Playing Your Cards Right: The Art of Josephus Dealing

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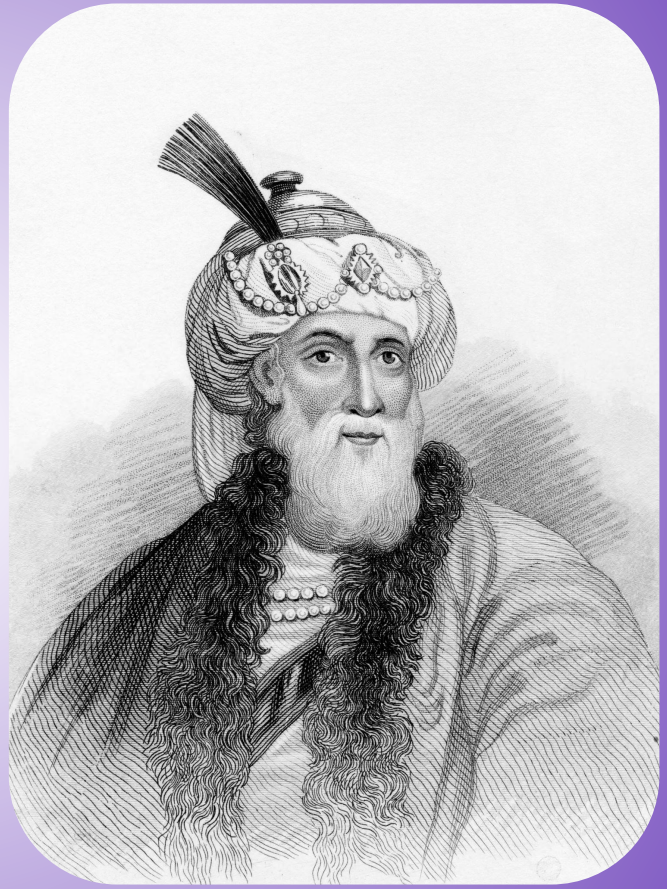
Josephus Problem

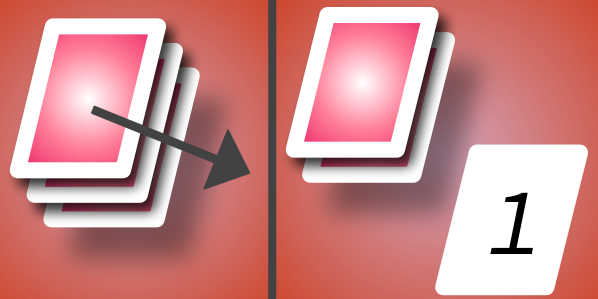
- named after soldier and historian Flavius Josephus
- unique premise which we will demonstrate



Order of Elimination

2, 4, 6, 3, 1, 5

People can be represented as cards.

