Introduction to Group Theory

Jianing Huang, Sylvia Lee

MIT PRIMES Circle

May 18, 2025

How Can We Understand the Symmetry of this Shape?

What is the set of all the symmetries of this square, and how can they be composed?

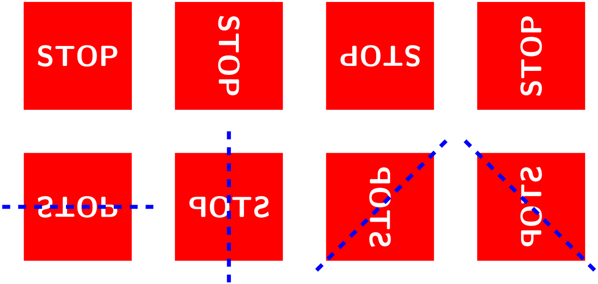

Figure: A beautiful red square

Definition of Groups

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3. *Identity*. There exists an identity $e \in G$ such that $ae = ea = a$.
4. *Inverse*. For all $a \in G$, there is an element $b \in G$ such that $ab = ba = e$. b is the inverse of a, denoted $b = a^{-1}$.

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The **order of an element** a, denoted $|a|$, is the smallest positive integer n such that $a^n = e$.

Examples of Groups

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The group $\mathbb{Z}$ has infinite order.

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The dihedral group D_4 is the set of all symmetries of a square $\{I, V, H, D_1, D_2, R_{90}, R_{180}, R_{270}\}$ with composition as the operation.

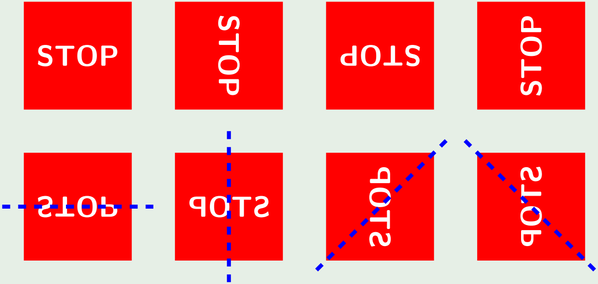

Figure: D_4 has order 8.

Examples of Groups

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The dihedral group D_4 is the set of all symmetries of a square $\{I, V, H, D_1, D_2, R_{90}, R_{180}, R_{270}\}$ with composition as the operation.

- Closure: The composition of any two transformations always results in another transformation that preserves the square's shape.

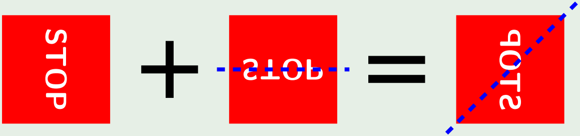

Figure: $R_{90} + H = D_1$.

Examples

- Associativity: Composition is associative.

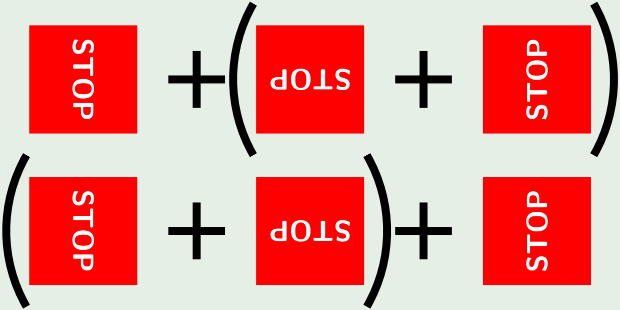

Figure: $R_{90} + (R_{180} + R_{270}) = (R_{90} + R_{180}) + R_{270} = I$.

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Figure: The "do nothing" operation.

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Figure: The “do nothing” operation.

- Inverse: Each transformation can be “undone” by the opposite transformation.

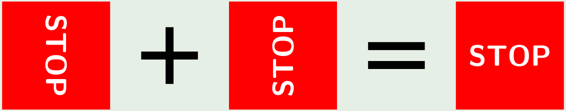

Figure: $R_{90} + R_{270} = I$, so $(R_{90})^{-1} = R_{270}$.

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The set of rotations $\{I, R_{90}, R_{180}, R_{270}\}$ in the group of transformations $D_4 = \{I, V, H, D_1, D_2, R_{90}, R_{180}, R_{270}\}$ on a square is a subgroup, where the operation is composition.

