Graph Theory

An Algorithmic Exploration in Graph Theory

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Outline

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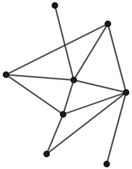
Basic Definitions

Definition (Graph)

- A *graph* is a pair (V, E)
- V is the vertex set. *Vertices* are points on a graph.
- E is the set of edges. *Edges* are lines connecting any two of these vertices
- An edge e in E is a set $\{x, y\}$ where vertex x is not equal to vertex y, and x and y are both in V.



Basic Definitions

Definition (Isomorphism)

Two graphs are considered *isomorphic* to each other if they preserve the same number of vertices and edges, as well as all edge connections, even if they are not structurally identical.
In graph theory, isomorphic graphs can essentially be considered as the same graph.

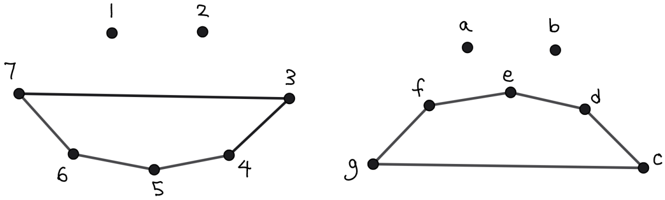


Figure: Example of isomorphic graphs

Basic Definitions

Definition (Neighbor)

Two vertices x, y in G are *neighbors* if they are adjacent.
The set of all neighbors of a vertex v is called its *neighborhood* $(N(v))$.

Definition (Degree)

The number of neighbors of vertex v. This is denoted as $\deg(v)$.
Can also be defined as $\deg(v) = |N(v)|$.

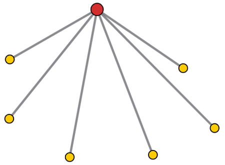


Figure: The neighborhood of the red vertex are the yellow vertices.

Basic Definitions

Definition (Path)

A graph P with $V = \{v_0, v_1, ... v_k\}$ and $E = \{v_0v_1, v_1v_2, ..., v_{k-1}v_k\}$.

Definition (Cycle)

A graph K with $V = \{v_0, v_1, ... v_k\}$ and $E = \{v_0v_1, v_1v_2, ..., v_{k-1}v_k, v_kv_0\}$.
Cycles can be denoted as C^n for n-cycle for a cycle of length n.

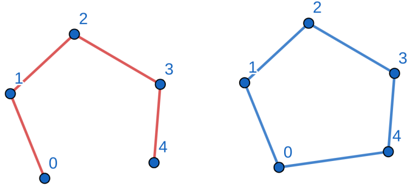


Figure: Examples of a path and a cycle

Trees

Definition (Tree)

A *tree* is a connected graph that is acyclic (not containing any cycles).

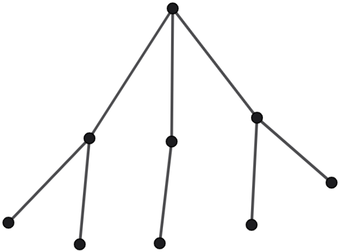


Figure: Example of a tree

Leafs

Definition (Leaf)

A *leaf* is a vertex on a tree that has degree 1. Every tree has at least 1 leaf.

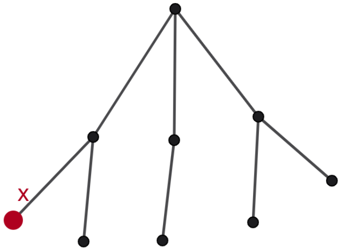


Figure: Vertex X is a leaf

Bridges

Definition (Bridge)

Bridges are edges on a connected graph G such that removing them would cause G to become disconnected. In other words, bridges are edges that are necessary to form certain paths between pairs of vertices.

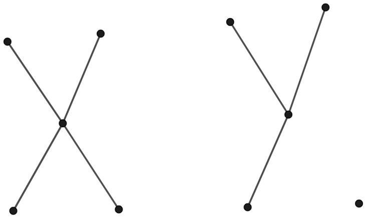


Figure: A connected graph and the resulting disconnected graph when a bridge is removed

Features of Trees

Lemma (Trees)

Trees have 3 defining features:

1. *All trees are connected graphs with n vertices that have $n - 1$ edges.*
2. *Trees contain no cycles*
3. *There is a unique path between any two vertices on a tree.*

Spanning Trees

Definition (Spanning Tree)

A spanning tree T of a connected graph G is a subgraph of G such that T is a tree and contains all of G's vertices ($V(T) = V(G)$).

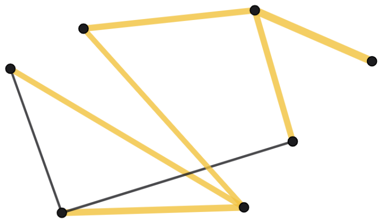


Figure: An example of a spanning tree on a connected graph (highlighted in yellow)

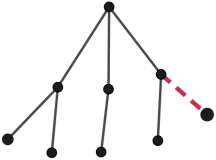
Bridge Lemma

Lemma

Trees consist entirely of bridges

Proof.