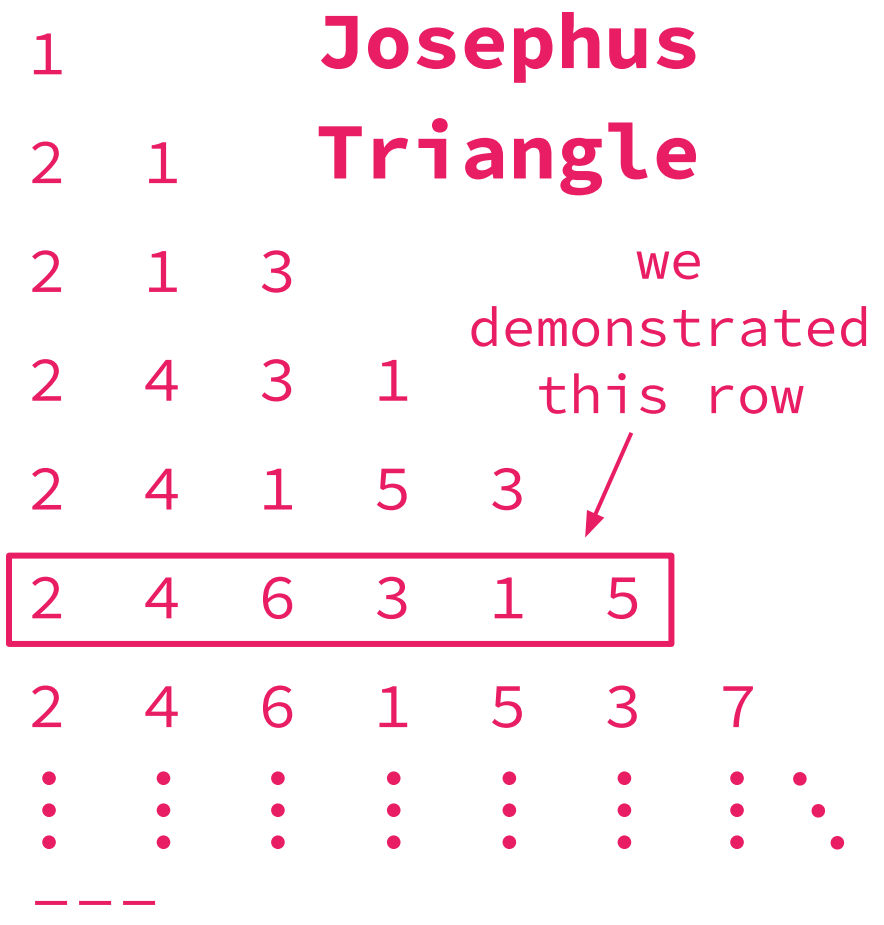
Josephus Problem	Related Letter	Dealing	Picture
Skipping a person	U (under)	Putting the top card to the bottom of the deck	
“Killing” a person	D (down)	Putting the top card on the table (face up)	

Therefore,

the Josephus Problem is equivalent to UD dealing.

Josephus Triangle

- order -> person/card
- row represents the order in which people are eliminated in the Josephus problem



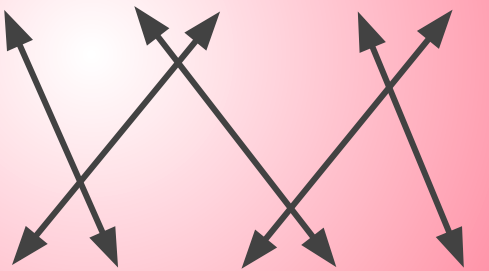
How We Arrange The Cards

- person/card -> order
- to arrange *n* cards, we look at the *n*th row of the triangle

Connection

The Josephus and dealing triangles are connected: they are **inverse permutations**

Josephus Triangle 2 4 1 5 3



UD Triangle 3 1 5 2 4

Index: 1 2 3 4 5

The index and number are swapped.

Other Dealing Patterns

- there are dealings other than UD
- more triangles and patterns from other dealings

First 5 rows of DU(🇦🇺) dealing triangle

```
1
1  2
1  3  2
1  3  2  4
1  5  2  4  3
```

First 5 rows of UUD dealing triangle

```
1
1  2
2  3  1
4  2  1  3
2  4  1  5  3
```

Magic Trick

One of Our Sequences in the Online Encyclopedia of Integer Sequences

sequence number

description

sequence data

comments

formulae

example

authors (us!)

A378635 Triangle T(n,k) read by rows, where row n is a permutation of numbers 1 through n, such that if the deck of n cards is prepared in this order, and under-down dealing is used, then the resulting cards are put down in increasing order.

