Pick's Theorem

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MIT PRIMES Circle

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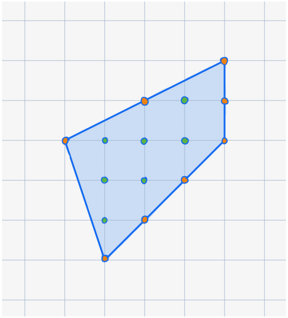

Figure 1: We can calculate the area of this shape using Pick's Theorem.

What's the area of this triangle?

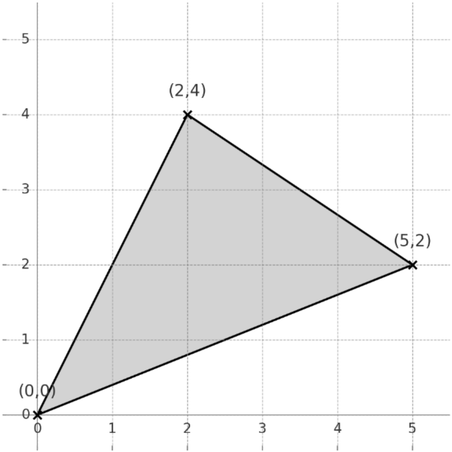

Figure 2: The depicted triangle has the following vertices: $(0, 0)$, $(2, 4)$, and $(5, 2)$.

Strategy 1: Enclose in a box

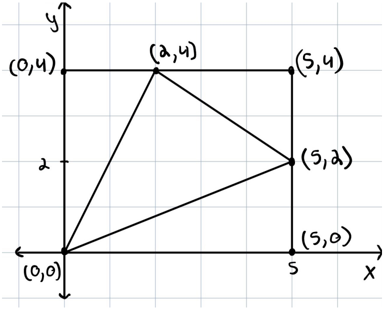

Figure 3: We can find the box's area and the areas of three three small triangles and subtract the sum of their areas from the box's area.

- The box has an area of 20, the triangle on the left has an area of 4, the one on the top right is 3, and the bottom one is 5
- By adding their areas up, we get $4 + 3 + 5 = 12$
- We subtract 12 from the area of the entire box, $20 - 12 = 8$, to get that the area of the original triangle is 8

Strategy 2: Heron's Formula

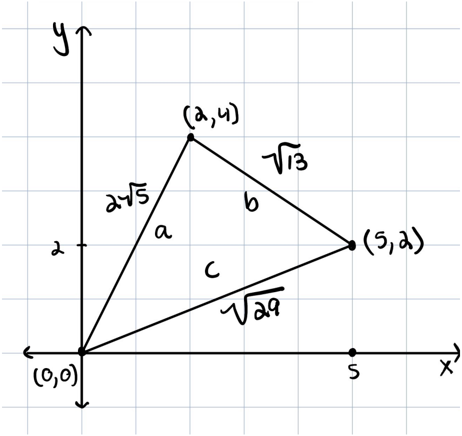

Figure 4: A triangle with side lengths a, b, c will have a perimeter of $a + b + c$ and a semiperimeter of $\frac{a+b+c}{2}$, which we will define as s. Using the values of the variables for this particular triangle with Heron's formula $\sqrt{s(s-a)(s-b)(s-c)}$ we find that the area is 8.

Strategy 3: Cut the Triangle in Half

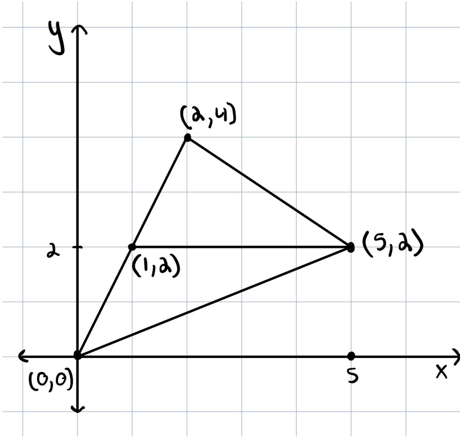

Figure 5: The area of each of the smaller triangles is $\frac{4*2}{2}$, which equals 4; so, when adding the areas together, we can deduct that the area of the original triangle is 8.

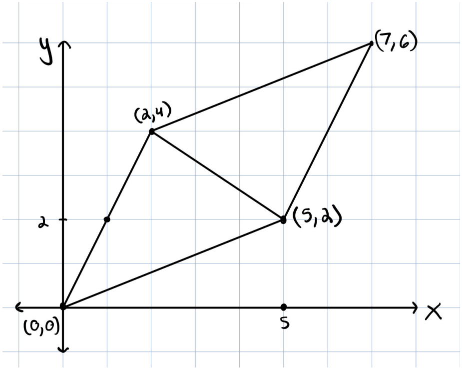

Figure 6: If we form a matrix $\begin{pmatrix} 5 & 2 \\ 2 & 4 \end{pmatrix}$ using the vertices (5,2) and (2,4), with the determinant formula $ad - bc$ for a 2 by 2 matrix $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ we get $20 - 4 = 16$. The area of a polygon is half the determinant, so the area of this triangle is 8.