Since there is only one unique path between every pair of vertices in a tree, each edge is part of at least one path between vertices. Removing an edge from a tree would break the path between some pair of vertices and disrupt the tree's connectivity. Thus, we can conclude that all edges in a tree are bridges. □



Spanning Tree Algorithm

Lemma (Spanning Tree Algorithm)

A spanning tree can be generated from a connected graph by removing edges without disconnecting the graph.
Let $G = (V, E)$ *be a connected graph with* n *vertices.*

1. *Set* F *equal to* E
2. *While there is an edge* α *of* F *such that* α *is not a bridge of the graph* $T = (V, F)$*, remove* α *from* F *The terminal graph* $T = (V, F)$ *is a spanning tree of* G*.*

Algorithm Diagram

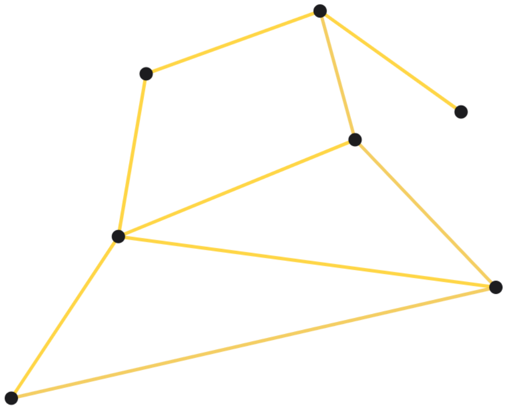


Figure: A connected graph

Algorithm Diagram

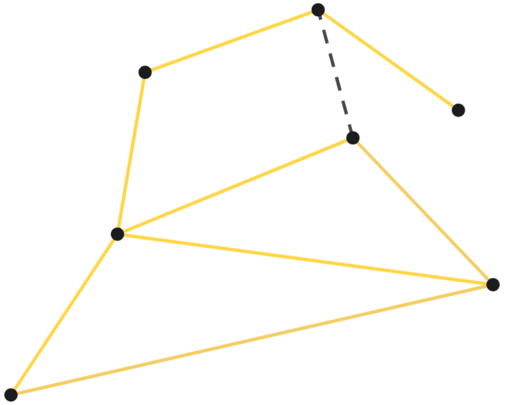


Figure: A connected graph with one non-bridge removed (denoted by the dashed edge)

Algorithm Diagram

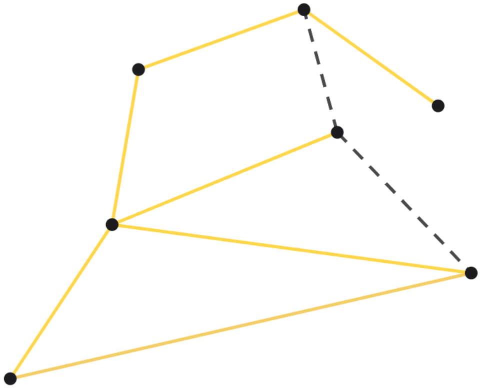


Figure: A connected graph with two non-bridges removed

Algorithm Diagram

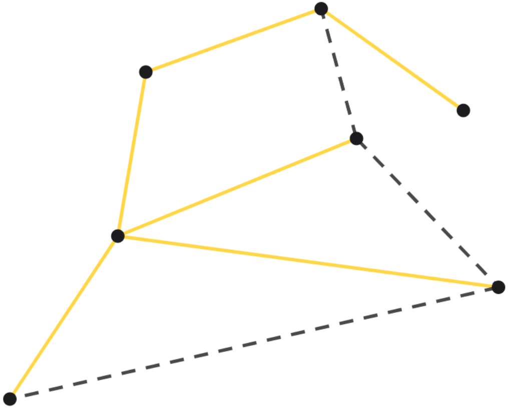


Figure: A connected graph with three non-bridges removed, resulting in a spanning tree

Coloring Definitions

Vertex coloring

Vertex coloring is the process of assigning colors to the vertices of a graph G such that no two adjacent vertices are the same color.

Chromatic Number

$\chi(G)$ is the minimum number of colors required to do a vertex coloring of graph G.

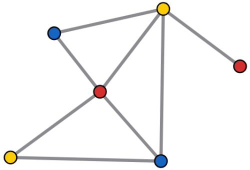


Figure: A example of a vertex coloring

Coloring Lemmas

Lemma: Even Cycle

For any even number n, $\chi(C^n) = 2$.

Lemma: Odd Cycle

For any odd number n, $\chi(C^n) = 3$.

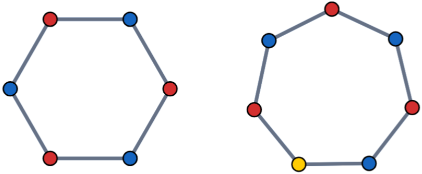


Figure: Examples of cycle colorings

Tree Coloring

Lemma

All trees can be colored with just black and white so that neighboring vertices are different colors.

