathrm{SL}_2}(\gamma) = p^{-\delta} \cdot (1-\chi p^{-1})^{-1} \cdot O(\gamma).$$

Corollary 7.9. *We have*
$$\nu^{\mathrm{SL}_2}(\gamma) = p^{-\delta} \cdot (1-\chi p^{-1})^{-1} \cdot \begin{cases} p^\delta, & \chi = 1, \\ \frac{1}{2} \cdot \frac{p^{\delta+1}-1}{p-1}, & \chi = 0, \\ \frac{p^\delta - 1}{p-1} \text{ or } \frac{p^{\delta+1}-1}{p-1}, & \chi = -1. \end{cases}$$

8. Convergence of local ratios

In this section, we let $F = \mathbb{Q}_p$ (although everything works with a more general p-adic field). In this context, the local ratios we defined can be rephrased as a valuation (and therefore open) condition on F.

By the theory of Igusa zeta functions, we know that given a variety $\mathcal{X}$ defined over $\mathrm{Spec}(\mathbb{Z}_p)$ as the zero locus of a function $f \in \mathbb{Z}_p[X_1, \ldots, X_\ell]$, the ratios
$$\frac{\#(\mathcal{X} \times_{\mathrm{Spec}(\mathbb{Z}_p)} \mathrm{Spec}(\mathbb{Z}_p/p^k\mathbb{Z}_p))}{p^{k \dim \mathcal{X}}} = \frac{\#(f^{-1}(p^k\mathbb{Z}_p)/p^k\mathbb{Z}_p)}{p^{k(\ell-1)}}$$
do not necessarily stabilize when $\mathcal{X}$ is singular. Note that on the left-hand side we use # to mean the number of rational points.

Example 8.1. Let $f(x) = (x+y)^2$. We have that $\#(f^{-1}(p^k\mathbb{Z}_p)/p^k\mathbb{Z}_p) = p^{k+\lfloor k/2 \rfloor}$ and therefore
$$\frac{\#(f^{-1}(p^k\mathbb{Z}_p)/p^k\mathbb{Z}_p)}{p^k} = p^{\lfloor k/2 \rfloor},$$
which never stabilizes.

We wish to show that the orbit of a regular semisimple $\gamma \in \mathrm{GL}_n(F)$ has a very "tame" singularity, and not only do ratios converge, they stabilize for k large enough.

8.1. **A low-dimensional example.** Let $X = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \in M_2(F)$ where $\lambda \neq \mu$ and $\min(\mathrm{val}(\lambda), \mathrm{val}(\mu)) = 0$.

The local ratios boil down to counting matrices of the form

$$\begin{pmatrix} x & y \\ s & t \end{pmatrix} \in M_2(\mathbb{Z}/p^k\mathbb{Z})$$

such that $x + t \equiv \lambda + \mu \pmod{p^k}$ and $xt - ys \equiv \lambda\mu \pmod{p^k}$. Using the trace condition, this is equivalent to counting

$$\begin{pmatrix} x & y \\ z & \lambda+\mu-x \end{pmatrix} \in M_2(\mathbb{Z}/p^k\mathbb{Z}) \text{ such that } x\lambda + x\mu - x^2 - zy \equiv \lambda\mu \pmod{p^k}.$$

Let $f\colon \mathbb{Z}_p^3 \to \mathbb{Z}_p$ be defined by $f(x, y, z) = x\lambda + x\mu - x^2 - zy - \lambda\mu$. The zero locus of f is exactly the $\mathrm{GL}_2(F)$-conjugacy class of X and $f^{-1}(p^k\mathbb{Z}_p) = V_k(X)$ as defined in (3.1). Write $S_k := \{M \in M_2(\mathbb{Z}/p^k\mathbb{Z})\colon f(M) \equiv 0 \pmod{p^k}\}$. We want to check that $\frac{\#S_k}{p^{2k}}$ stabilizes.

Example 8.2. We give an example illustrating the method in [Gek03]. Take $\lambda = 1 + p^m$ and $\mu = 1 - p^m$ for $m \geq 1$. Then $X \equiv I_2 \pmod{p}$, which is its own conjugacy class. The set of elements in $M_2(\mathbb{Z}/p\mathbb{Z})$ with the same characteristic polynomial as X modulo p corresponds to solutions of $x(2-x) - yz \equiv 1 \pmod{p}$, or equivalently,

$$yz \equiv -(x-1)^2 \pmod{p}.$$

If $x = 1$ then either y or z are trivial modulo p, so there are $2p - 1$ solutions. If $x \neq 1$ then the set of solutions $\{(x, y, -\frac{(x-1)^2}{y}) : x \not\equiv 1 \pmod{p}\}$ which has cardinality $(p-1)^2$. This yields $\#S_1 = p^2$.

The set of solutions split into two conjugacy classes: $\{I_n\}$, and the remaining p^2-1 solutions. We can verify the point count through the orbit-stabilizer theorem, the (Z) of any element of the second class has $p(p-1)$ elements and we have $p(p-1) \times (p^2-1) = (p^2-1)(p^2-p) = \#\mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$.

If $k > 1$ then again, all elements that are nonscalar modulo p are conjugate and their centralizer has smooth reduction modulo p hence cardinality $p^{2(k-1)}\#Z = p^{2k-1}(p-1) = p^{2k} - p^{2k-1}$. We will treat the remaining cases in Examples 8.3 and 8.5, after a few general observations. By the orbit stabilizer theorem, the number of nonscalar matrices with the prescribed characteristic polynomial is $p^{2k} - p^{2(k-1)}$.

We have

$$\nabla f(x, y, z) = \begin{pmatrix} \lambda + \mu - 2x \\ -z \\ -y \end{pmatrix}.$$

Note that $f(\lambda, 0, 0) = f(\mu, 0, 0) = 0$ and $\nabla f(\lambda, 0, 0) = \begin{pmatrix} \lambda - \mu \\ 0 \\ 0 \end{pmatrix} = -\nabla f(\mu, 0, 0)$. Since $\lambda \neq \mu$ we do have

$$|f(\lambda, 0, 0)|_p = 0 < \|\nabla f(\lambda, 0, 0)\|_p^2 = |\lambda - \mu|_p^2.$$

Example 8.3. Continuing Example 8.2. Modulo p, the solution $(1, 0, 0)$ is the only element of S_1 so that $\nabla f(x, y, z) = 0 \pmod{p}$. For all other solutions, either z or y is nonzero modulo p hence any lift of a mod p solution $\mathbb{Z}_p$ will satisfy $|\nabla f| = 1$.