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Example

The set $\{1, 2, 4\}$ under multiplication is a subgroup of $\mathbb{Z}/7\mathbb{Z}^{\times} = \{1, 2, 3, 4, 5, 6\}$.

Definition of Cyclic Groups

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A group G is **cyclic** if $G = \{a^n \mid a \in G, n \in \mathbb{Z}\}$, i.e. the powers of one element a in G covers the whole group. We can denote this as $G = \langle a \rangle$, where element a is a **generator** of G.

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Example

The integers $\mathbb{Z}$ under addition is an infinite cyclic group $\mathbb{Z} = \langle 1 \rangle$. Negative integers can be generated by the element $1^{-1} = -1$. The order of $\mathbb{Z}$ is infinite.

Examples of Cyclic Groups

Example

The set of nonzero remainders mod 7, $\mathbb{Z}/7\mathbb{Z}^{\times}$, is the cyclic group $\langle 3 \rangle$ with generator 3.

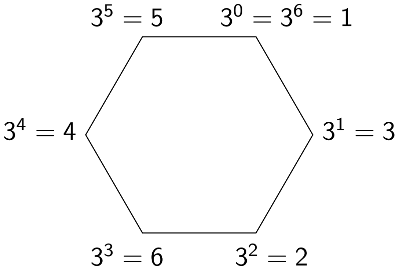

Figure: All nonzero remainders mod 7 are generated by every set of six consecutive powers of 3.

Main Theorem

Theorem (Fundamental Theorem of Cyclic Groups)

Let G be a finite cyclic group $\langle a \rangle$ with order n, then every subgroup H of G must satisfy the following:

1. *H is also cyclic. Specifically, $H = \langle a^m \rangle$.*
2. *$|H|$ is a divisor of n. In particular, if $H = \langle a^m \rangle$, then $|H| = \frac{n}{m}$.*
3. *If the order of H is known, then H is unique.*

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Proof.

Let $G = \langle a \rangle$, then H must contain some power of a.
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Express $k = mq + r$, where $r < m$.

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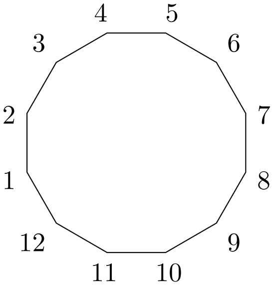
Wait! m was the smallest positive number such that $a^m \in H$.
Thus, $r = 0$ and $k = mq$.
So for all $b \in H$, $b = a^k = a^{mq} = (a^m)^q$ for some q.
i.e. all elements in H is a power of a^m, so $H = \langle a^m \rangle$. $\square$

Theorem in Application - D_{12}

Example

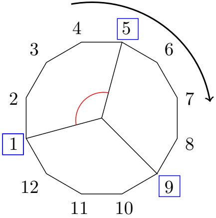
Consider a regular dodecagon (12 sides) and its dihedral group D_{12}, which includes all potential transformations, where the subgroup of rotations is a cyclic group of order 12, let's call this G.



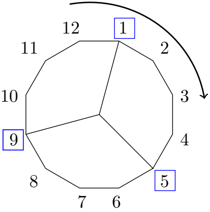
Theorem in Application - D_{12}

Example

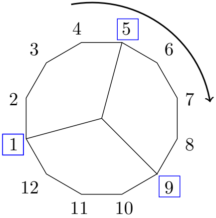
Can you find a subgroup of G with order 3? Let's verify that it is also cyclic, and that it is unique.



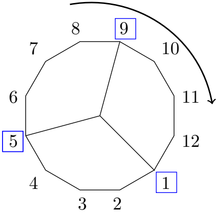
I



R_{120}



$I = (R_{120})^3$



$R_{240} = (R_{120})^2$

This is the cyclic subgroup $\langle R_{120} \rangle$. Note that R_{240} would generate the same group. $R_{120} = (R_{30})^4$, where R_{30} is the generator of the whole group, and $4 = 12 \div 3$.

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The answer is no!

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Can you find a subgroup of G with order 5?

The answer is no!
Since 5 is not a divisor of 12, you cannot find a subgroup with order 5 in a cyclic group with order 12, as shown by the Fundamental theorem.

Works Cited

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Credits

Thank you to the PRIMES CIRCLE program and to our mentor June Kayath for providing us with this opportunity!
Thank you all for listening!