Strategy 5: Pick's Theorem

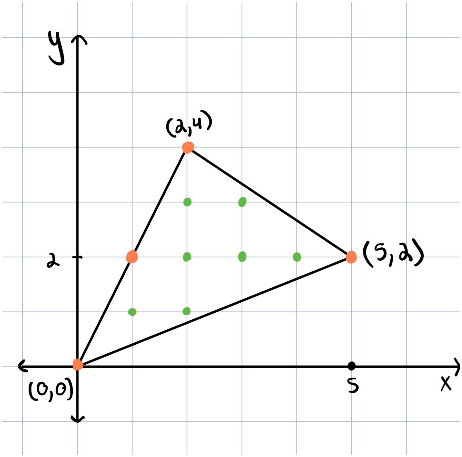

Figure 7: Pick's Theorem uses the formula $A = i + \frac{b}{2}$-1, and $b = 4$, the number of boundary (orange) points, and $i = 7$, the number of interior (green) points. So, when we plug in b and i into the equation, $A = 7 + \frac{4}{2}$-1, we get that $A = 8$.

Statement of Pick's Theorem

Theorem

Consider a simple polygon with lattice points as vertices, i interior lattice points, and b boundary lattice points, the area A of the polygon is $A = i + \frac{b}{2} - 1$.

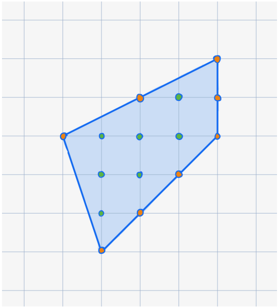

Figure 8: Depicted above is a polygon with 8 boundary lattice points shown in orange and 7 interior lattice points shown in green, so the area is $A = 7 + \frac{8}{2} - 1 = 10$.

The lattice point hypothesis in Pick's Theorem

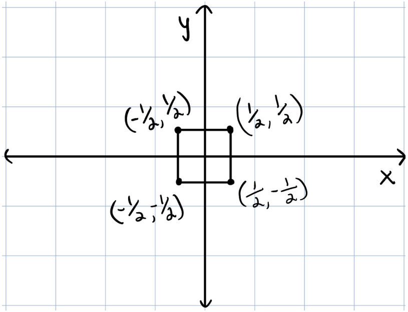

Figure 9: A square with vertices $(-1/2, -1/2)$, $(1/2, -1/2)$, $(1/2, 1/2)$, and $(-1/2, 1/2)$, and an area of 1.

- Has 1 interior lattice point, so $i = 1$, and 0 boundary lattice points, so $b = 0$
- Applying Pick's Theorem: area is $i + b/2 - 1 = 0$, but this statement is false
- Therefore important for the polygon to have lattice points as vertices

The simple polygon hypothesis in Pick's Theorem

Theorem

Consider a polygon with lattice points as vertices, with i interior lattice points, b boundary lattice points, and h holes, then the area A of the polygon is $A = i + \frac{b}{2} + h - 1$.

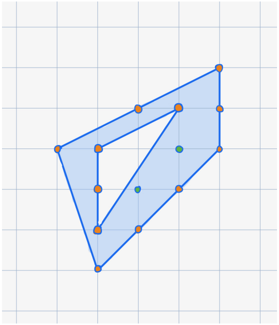

Figure 10: Depicted above is a polygon with a hole. The number of interior lattice points i=2, boundary lattice points b=12, and holes h=1. So, $A = i + \frac{b}{2} + h - 1 = 2 + 6 = 8$.

Proof of Pick's theorem

Theorem

Let Δ be a triangle that contains no boundary or interior lattice points except for the three vertices. Then Δ has area $1/2$. We call Δ a special triangle.

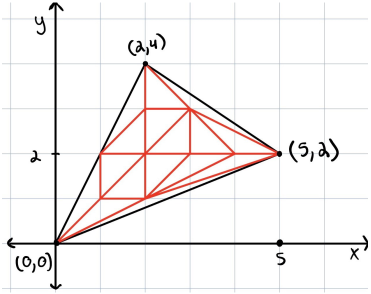

Figure 11: A triangle can be subdivided into 16 special triangles and regarded as a graph.

Proof of Pick's theorem (Cont.)

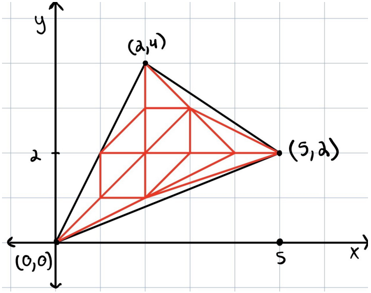

Figure 12: A triangle can be subdivided into 16 special triangles and regarded as a graph.

Theorem

In a planar graph G, let v be the number of vertices, e the number of edges, and f the number of faces. Then G has Euler characteristic $\chi = v - e + f = 2$.

Using Euler's characteristic $v - e + f = 2$ and the expressions $v = i + b$, $f = 2A + 1$, and $e = \frac{6A+b}{2}$ from $6A = 2e - b$, we can find Pick's theorem.

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