We can then use Hensel's lemma to say that there are exactly p^{2k} lifts of nonscalar matrices mod p to roots of p modulo p^k. Write $S_k = A_k \sqcup B_k$ where A_k are elements of S_k that are scalars modulo p, and B_k the rest. We get that $\#B_k = p^{2k}\#B_1$ for all $k \geq 2$. In particular $\frac{\#B_k}{p^{2k}}$ is constant. In Example 8.5, we finish the analysis by studying A_k.

Remark 8.4. Although one might be tempted to use Hensel's Lemma in its more general forms when $0 < \|\nabla f\|_\infty < 1$, we must observe that not every matrix $M \in M_2(\mathbb{Z}/p^k\mathbb{Z})$ (for $k > 1$) lifts to a root of f over $\mathbb{Z}_p$. Indeed, for all $e > 1$ one may define

$$M_e = X + \begin{pmatrix} 0 & p^e \\ p^e & 0 \end{pmatrix}.$$

If $e < k$ and $2e \geq k$ then $f(M_e) \equiv 0 \pmod{p^k}$. Indeed, any lift of M_e to $\mathbb{Z}_p$ is of the form $X + \begin{pmatrix} xp^k & p^e + yp^k \\ p^e + sp^k & tp^k \end{pmatrix}$. By linearity of the trace, the trace of such a matrix is equal to the one of X if and only if the trace of the right summand is zero, or in other terms, we have $x + t = 0$. There is however a problem with the determinant, the determinant of this lift is $\lambda\mu + p^{2e} \pmod{p^{2e+1}}$ and therefore no lift of M_e to $\mathbb{Z}_p$ is conjugate to X.

Example 8.5. We continue Examples 8.2 and 8.3. We have shown in Example 8.2 that $A_1 = 1$. Therefore, elements of A_k are of the form $I_2 + pM$ where $M \in M_2(\mathbb{Z}/p^{k-1}\mathbb{Z})$. Computing the trace and determinant, we also get the conditions

$$p\operatorname{tr}(M) \equiv 0 \pmod{p^k}, \quad 1 + p\operatorname{tr}(M) + p^2\det(M) \equiv 1 - p^{2m} \pmod{p^k}.$$

If $k = 2$ then we only have one condition, namely $\operatorname{tr}(M) \equiv 0 \pmod p$ and we get $\#A_2 = p^3$. Now assume that $k > 2$. Simplifying the equations, we get $\operatorname{tr}(M) \equiv 0 \pmod{p^{k-1}}$ and $\det(M) \equiv -p^{2m-2} \pmod{p^{k-2}}$.

The idea is that the condition above gives us a condition on the characteristic polynomial of $M \in M_2(\mathbb{Z}/p^{k-1}\mathbb{Z})$, which lets us proceed by induction. If $m = 1$ then M cannot be a scalar modulo p, therefore we are reduced to a counting of the form $\#B_k$ (for a different characteristic polynomial). We make this explicit in the proposition below.

Proposition 8.6. *For all $t \in \mathbb{Z}_p$ and $d \in \mathbb{Z}_p^\times$ define*

$$S_k(t,d) = \{M \in M_2(\mathbb{Z}/p^k\mathbb{Z}) : \operatorname{tr}(M) \equiv t \pmod{p^k},\ \det(M) \equiv d \pmod{p^k}\}.$$

Write $B_k(t,d)$ for the subset of $S_k(t,d)$ consisting of matrices whose reduction modulo p are scalars, and $A_k(t,d) = S_k(t,d)\backslash B_k(t,d)$. Let $X = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ where $\lambda \neq \mu$ and $\min(\operatorname{val}(\lambda), \operatorname{val}(\mu)) = 0$ and define $\delta = \operatorname{val}(\lambda - \mu)$. We have the following:

(1) $\#A_k(\lambda+\mu, \lambda\mu) = p^{2k-1}\#A_1(\lambda+\mu, \lambda\mu)$,
(2) If $\delta = 0$ then $S_k(\lambda+\mu, \lambda\mu) = A_k(\lambda+\mu, \lambda\mu)$,
(3) If $k > 2\delta$ then $p^{-2k}\#B_k(\lambda+\mu, \lambda\mu)$ and $p^{-2k}\#S_k(\lambda+\mu, \lambda\mu)$ are constant.

Proof. For part (1), note that $A_k(t,d)$ is always a single orbit. Let Z_k denote the stabilizer of an element of the orbit. Since elements of $A_k(t,d)$ are regular, their stabilizer is a space of dimension 2 (a torus or a product of a torus and an affine space), hence $Z_{k+1} = p^2 Z_k$ and the result follows.

Part (2) is immediate since if $\delta = 0$ then $B_1(\lambda+\mu, \lambda\mu) = \emptyset$

To show part (2) we proceed by induction on $\delta \geq 1$.

($\delta = 0$) This follows from (1) and (2).

($\delta \geq 1$) Write $\mu = \lambda + p^\delta u$ with $u \in \mathbb{Z}_p^\times$. We have $B_1 = \{\lambda I_2 \pmod p\}$. Write $N = \lambda I_2 + pM \in A_k(\lambda+\mu, \lambda\mu)$. Computing the characteristic polynomial of N, we get the necessary and sufficient conditions

$$p \operatorname{tr} M \equiv p^\delta u \pmod{p^k}, \quad p^2 \det(M) \equiv 0 \pmod{p^k}.$$

We get that

$$B_k(\lambda+\mu, \lambda\mu) \cong \bigsqcup_{i=0}^{p-1} S_{k-1}(p^{\delta-1}u, ip^{k-2}).$$

The valuation of the discriminant of a matrix of trace $up^{\delta-1}$ and determinant ip^{k-2} with $u, i \in \mathbb{Z}_p^\times$ is

$$\frac{1}{2} \operatorname{val}\left(u^2 p^{2(\delta-1)} - 4ip^{k-2}\right) = \delta - 1$$

since $k > 2\delta$. Therefore we can use the induction hypothesis to determine that

$$\frac{\#B_k(\lambda+\mu, \lambda\mu)}{p^{2k}} = \sum_{i=0}^{p-1} \frac{\#S_{k-1}(p^{\delta-1}u, ip^{k-2})}{p^{2k}}$$

is constant. □

Remark 8.7. Note that this approach extends verbatim to nondiagonal matrices and the counts of S_k can be written explicitly. This is done in [Gek03, §4]. However, we only care about the convergence, hence we could simplify the arguments and motivate the next section.