1, 2, 1, 2, 1, 3, 4, 1, 3, 2, 3, 1, 5, 2, 4, 5, 1, 4, 2, 6, 3, 4, 1, 6, 2, 5, 3, 7, 8, 1, 5, 2, 7, 3, 6, 4, 5, 1, 9, 2, 6, 3, 8, 4, 7, 8, 1, 6, 2, 10, 3, 7, 4, 9, 5, 6, 1, 9, 2, 7, 3, 11, 4, 8, 5, 10, 11, 1, 7, 2, 10, 3, 8, 4, 12, 5, 9, 6, 7, 1, 12, 2, 8, 3, 11, 4, 9, 5, 13, 6, 10, 11, 1, 8, 2, 13, 3, 9, 4

(list; table; graph; refs; listen; history; text; internal format)

OFFSET 1,2

COMMENTS Under-down dealing is a dealing pattern where the top card is put on the bottom of the deck, and the next card is dealt. Then, this pattern repeats until all cards are dealt.
This card dealing is related to the Josephus problem. The card in row n and column k is x if and only if in the Josephus problem with n people, the person number x is the k-th person eliminated. Equivalently, each row of Josephus triangle A321298 is an inverse permutation of the corresponding row of this triangle.
The total number of moves for row n is 2n.
The first column is A225381, the order of elimination of the first person in the Josephus problem.

FORMULA T(1,1) = 1, for n > 1, T(n,1) = T(n-1,n-1) + 1 and T(n,2) = 1. For n > 1 and k > 2, T(n,k) = T(n-1,k-2) + 1.

EXAMPLE Suppose there are four cards arranged in order 4,1,3,2. Card 4 goes under, and card 1 is dealt. Now the deck is ordered 3,2,4. Card 3 goes under, and card 2 is dealt. Now the leftover deck is ordered 4,3. Card 4 goes under, and card 3 is dealt. Then card 4 goes under, and card 4 is dealt. The dealt cards are in order. Thus, the fourth row of the triangle is 4,1,3,2.
Triangle begins:
1;
2, 1;
2, 1, 3;
4, 1, 3, 2;
3, 1, 5, 2, 4;
5, 1, 4, 2, 6, 3;
4, 1, 6, 2, 5, 3, 7;
8, 1, 5, 2, 7, 3, 6, 4;
5, 1, 9, 2, 6, 3, 8, 4, 7;

KEYWORD nonn,tabl

AUTHOR Tanya Khovanova and the MIT PRIMES STEP junior group, Dec 02 2024

STATUS approved

Our Sequences in the OEIS

A378635, A378674, A378682, A380195,

A380201, A380202, A380204, A380246,

A380247, A380248, A381048, A381049,

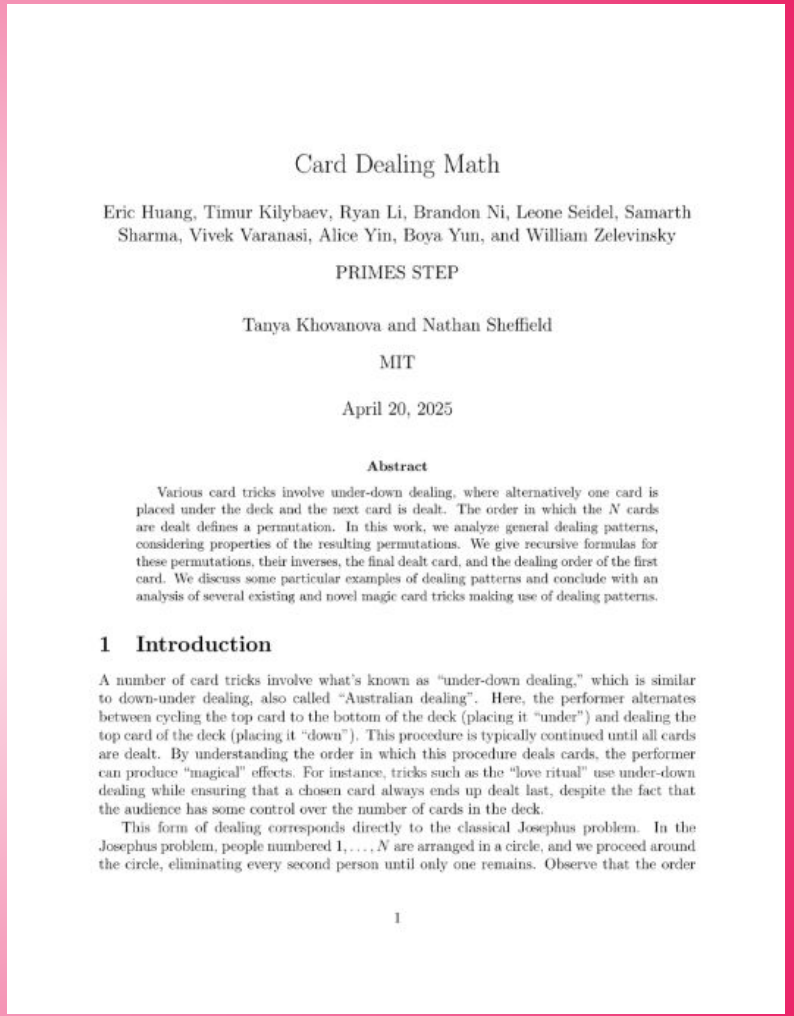
A381050, A381051, A381114, A381127, A381128, A381129,