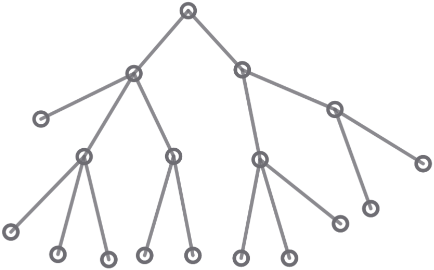
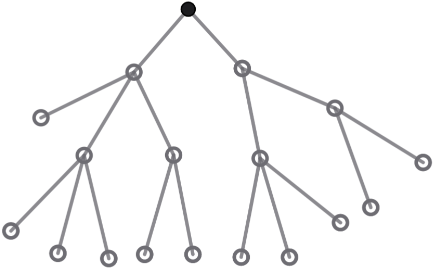
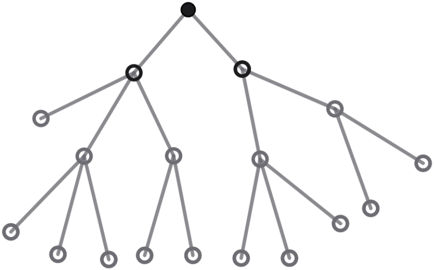


Figure: An example of a tree

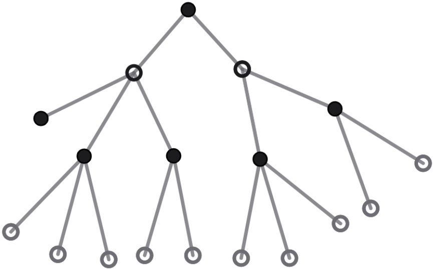
Tree Coloring continued



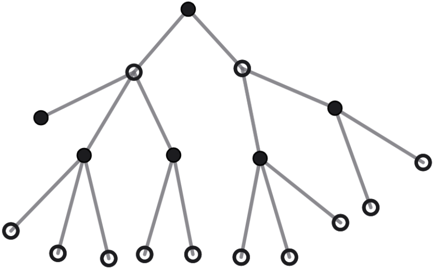
Tree Coloring continued



Tree Coloring continued



Tree Coloring continued



Remark

For any tree T, $\chi(T) = 2$.

Problem

Problem

Suppose we only have two airlines: a red airline and a blue airline.

- The red airline offers flights to different locations on a tree
- The blue airline offers flights on any odd cycle, but are more expensive

You want to go on a vacation, so you decide to take a round trip with an odd number of flights. Show that we only need one blue flight for the trip.

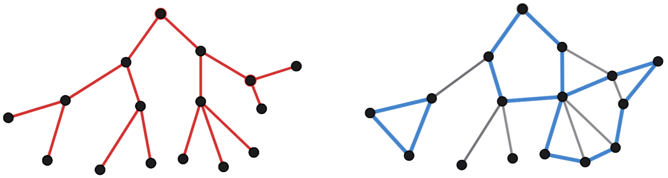
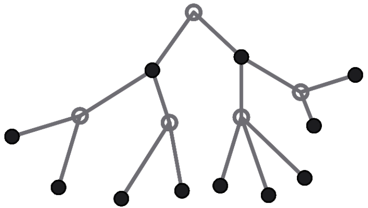


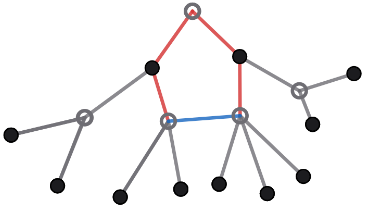
Figure: Examples of possible configurations

Solution continued

Lemma

We need at least one blue flight that is not included in the red airline tree, as you cannot travel on a cycle in any tree.

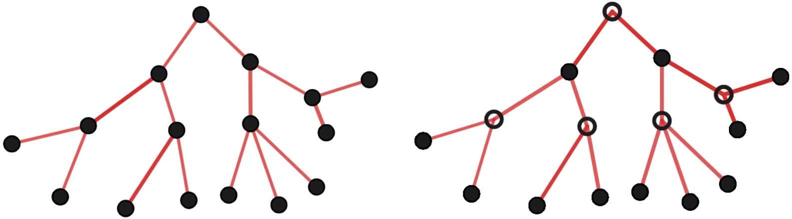




Solution continued

Lemma

If we do a vertex coloring of the red airline tree, every edge on this tree must have two different colored endpoint vertices.



Solution continued

Lemma

There must always be one pair of adjacent vertices with the same color when coloring an odd cycle with only 2 colors.

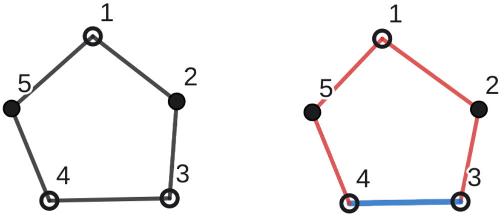


Figure: A possible configuration of a trip

Thank you!

Thank you for listening! Special thanks to our mentors Tomasz and Julia for teaching us Graph Theory and guiding us throughout this program.