8.2. **A general approach.** Let us write $\mathfrak{gl}_n = M_n$ and $\mathfrak{t} \subset \mathfrak{gl}_n$ its Cartan subalgebra of diagonal matrices.

Recall that $\mathfrak{c} : \mathfrak{gl}_n \to \mathbb{A}^n$ is the Steinberg quotient, mapping a matrix to the coefficients of its characteristic polynomial.

It is known ([Bou02, Chapter 5, §5]) that the Jacobian of $\mathfrak{c}$ evaluated at a $\mathfrak{t}$ has only one nonzero Jacobian, whose determinant

$$\det(J_{\mathfrak{c}}|_{\mathfrak{t}}) = \prod_{\alpha \in \Phi^+} \alpha,$$

up to a unit, where Φ^+ is the set of positive roots associated to $\mathfrak{t}$. Let $j_{\mathfrak{c}}$ be that determinant.

For a general X, by conjugation invariance, we have

$$|\det(J)_{\mathfrak{c}}(X)| = \prod_{\alpha \in \Phi^+} |\alpha(X_s)|.$$

A semisimple element $X \in \mathfrak{gl}_n(F)$ is regular whenever $j_{\mathfrak{c}}(X) \neq 0$.

Lemma 8.8. *Viewing $\mathfrak{gl}_n(\mathbb{Z}_p)$ as the points of a $\mathbb{Z}_p$-scheme, the smooth locus of $\mathfrak{c}$ is $j_{\mathfrak{c}}^{-1}(\mathbb{Z}_p^\times)$, i.e. the space of points whose reduction modulo p is regular.*

Proof. This is just the criterion for smoothness by looking at the special fiber. □

Remark 8.9. This result is more general than the case of $\mathfrak{gl}_n$. In our case, we have

$$|\det J_{\mathfrak{c}}(M)| = \prod_{1 \leq i \leq j \leq n} |\lambda_i - \lambda_j|,$$

where $\lambda_1, \dots, \lambda_n \in F^{\mathrm{sep}}$ are the eigenvalues of M.

Indeed, the only nonzero minor of $J_{\mathfrak{c}}(\mathrm{diag}(x_1, \dots, x_n))$

$$\begin{pmatrix} -1 & \cdots & -1 \\ s_1(x_2,\dots,x_n) & \cdots & s_1(x_1,\dots,x_{n-1}) \\ & \vdots & \\ (-1)^n s_{n-1}(x_2,\dots,x_n) & \cdots & (-1)^n s_{n-1}(x_1,\dots,x_{n-1}) \end{pmatrix}$$

where $s_i = \sum_{1\le k_1<\cdots<k_i\le n} \prod_{m=1}^{i} X_{k_m} \in \mathbb{Z}[X_1, \dots, X_n]$ is the degree i elementary symmetric polynomial in n variables. An easy induction (removing the last column to the others) shows that its determinant is the expected one.

Let us establish some notation: Given $\mathbf{x} \in \mathbb{Z}_p^n$ we let

- $\delta(\mathbf{x}) = \mathrm{val}\, j_{\mathfrak{c}}(X) \in \frac{1}{2}\mathbb{Z}$, where $X \in \mathfrak{c}^{-1}(\mathbf{x})$,
- $\varphi_{\mathfrak{x}}(M) = \mathfrak{c}(M) - \mathbf{x}$,
- $S_k(\mathbf{x}) = \{M \in \mathfrak{gl}_n(\mathbb{Z}/p^k\mathbb{Z}) \ : \ \varphi_{\mathbf{x}}(M) = 0 \pmod{p^k}\}$,
- $S_k^r(\mathbf{x}) = \{M \in S_k(\mathbf{x}) \ : \ r_{\overline{M}} = r\}$, where $\overline{M} = M \pmod p$.
- $S_k^{\geq r}(\mathbf{x}) = \bigsqcup_{k\geq r} S_k^r(\mathbf{x})$.

Let us list a few straightforward facts.

Lemma 8.10. *For any $\mathbf{x} \in \mathbb{Z}_p^n$ and $k \geq 1$ we have that $S_k^0(\mathbf{x})$ is a single conjugation class and $\#S_k^0(\mathbf{x}) = p^{(k-1)n(n-1)}\#S_1^0(\mathbf{x})$.*

Proof. By definition of S_k^0, any $M \in S_k^0(\mathbf{x})$ reduces to a regular element. Since $M \pmod p$ is regular, the centralizer Z_k of M is a space of fixed dimension n, so $\#Z_{k+1} = p^n\#Z_k$ hence $\#S_{k+1}^0(\mathbf{x}) = \frac{p^{n^2}}{p^n}\#S_k^0(\mathbf{x})$ as desired. □

Lemma 8.11. *Let $X \in \mathfrak{gl}_n(\mathbb{Z}_p)$ be a regular semisimple element and let $\mathbf{x} = \mathfrak{c}(X)$. The following are equivalent:*

(1) $\delta(\mathbf{x}) = 0$,
(2) *X is regular modulo p,*
(3) $S_1^{\geq 0}(\mathbf{x}) = \emptyset$,
(4) $S_k^{\geq 0}(\mathbf{x}) = \emptyset$ *for all k,*
(5) $S_1(\mathbf{x}) = S_1^0(\mathbf{x})$,
(6) $\#S_k(\mathbf{x}) = p^{(k-1)n(n-1)}\#S_1^0(\mathbf{x})$.