A381151, A381591, A381622, A381623, A381667, A382354,

A382355, A382356, A382358, A382528, A383076, A383845

In our paper, we also...

- Discuss other dealing patterns such as UDU
- Find and prove many formulas and patterns for dealing triangles
- Discuss some existing and novel card tricks involving dealing



Card Dealing Math

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Abstract

Various card tricks involve under-down dealing, where alternatively one card is placed under the deck and the next card is dealt. The order in which the N cards are dealt defines a permutation. In this work, we analyze general dealing patterns, considering properties of the resulting permutations. We give recursive formulas for these permutations, their inverses, the final dealt card, and the dealing order of the first card. We discuss some particular examples of dealing patterns and conclude with an analysis of several existing and novel magic card tricks making use of dealing patterns.

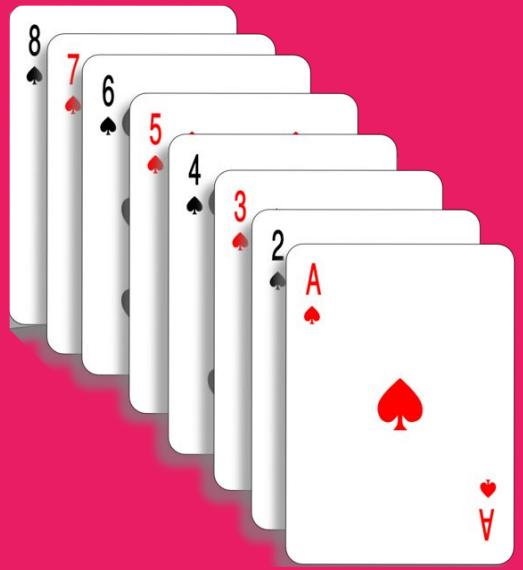
1 Introduction

A number of card tricks involve what's known as "under-down dealing," which is similar to down-under dealing, also called "Australian dealing". Here, the performer alternates between cycling the top card to the bottom of the deck (placing it "under") and dealing the top card of the deck (placing it "down"). This procedure is typically continued until all cards are dealt. By understanding the order in which this procedure deals cards, the performer can produce "magical" effects. For instance, tricks such as the "love ritual" use under-down dealing while ensuring that a chosen card always ends up dealt last, despite the fact that the audience has some control over the number of cards in the deck.

This form of dealing corresponds directly to the classical Josephus problem. In the Josephus problem, people numbered $1, \ldots, N$ are arranged in a circle, and we proceed around the circle, eliminating every second person until only one remains. Observe that the order

Josephus in Australia

Any Questions?



Thanks for listening!

Also thanks to Tanya Khovanova and Nathan Sheffield for supporting our research