Number Theory

Diving into the Difference Game

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How to play difference game

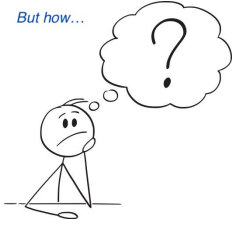
- Begin with two different positive integers
- Players take turns writing differences of numbers already on the board (new difference must be unique)
- Player loses game if unable to form new positive difference

Let's Play!!

How to Win Difference Game!

Suppose that the game begins with M and N, where $M \geq N$

1. Find the gcd of M and N
2. Divide M by the gcd(M,N)
   a. If resulting fraction is **odd**, the **first** player wins
   b. If resulting fraction is **even**, the **second** player wins
3. Choose who will start and set yourself up for success!

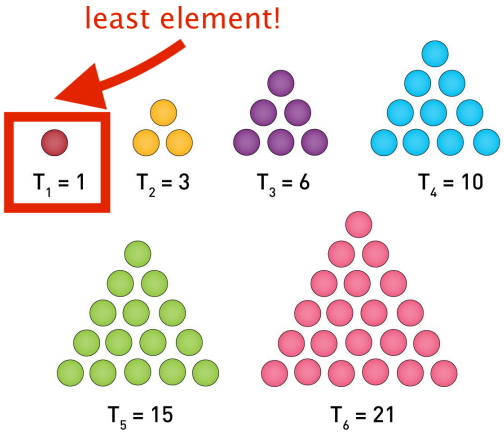


Five Lemmas

These are the five lemmas used to prove the winning strategy:

1. Well-Ordering Principle
2. Every number generated is a multiple of the gcd of the starting numbers.
3. Every number generated is a multiple of the smallest number generated.
4. The smallest number generated is the gcd of the starting numbers.
5. The numbers generated are the positive multiples of gcd(M, N) up to M

Lemma 1: Well-Ordering Principle



The Well-Ordering Principle states that every nonempty set of nonnegative integers must have a least element.

Lemma 2

Every a_i generated in the difference game is a multiple of the gcd of the starting numbers, X_m and X_n.

$$S = \{i \mid \gcd(X_M, X_N) \nmid a_i\}.$$

WANT TO PROVE: Set S is empty
Let ℓ be the **least element** of Set S
PLAN: Prove ℓ does not exist $\rightarrow$ Set S is empty
If ℓ is the least element in $S \rightarrow gcd \nmid x_\ell$
STARTING NUMBERS: x_m and x_n
x_l will be difference of x_m and x_n
l is the least element of Set $S \rightarrow$ all terms that come before x_l are all divisible by the *gcd*
So... $\gcd(X_M, X_N)$ divides all terms that are produced before ℓ.
Can factor gcd out of all differences generated $\rightarrow x_\ell$ is also divisible by gcd.
x_ℓ divisible by gcd $\rightarrow x_\ell$ is not in Set S.
BUT... x_ℓ was the least element?
No least element $\rightarrow$ Set S is empty!

Lemma 3

Every number generated is a multiple of the smallest number generated.
Set $S = \{A_1, A_2, A_3, \ldots, A_N\}$ where each A_i was generated in the Difference Game.
Let d be the least element of S
Starting Numbers: X_m and X_n, $\gcd(X_m, X_n) = g$
x is a difference generated, ASSUME: $d \nmid x$
Using the division algorithm,
$x = dq + r$
$1 \leq r < d$
$r = x - dq$

Lemma 3

$$x - d, x - 2d, x - 3d, ...$$

$$x - dq \longrightarrow \text{ VALID DIFFERENCE!}$$

$$x - dq = r$$

r is a valid difference, but...

Remember: $r < d$

$r < d$ AND r is a difference in our game $\longrightarrow$ d is not least element of S

Conclusion: Lemmas In Action

Goal: Determine who wins the Difference Game based on the starting numbers.

How the Lemmas Help:

- **Lemma 1 (Well-Ordering Principle):** Guarantees a least number exists — the game must end.
- **Lemma 2:** All numbers generated are multiples of the gcd — this controls the structure of the game.
- **Lemma 3:** The gcd is the smallest number generated — this becomes the "unit" of progress.
- **Lemma 4:** Confirms the gcd is actually generated — not just a theoretical idea.
- **Lemma 5:** Shows the full set of differences is: {gcd, 2·gcd, ..., M}

The End