Proof. The equivalences (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) $\Leftarrow$ (6) are clear since if $\delta(\mathbf{x}) = 0$ then any element in $\mathfrak{gl}_n(\mathbb{F}_p)$ with characteristic polynomial $\mathbf{x} \pmod p$ is conjugate to $X \pmod p$. Then note that $j_{\mathfrak{c}}(X) \in \mathbb{Z}_p^\times$ for any lift of M to $\mathfrak{gl}_n(\mathbb{Z}_p)$. This lets us use part (1) of Hensel's lemma with $f = \varphi_{\mathbf{x}}$ stated in Theorem 2.18. □

Proposition 8.12. *Let $\gamma \in \mathrm{GL}_n(\mathbb{Z}_p)$ be a regular semisimple element. Let $k = \#\{\alpha \in \Phi^+ \ : \ |\alpha(\gamma)| < 1\}$ and $\mathbf{x} = \mathfrak{c}(X)$. We have*

$$S_k(\mathbf{x}) = \bigsqcup_{i=0}^{k} S_k^i(\mathbf{x}).$$

Proof. For any $X \in S_1(\mathbf{x})$, we have $G_X \subset G_{X_s}$ and since X_s is semisimple, we get $\dim(G_X) \leq \dim(G_{X_s}) = n + \#\{\alpha \in \Phi \ : \alpha(X_s) = 0\} = n + 2\#\{\alpha \in \Phi^+ \ : \alpha(X_s) = 0\}$. □

Let us define the *residual regularity* of $X \in M_n(\mathbb{Z}_p)$ the integer $r_{\overline{X}}$ where $\overline{X} = X \pmod p$.

Proposition 8.13. *Let $\gamma \in \mathrm{GL}_n(\mathbb{Z}_p)$ be a regular semisimple element.*
For all $k > 2\delta(\mathfrak{c}(\gamma)) + n$. We may decompose

$$S_k^{n(n-1)/2}(\mathfrak{c}(\gamma)) \cong \bigsqcup_i S_k(\mathbf{x}_i),$$

where the number of sets is independent of k and $S_k^{n(n-1)/2}(\mathbf{x}_i) = \emptyset$ for all i

Proof. We may assume that the residual regularity of γ is $n(n-1)/2$ since otherwise $S_k^{n(n-1)/2}(\mathfrak{c}(\gamma)) = \emptyset$. Since γ is regular and its residue modulo p must be scalar, we may write $\gamma = \lambda I_n + p^\ell Y$ where $\ell = \min(\mathrm{val}(\alpha(\gamma))) \geq 1$. The residual regularity of Y is strictly smaller than the one of X.

Let $\mathbf{x} = \mathfrak{c}(\gamma)$ and $\lambda I_n + pM \in S_k^{n(n-1)/2}(\mathbf{x})$. By definition of S_k, we have

$$\chi_{\lambda I_n + M}(t) = \chi_{pM}(t-\lambda) \equiv \chi_\gamma(t) = \chi_{p^\ell Y}(t-\lambda) \pmod{p^k}.$$

Call $\mathfrak{c}_i$ the map giving the $n-i$th coefficient of the characteristic polynomial. We have $\mathfrak{c}_i(pM) = p^i \mathfrak{c}(M)$ By previous equation we get

$$\mathfrak{c}_i(M) = \mathfrak{c}_i(p^{\ell-1}Y) \pmod{p^{k-i}},\ 1 \leq i \leq n.$$

We get

$$\mathfrak{c}(M) \in \left\{ \left(\mathfrak{c}_1(p^{\ell-1}Y), \mathfrak{c}_2(p^{\ell-1}Y) + \alpha_1 p^{k-2}, \dots\right) : \alpha_i \in \mathbb{Z}/p^i\mathbb{Z} \right\} \pmod{p^{k-1}},$$

and therefore

$$S_k^{n(n-1)/2}(\mathbf{x}) \cong \bigsqcup_{\alpha_1,\dots,\alpha_{n-1}} S_{k-1}\left((\mathfrak{c}_i(p^{\ell-1}Y) + \alpha_i p^{k-i})_{i=1}^n \right)$$

We use $\alpha_0 = 0$ above. If $\ell = 1$ then the residual regularity of any element of $S_{k-1}\left((\mathfrak{c}_i(Y) + \alpha_i p^{k-i})_{i=1}^n\right)$ is at most the residual regularity as Y which is strictly smaller than $n(n-1)/2$.

If $\ell \geq 2$ then we may apply the induction hypothesis on each set in the disjoint union above and conclude. □

Proof. The second part follows from the fact that $\#S_k(\mathbf{x}) = \#A_k(\mathbf{x}) + \#B_k(\mathbf{x})$ and $p^{-kn(n-1)}\#A_k(\mathbf{x})$ is constant by Lemma 8.10.

For the first part, we will prove it by induction on $\delta(\mathbf{x})$.

($\delta(\mathbf{x}) = 0$) $\#B_k(\mathbf{x}) = 0$ for all k.

($\delta(\mathbf{x}) \geq 1$) Let $N \in A_k(\mathbf{x})$ and $\overline{N} \in A_1(\mathbf{x})$ its residue modulo p. We can write $N = \overline{N} + pM$ with $M \in \mathfrak{gl}_n(\mathbb{Z}/p^{k-1}\mathbb{Z})$. □

Corollary 8.14. *Let $\gamma \in \mathrm{GL}_n(\mathbb{Z}_p)$ be a regular semisimple element. If $r = r(\overline{\gamma}) > 0$ and $k \geq 2\delta + n$ then $S_k^r(\mathfrak{c}(\gamma)) \cong \bigsqcup_i S_k(x_i)$, where $S_k^{\geq r}(x_i) = \emptyset$, and the number of x_i's is independent of k.*

Proof. This is obtained by induction using the previous proposition. If $r = n(n-1)/2$ then we use the previous proposition, otherwise decompose $\gamma = X + p^\ell Y$ where $X \pmod p$ is a "block-scalar" matrix. Then use Proposition 8.13 on each block. □

This means that we can decompose the local density sets S_k in a union of regular sets, and we obtain the following.

Theorem 8.15. *Let $\gamma \in \mathrm{GL}_n(\mathbb{Z}_p)$ be a regular semisimple element, and let $\delta = \delta(\gamma)$. Then for all $k \geq 2\delta + n$ we have*

$$\frac{\#S_k(\mathfrak{c}(\gamma))}{p^{kn(n-1)}} = \nu(\gamma) = \frac{p^{n^2-1}}{\#\,\mathrm{SL}_n(\mathbb{F}_p)} \cdot O^{\mathrm{geom}}_{\mathrm{GL}_n}(\gamma, \mathbb{1}_{\mathrm{GL}_n(\mathbb{Z}_p)}).$$

In particular,

$$O^{\mathrm{geom}}_{\mathrm{GL}_n}(\gamma, \mathbb{1}_{\mathrm{GL}_n(\mathbb{Z}_p)}) = \frac{\prod_{i=2}^{n}(1-p^{-i})}{p^{(2\delta+n)n(n-1)-1}} \#S_k(\mathfrak{c}(\gamma)).$$

Proof. This is just a restatement of Theorem 3.3 with the knowledge of stabilization of ratios. □

Remark 8.16. Note that we can adapt the formula above to integrals of characteristic functions of double cosets $\mathrm{GL}_n(\mathbb{Z}_p)p^{\lambda}\mathrm{GL}_n(\mathbb{Z}_p)$ by increasing k by $|\lambda|$ (the sum of the entries of the weight λ).

