Set Theory and Logic

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Sentential Logic

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Sentential Logic

Sentential logic, also called *propositional logic*, studies how prime sentences (atomic) combine using logical connectives.

- A prime sentence is a basic statement that is either true or false.
  Example:
- Let A = T, where A could be "Dogs normally have 4 legs", a True statement.
- Let B = F could be "Dogs are blue", a False statement.

Logical Connectives

Logical connectives are symbols used to build compound sentences from the prime ones. The most common are the following:
Taking into mind that $A = \mathrm{T}$ and $B = \mathrm{F}$

- $\neg A$ — not A (negation) is False
- $A \wedge B$ — A and B (conjunction) is False
- $A \vee B$ — A or B (disjunction) is True
- $A \rightarrow B$ — if A, then B (implication) is False
- $A \leftrightarrow B$ — A if and only if B (bi-conditional) is False

Truth Tables

Purpose: Evaluate logical expressions systematically.

Example 1: $A \vee B$

A	B	$A \vee B$
T	T	T
T	F	T
F	T	T
F	F	F

Example 2: $A \vee \neg A$ *(a tautology)*

Key Term: Tautology — A statement that is always true, e.g., $A \vee \neg A$.

A	$\neg A$	$A \vee \neg A$
T	F	T
F	T	T

Logical Laws – Identity

$\top$: A Verum, statement that is always true.
$\bot$: A Falsum, statement that is always false.
$\equiv$: Logical Equivalence, where two composites have the same truth table.

Identity Laws:

- $A \vee \bot \equiv A$
- $A \wedge \top \equiv A$

Example Table: $A \vee \bot$

A	$\bot$	$A \vee \bot$
T	F	T
F	F	F

Logical Laws – Domination

Domination Laws:

- $A \vee \top \equiv \top$
- $A \wedge \bot \equiv \bot$

Example Table: $A \wedge \bot$

A	$\bot$	$A \wedge \bot$
T	F	F
F	F	F

Summary

Key Concepts Reviewed:
We touched base on the following:

- Sentential Logic
- Logical Connectives
- Truth Tables (Tautologies)
- Logical Laws

Set Theory and Proof Techniques

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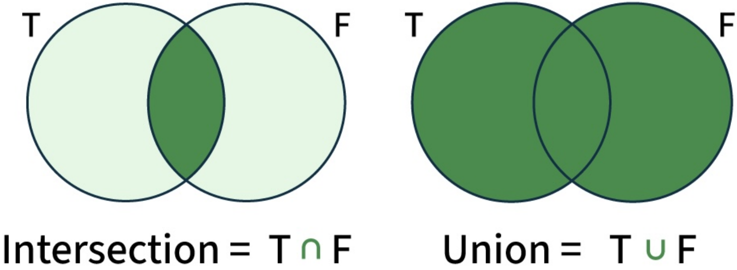
Set Theory

Sets a well-defined collection of distinct objects, which are called *elements* of a set.
Subset A set A is a subset of another set B when all elements of A are within B, all $a \in B$
Inclusion A set A is a subset of a set B and every element of A is also in B, meaning A can be equal to B. Denoted as $A \subseteq B$ and read as "A is included in B."
Proper Inclusion Every element of a set A is included within a set B but they are not equal, B has additional other elements, denoted as $A \subset B$ and read as "A is properly included in B."

Set Theory

Intersection the set containing all elements that in both sets A and B, denoted as $A \cap B$
Union in two sets A and B, a union is the set containing all elements that are in either A or B or both, denoted as $A \cup B$



Proof Techniques

Definition

Proof by contradiction assumes the negation of the desired conclusion and derives a logical inconsistency from this assumption.

Proof by Contradiction

Example

Prove that, if $A \cap B = A$, then $A \subseteq B$.
Proof: Suppose for the sake of contradiction that $A \nsubseteq B$. Then there exists some $a \in A$ such that $a \notin B$. This implies that $a \notin A \cap B$. But then, since $a \in A$, $A \cap B \neq A$, which is a contradiction, indicating that in fact $A \subseteq B$.

Sudoku as Sets and Predicate Logic

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Why Logic? Why Sudoku?

- **More than just a puzzle.** It challenges players to use logic, deduction, and structure.
- Because we can use **set theory** and **predicate logic** to formally describe the rules and valid solutions.
- Translate human reasoning into mathematical language
- It shows how mathematics can model real world problem solving.
- Using logic to understand Sudoku is an example of how logic helps us structure and automate reasoning in various fields.

Key Mathematical Tools

- **Set**: A group or collection of distinct items. In Sudoku, a row, column, or box can be seen as a set of numbers from 1 to 9.
- **Predicate**: A logical sentence that becomes true or false depending on the input. Example: "x is greater than 5"
- **Quantifiers**: Symbols that help us talk about "how many" things something applies to.
  - $\forall$ means "for all"
  - $\exists$ means "there exists"
- **Satisfiability**: Are all constraints consistent?

Formalizing the Grid

- A Sudoku grid has cells. We label each one by row and column: $G = \{(i,j) \mid 1 \leq i,j \leq 9\}$
- The digits we can place: $D = \{1, 2, \ldots, 9\}$
- The relation $R \subseteq G \times D$: This matches each grid cell with a digit like $((3,4),7)$ for placing 7 at (3,4).
- The predicate $P(i,j,d)$: True if digit d is placed in row i, column j. $P(i,j,d) \iff ((i,j),d) \in R$

						9		
3	4				2		1	5
		9		5	7			
1			3		5			7
		4					9	
7			6		1			8
		3		9	9			
4	6	9			9		5	1
						2		

Sudoku Rules as Set Theory and Logic

Row constraint (no repeated digit in a row):

$$\forall i, d,\ j_1 \neq j_2 :\ \neg(P(i, j_1, d) \wedge P(i, j_2, d))$$

Column constraint:

$$\forall j, d,\ i_1 \neq i_2 :\ \neg(P(i_1, j, d) \wedge P(i_2, j, d))$$

Subgrid constraint:
Let

$$S_{k,l} = \{(i, j) \mid 3k-2 \leq i \leq 3k,\ 3l-2 \leq j \leq 3l\}$$

Each subgrid must contain all digits once.

How Logic Solves Sudoku (Part 1)

- **Starting Point:** We begin with a few numbers already filled in. Call this set R_0, part of the full solution set R, $R_0 \subseteq R$.
- **Translate to Logic:** Each given number becomes a statement like $P(i, j, d)$: "Cell (i, j) has digit d."
- **Use Logic Rules:**
  - If we know a cell has digit d, then it can't have any other digit $d' \neq d$.
  - That digit d also can't appear again in the same row, column, or box.

How Logic Solves Sudoku (Part 2)

- **Make Deductions:**
  - If only one digit can go in a cell, we can safely place it.
  - If a digit can only go in one spot in a row, column, or box, we place it there.
- **Repeat:** Keep applying these logical steps and updating possibilities.
- Continue until every cell is filled (a solution), or no choices are left (no solution).

						9		
3	4				2		1	5
		9		5	7			
1			3		5			7
		4					9	
7			6		1			8
		3		9	9			
4	6	9			9		5	1
						2		

Summary: Sudoku and Logic

- The way we solve Sudoku using logic is like following an **algorithm**, a set of rules to solve a problem.
- Every move is a deduction, where we use the rules to figure out what must be true.
- Sudoku turns into a puzzle made of logical constraints
- Shows how math can model the way we think and solve problems.
- The logic and methods we used apply to important concepts and tools in computer science, science, and even everyday reasoning.

Thank You

Questions?