9. Conclusion

In this paper, we extended the methods of Achter and Gordon [AG17] for computing orbital integrals using local ratios. Our contributions are fourfold:

(1) We generalized the method from GL_2 to GL_n for regular semisimple elements in the maximal compact subgroup $\mathrm{GL}_n(\mathbb{Z}_p)$, showing that their geometric orbital integral can be computed as a limit of finite counting problems.
(2) We extended the class of test functions from the characteristic function of $\mathrm{GL}_n(\mathbb{Z}_p)$ to that of any double coset in the Cartan decomposition, allowing for the computation of orbital integrals for nonintegral elements.
(3) We initiated the study of orbital integrals in $\mathrm{SL}_n(\mathbb{Q}_p)$. We use a lattice method to prove a set of criteria for conjugacy in $\mathrm{SL}_n(\mathbb{Q}_p)$. Based on this, we defined a new local ratio for SL_n.
(4) We related the SL_2 local ratios explicitly to SL_2 orbital integrals.

With the case of linear groups fully covered, it would now be interesting to adapt these methods to other split classical groups, or forms of the linear groups such as unitary groups.

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Appendix A. Computing local ratios

Below, we provide samples of Python code used to compute local ratios using a brute-force approach.

```python
import itertools
import math
import numpy as np

p = 3
n = 2
gamma = ((44, 27), (57, 35))

def coeffs(matrix, mod):
  return [int(c % mod) for c in np.rint(np.poly(np.array(matrix, dtype=float
      ))).astype(int)]

def conjugate(matrix, gamma_coeffs, mod):
  # first check trace
  trace = sum([matrix[i][i] for i in range(n)]) % mod
  if (trace + gamma_coeffs[1]) % mod != 0:
    return False

  return coeffs(matrix, mod) == gamma_coeffs

for k in range(1, 6):
  mod = p ** k
  print(f"mod: {mod}")
  ring = set(range(mod))
  gamma_coeffs = coeffs(gamma, mod)
  count = 0
  for matrix in itertools.product(itertools.product(ring, repeat=n), repeat=
      n):
    if conjugate(matrix, gamma_coeffs, mod):
      count += 1
  num_GLn = p ** ((k - 1) * n * n) * math.prod([p**n - p**i for i in range(n
      )])
  if num_GLn % (p**k - p**(k - 1)) != 0:
    raise Exception
  num_SLn = num_GLn // (p**k - p**(k - 1))
  if num_SLn % (p**k) != 0:
    raise Exception
  denominator = num_SLn // (p**k)
  print(f"GLn ratio: {count/denominator}")
```

Listing 1. Python code for computing GL_n ratios

```python
from functools import lru_cache

@lru_cache(maxsize=None)
def hilbert_symbol(a: int, b: int, mod: int) -> int:
    a %= mod
    b %= mod

```

```python
    for x in range(mod):
        for y in range(mod):
            for z in range(mod):
                if x % p == 0 and y % p == 0 and z % p == 0:
                    continue
                if (a * x ** 2 + b * y ** 2 - z ** 2) % mod == 0:
                    return 1
    return -1

def is_same_norm_class(a, b, D, mod):
    return hilbert_symbol(a, D, mod) == hilbert_symbol(b, D, mod)

for k in range(1, 6):
    mod = p ** k
    print(f"mod: {mod}")
    ring = set(range(mod))
    gamma_coeffs = coeffs(gamma, mod)
    trace = gamma_coeffs[1]
    discriminant = (trace ** 2 - 4) % mod
    print("discriminant: ", discriminant)

    for i in range(mod):
        print(f"hilbert symbol for ({i}, {discriminant}): {hilbert_symbol(i,
             discriminant, mod)}")

    count = 0
    for matrix in itertools.product(itertools.product(ring, repeat=n),
         repeat=n):
        if conjugate(matrix, gamma_coeffs, mod) and is_same_norm_class(
             matrix[0][1], gamma[0][1], discriminant, mod):
            count += 1
    num_GLn = p ** ((k - 1) * n * n) * math.prod([p ** n - p ** i for i in
         range(n)])
    if num_GLn % (p ** k - p ** (k - 1)) != 0:
        raise Exception
    num_SLn = num_GLn // (p ** k - p ** (k - 1))
    if num_SLn % (p ** k) != 0:
        raise Exception
    denominator = num_SLn // (p ** k)
    print(f"SLn ratio: {count / denominator}")
```

Listing 2. Python code for computing SL_n ratios (must follow Listing 1)

The above brute-force approach of computing the Hilbert symbol can potentially be sped up using the following:

Theorem A.1 ([Ser73, Theorem 1, p. 20])**.** *Let $a, b \in \mathbb{Q}_p^\times$ where $p > 2$ is prime. Write $a = p^m u$ and $b = p^n w$ for some $m, n \in \mathbb{Z}$ and $u, w \in \mathbb{Z}_p^\times$. Then,*

$$(a, b)_p = \left(\frac{-1}{p}\right)^{mn} \left(\frac{u}{p}\right)^{n} \left(\frac{w}{p}\right)^{m},$$

where $\left(\frac{\cdot}{p}\right)$ denotes the Legendre symbol.

Appendix B. Detecting rational conjugacy in SL_2

This section provides an elementary proof of criteria for $\mathrm{SL}_2(\mathbb{Q}_p)$ conjugacy also mentioned in Corollary 5.20. As such, we use ν_p to denote the p-adic valuation. This is not to be confused with the notation $\nu_p(\gamma)$ for the local ratio.

B.1. Square classes, norm classes, and the Hilbert symbol.

For an odd prime p, the structure of the multiplicative group $\mathbb{Q}_p^\times$ modulo squares is well-understood. Every element $x \in \mathbb{Q}_p^\times$ can be uniquely written as $x = p^k u$, where $k = \nu_p(x)$ and $u \in \mathbb{Z}_p^\times$ is a p-adic unit. An element is a square in $\mathbb{Q}_p^\times$ if and only if its valuation k is even and its unit part u is a quadratic residue modulo p.

This gives rise to the group of square classes $\mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2 \cong (\mathbb{Z}/2\mathbb{Z})^2$. This group has four elements, for which we can choose the representatives $\{1, \epsilon, p, p\epsilon\}$, where $\epsilon \in \mathbb{Z}_p^\times$ is a unit whose reduction modulo p is a quadratic nonresidue. Consequently, we will refer to elements of $\mathbb{Q}_p^\times$ as being squares, ϵ-type, p-type, or $p\epsilon$-type, respectively.

Next, we explore norm classes as they relate to SL_2. Let $\gamma \in \mathrm{SL}_n(\mathbb{Q}_p)$. We write $\mathbb{Q}_p[\gamma]$ for the smallest subring of the matrix ring $M_n(\mathbb{Q}_p)$ containing $\mathbb{Q}_p$ and γ. We have

$$\mathbb{Q}_p[\gamma] \cong \mathbb{Q}_p[X]/m_\gamma(X),$$

where m_γ is the minimal polynomial of γ.

Now, restricting our focus to $n = 2$, we can examine $\mathbb{Q}_p[\gamma]$ with more specificity. We also restrict our scope to regular semisimple matrices, so the minimal polynomial always equals the characteristic polynomial.

Let t be the trace of γ. If $t^2 - 4$ is a square in $\mathbb{Q}_p$, then the minimal polynomial splits over $\mathbb{Q}_p$, meaning

$$\mathbb{Q}_p[\gamma] \cong \mathbb{Q}_p \oplus \mathbb{Q}_p.$$

Furthermore, the norm map $N\colon \mathbb{Q}_p[\gamma]^\times \to \mathbb{Q}_p^\times$ given by $(x, y) \mapsto xy$ is surjective.

However, if $t^2 - 4$ is not a square, then the minimal polynomial is irreducible, and

$$\mathbb{Q}_p[\gamma] \cong \mathbb{Q}_p\left(\sqrt{t^2-4}\right).$$

It can be shown that the norm map $N\colon \mathbb{Q}_p[\gamma]^\times \to \mathbb{Q}_p^\times$ given by

$$a + b\sqrt{t^2-4} \mapsto a^2 - b^2(t^2-4)$$

has cokernel isomorphic to $\mathbb{Z}/2\mathbb{Z}$. The cosets of the image of this norm map will be referred to as "norm classes" in $\mathbb{Q}_p^\times$.

The Hilbert symbol can be used to represent the norm class of an element:

Definition B.1. For a local field K, the *Hilbert symbol* $(a, b)_K$ for $a, b \in K^\times$ is defined to be $+1$ if the equation $z^2 = ax^2 + by^2$ admits a nontrivial solution $(x, y, z) \in K^3$, and -1 otherwise. Equivalently, the symbol $(a, b)_K$ equals 1 precisely when b is the norm of an element of $K[\sqrt{a}]$. In the case that $K = \mathbb{Q}_p$, we write $(a, b)_p \coloneqq (a, b)_{\mathbb{Q}_p}$.

The notion of the Hilbert symbol is used in the paper for describing local ratios for $\mathrm{SL}_2(\mathbb{Q}_p)$.

B.2. **An elementary view of conjugacy in** SL_2**.** Next, we develop criteria to ascertain whether two regular semisimple matrices in $\mathrm{SL}_2(\mathbb{Q}_p)$ are conjugate in $\mathrm{SL}_2(\mathbb{Q}_p)$. In general, two regular semisimple matrices are conjugate in $\mathrm{GL}_n(\mathbb{Q}_p)$ if they have the same characteristic polynomial. Unfortunately, this is not true when considering conjugacy in $\mathrm{SL}_2(\mathbb{Q}_p)$, as we can see in the following example.

Example B.2. The matrices $A = \begin{pmatrix} 0 & \frac{1}{2} \\ -2 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & \frac{3}{2} \\ -\frac{2}{3} & 0 \end{pmatrix}$ are not conjugate in $\mathrm{SL}_2(\mathbb{Q}_3)$. To see why, if we let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $MAM^{-1} = B \implies MA = BM$ yields $c = -\frac{4}{3}b$ and $d = \frac{1}{3}a$ after expanding and solving. Plugging this into $ad - bc = 1$, we eventually get

$$a^2 + (2b)^2 = 3.$$

Note that a^2 and $(2b)^2$ have even 3-adic valuations. Since the 3-adic valuation of the right side is odd, the 3-adic valuations of a^2 and $(2b)^2$ must be equal, so we can write $\nu_3(a^2) = \nu_3((2b)^2) = 2k$ for some integer k. From this, we get $\nu_3(a) = \nu_3(2b) = k$, so we can write $a = 3^k u$ and $2b = 3^k v$ where u and v are units in $\mathbb{Z}_p$. Now,

$$a^2 + (2b)^2 = 3^{2k}u^2 + 3^{2k}v^2 = 3^{2k}(u^2 + v^2).$$

Since u and v are units, we have $u^2 + v^2 \equiv 1 + 1 \equiv 2 \pmod 3$. So,

$$\nu_3(a^2 + (2b)^2) = \nu_3(3^{2k}(u^2 + v^2)) = 2k = \nu_3(3) = 1,$$

which is a contradiction.

Since $\mathrm{SL}_2(\mathbb{Q}_p)$ is a subset of $\mathrm{GL}_2(\mathbb{Q}_p)$, two matrices that are conjugate in $\mathrm{SL}_2(\mathbb{Q}_p)$ must have the same trace (and determinant). However, as the above example shows, there must be an additional criterion placed on the two matrices to ensure conjugacy. It turns out that the criteria for conjugacy depend on the shared trace of the matrices.

Lemma B.3. *Let M and N be matrices in $\mathrm{SL}_2(\mathbb{Q}_p)$ with common trace t. If $D = t^2 - 4$ is a nonzero square in $\mathbb{Q}_p$, then M and N are conjugate in $\mathrm{SL}_2(\mathbb{Q}_p)$.*

Proof. Since D is the discriminant of the characteristic polynomial and is a nonzero square, we know that the characteristic polynomial splits over $\mathbb{Q}_p$. Thus, it must have two roots λ and λ^{-1} in $\mathbb{Q}_p$ (since the roots multiply to 1). Let v and w be eigenvectors corresponding to these roots. Let $P = \begin{pmatrix} v & w \end{pmatrix}$ be the 2×2 matrix with columns v and w. Then,

$$M = P \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} P^{-1}.$$

Since scalar multiples of eigenvectors are still eigenvectors, we can scale v in such a way that P has determinant 1, and thus M is conjugate to $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ in $\mathrm{SL}_2(\mathbb{Q}_p)$. Similarly, the matrix N is also conjugate to $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$, so M and N are conjugate. $\square$

If $t^2 - 4 \neq 0$ is not a square, then the stable orbit (the set of matrices with trace t) splits into two conjugacy classes. This can be seen via the following result:

Lemma B.4. *Let $t \in \mathbb{Q}_p$ such that $t^2 - 4$ is not a square in $\mathbb{Q}_p$. Let $\alpha \in \mathbb{Q}_p$ such that α is not a square in $\mathbb{Q}_p(\sqrt{t^2-4})$. Let $s = (-1)^{\nu_p(\alpha)}$, i.e., it is 1 if $\nu_p(\alpha)$ is even and -1 if it is odd. Then, the matrices $\frac{1}{2}\begin{pmatrix} t & 1 \\ t^2-4 & t \end{pmatrix}$ and $\frac{1}{2}\begin{pmatrix} t & s\alpha \\ \frac{t^2-4}{s\alpha} & t \end{pmatrix}$ are $\mathrm{GL}_2(\mathbb{Q}_p)$-conjugate but not $\mathrm{SL}_2(\mathbb{Q}_p)$-conjugate.*

Proof. If we let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $MAM^{-1} = B \implies MA = BM$ yields the following system of equations after expanding:

$$\frac{t}{2}a + \frac{t^2-4}{2}b = \frac{t}{2}a + \frac{s\alpha}{2}c$$
$$\frac{1}{2}a + \frac{t}{2}b = \frac{t}{2}b + \frac{s\alpha}{2}d.$$

Solving yields $c = \frac{t^2-4}{s\alpha}b$ and $d = \frac{a}{s\alpha}$.

Plugging this into $ad - bc = 1$ yields

$$\frac{a^2}{s\alpha} - \frac{(t^2-4)b^2}{s\alpha} = 1.$$

We can rearrange to get

$$a^2 - (t^2-4)b^2 = s\alpha. \tag{B.1}$$

The remainder of the proof aims to show that this equation has no solutions (a, b) (i.e. $s\alpha$ is not a norm in $\mathbb{Q}_p(\sqrt{t^2-4})$), and thus there is a contradiction.

Note that squares and ϵ-type elements (see Section B.1) have even p-adic valuations, while p-type and $p\epsilon$-type elements have odd p-adic valuations.

Continuing with the proof, we do casework on the square class of $t^2 - 4$.

- If $t^2 - 4$ is ϵ-type, then for α to not be a square in $\mathbb{Q}_p(\sqrt{t^2-4})$, it must be either p-type or $p\epsilon$-type. This means $s\alpha = -\alpha$ has odd p-adic valuation. Since t^2-4 has even p-adic valuation, $(t^2-4)b^2$ and a^2 both have even p-adic valuation. Since the p-adic valuation of the right side of (B.1) is odd, we must have
$$\nu_p(a^2) = \nu_p((t^2-4)b^2) = 2k$$
for some integer k. From this, we can write $a^2 = p^{2k}u$ and $(t^2-4)b^2 = p^{2k}v$ where u and v are units in $\mathbb{Z}_p$. Note that u is a quadratic residue while v is a nonquadratic residue. Our equation is now
$$p^{2k}(u - v) = -\alpha.$$
Noting that the right side has a p-adic valuation greater than $2k$, we can divide both sides by p^{2k} and take the equation mod p to get that $u - v \equiv 0 \pmod{p}$, or $u \equiv v \pmod{p}$. However, only u is a quadratic residue, so this is impossible.
- If $t^2 - 4$ is p-type, then α must be either ϵ-type or $p\epsilon$-type. If α is ϵ-type, then $s = 1$. Consider the p-adic valuation of the terms of (B.1). We have that $\nu_p(a^2)$ is even, $\nu_p((t^2-4)b^2)$ is odd, and $\nu_p(\alpha)$ is even. Thus,
$$\nu_p((t^2-4)b^2) > \nu_p(a^2) = \nu_p(\alpha) = 2k$$

for some integer k. Doing a similar thing to before, we eventually get to

$$\frac{a^2}{p^{2k}} \equiv \frac{\alpha}{p^{2k}} \pmod{p}.$$

The left side is a quadratic residue, but the right side is not. Thus, this is impossible.

The other case is if α is $p\epsilon$-type. Then $s = -1$. Similarly, we get

$$\nu_p(a^2) > \nu_p((t^2-4)b^2) = \nu_p(-\alpha) = 2k+1$$

for some integer k, and then

$$\frac{(t^2-4)b^2}{p^{2k+1}} \equiv \frac{\alpha}{p^{2k+1}} \pmod{p}.$$

The left side is a quadratic residue, but the right side is not. So, this case is also impossible.

- If t^2-4 is $p\epsilon$-type, then by an argument extremely similar to the one for the case where t^2-4 is p-type, we reach a contradiction.

So, no matter the square class of t^2-4, there are no solutions (a, b, c, d). Thus, the matrices are not conjugate by an element of $\mathrm{SL}_2(\mathbb{Q}_p)$. $\square$

When t^2-4 is not a square, the two conjugacy classes are closely linked to the two norm classes in $\mathbb{Q}_p$ with respect to $K = \mathbb{Q}_p(\sqrt{t^2-4})$ (also see Section B.1). The elements that are norms are of the form $w^2 - x^2(t^2-4)$ for some $w, x \in \mathbb{Q}_p$. The following lemmas explain why the norm class is relevant.

Lemma B.5. *Suppose t^2-4 is not a square in $\mathbb{Q}_p$. A matrix*

$$M = \begin{pmatrix} a & b \\ c & t-a \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Q}_p)$$

is conjugate to $\frac{1}{2}\begin{pmatrix} t & 1 \\ t^2-4 & t \end{pmatrix}$ if and only if $2b$ is a norm in $\mathbb{Q}_p(\sqrt{t^2-4})$.

Proof. To show that the condition is necessary, note that conjugating $\frac{1}{2}\begin{pmatrix} t & 1 \\ t^2-4 & t \end{pmatrix}$ by an arbitrary matrix $N = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$ with determinant 1 yields

$$N\left[\frac{1}{2}\begin{pmatrix} t & 1 \\ t^2-4 & t \end{pmatrix}\right]N^{-1} = \frac{1}{2}\begin{pmatrix} t - wy + xz(t^2-4) & w^2 - x^2(t^2-4) \\ z^2(t^2-4) - y^2 & t + wy - xz(t^2-4) \end{pmatrix}.$$

So, if M is conjugate to $\frac{1}{2}\begin{pmatrix} t & 1 \\ t^2-4 & t \end{pmatrix}$, we must have

$$b = \frac{1}{2}[w^2 - x^2(t^2-4)] \implies 2b = w^2 - x^2(t^2-4)$$

for some $w, x \in \mathbb{Q}_p$.

To show that the condition is sufficient, suppose $2b = w^2 - x^2(t^2-4)$ for some $w, x \in \mathbb{Q}_p$. Since t^2-4 is not a square, the matrix M is not diagonalizable over $\mathbb{Q}_p$, so $b \neq 0$. If we define

$$T = \begin{pmatrix} w & x \\ \frac{x(t^2-4)-w(2a-t)}{2b} & \frac{w-x(2a-t)}{2b} \end{pmatrix},$$

then explicit computation yields

$$T\left[\frac{1}{2}\begin{pmatrix} t & 1 \\ t^2-4 & t \end{pmatrix}\right]T^{-1} = \begin{pmatrix} a & b \\ c & t-a \end{pmatrix}.$$

We can also note that T has determinant 1, thus concluding the proof. ☐

Lemma B.6. *Using the notation from Lemma B.4, a matrix*

$$M = \begin{pmatrix} a & b \\ c & t-a \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Q}_p)$$

is conjugate to $\frac{1}{2}\begin{pmatrix} t & s\alpha \\ \frac{t^2-4}{s\alpha} & t \end{pmatrix}$ *if and only if* $2b$ *is not a norm in* $\mathbb{Q}_p(\sqrt{t^2-4})$.

Proof. To show that the condition is necessary, let $k = s\alpha$ and note that conjugating $\frac{1}{2}\begin{pmatrix} t & k \\ \frac{t^2-4}{k} & t \end{pmatrix}$ by an arbitrary matrix $N = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$ with determinant 1 yields

$$N\left[\frac{1}{2}\begin{pmatrix} t & k \\ \frac{t^2-4}{k} & t \end{pmatrix}\right]N^{-1} = \frac{1}{2}\begin{pmatrix} t - wyk + xz\frac{t^2-4}{k} & w^2k - x^2\frac{t^2-4}{k} \\ z^2\frac{t^2-4}{k} - y^2k & t + wyk - xz\frac{t^2-4}{k} \end{pmatrix}.$$

So, if M is conjugate to $\frac{1}{2}\begin{pmatrix} t & k \\ \frac{t^2-4}{k} & t \end{pmatrix}$, we must have

$$b = \frac{1}{2}\left[w^2k - x^2\frac{t^2-4}{k}\right].$$

Rearranging this equation gives

$$\frac{2b}{k} = w^2 - \left(\frac{x}{k}\right)^2(t^2-4).$$

This means that $\frac{2b}{k}$ is a norm in $\mathbb{Q}_p(\sqrt{t^2-4})$. From the proof of Lemma B.4, we know that k is not a norm in $\mathbb{Q}_p(\sqrt{t^2-4})$, so this means that $2b$ is not a norm.

To show that the condition is sufficient, suppose $2b$ is not a norm. Since k is not a norm, we know $\frac{2b}{k}$ is a norm, so we can write $\frac{2b}{k} = x_1^2 - x_2^2(t^2-4)$ for some $x_1, x_2 \in \mathbb{Q}_p$. Since t^2-4 is not a square, the matrix M is not diagonalizable over $\mathbb{Q}_p$, so $b \neq 0$. If we define $w = x_1$ and $x = kx_2$, and construct the matrix

$$T = \begin{pmatrix} w & x \\ \frac{x\frac{t^2-4}{k} - w(2a-t)}{2b} & \frac{wk - x(2a-t)}{2b} \end{pmatrix},$$

then explicit computation yields

$$T\left[\frac{1}{2}\begin{pmatrix} t & k \\ \frac{t^2-4}{k} & t \end{pmatrix}\right]T^{-1} = \begin{pmatrix} a & b \\ c & t-a \end{pmatrix}.$$

We can also note that T has determinant 1, thus concluding the proof. ☐

This leads us to the following conclusion:

Theorem B.7 (Criterion for $\mathrm{SL}_2(\mathbb{Q}_p)$ conjugacy)**.** *Let p be an odd prime, and let $M, N \in \mathrm{SL}_2(\mathbb{Q}_p)$ be regular semisimple matrices, both with trace t. Assume $D = t^2 - 4 \neq 0$. If D is a square in $\mathbb{Q}_p$, then M and N are conjugate in $\mathrm{SL}_2(\mathbb{Q}_p)$. If D is not a square in $\mathbb{Q}_p$, then M and N are conjugate in $\mathrm{SL}_2(\mathbb{Q}_p)$ if and only if the quotient of the top-right elements is a norm in $\mathbb{Q}_p(\sqrt{t^2-4})$.*

Proof. Since the matrices are regular semisimple, we must have $D \neq 0$. If D is a square, then we are done by Lemma B.3. Assume D is not a square. Based on Lemma B.5 and Lemma B.6, the norm class of twice the top-right element determines which of the two representatives in Lemma B.4 the matrix is conjugate to.

If the quotient of the top-right elements of M and N is a norm, then the top-right elements are in the same norm class, and so is twice the top-right elements. So, they are both conjugate to the same representative, and thus conjugate to each other.

If the quotient of the top-right elements of M and N is not a norm, then they are not in the same norm class. Each is then conjugate to a different representative, and by Lemma B.4, they are not conjugate to each other. $\square$

Therefore, Theorem B.7 details the additional criterion that must be checked when determining conjugacy in $\mathrm{SL}_2(\mathbb{Q}_p)$.