MIT PRIMES STEP Senior Group

Chip-Firing On Infinite k-ary Trees

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Table of contents

01

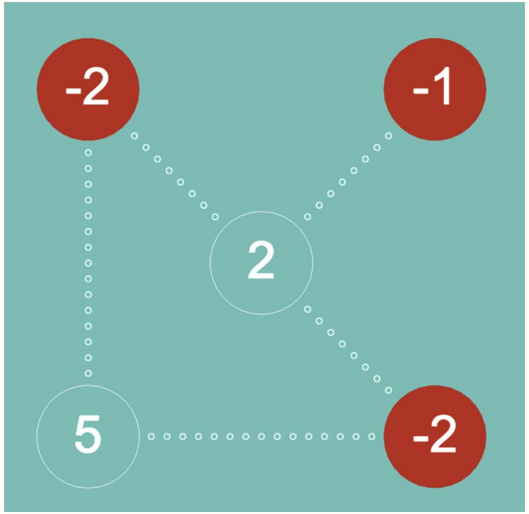
Intro to Chip-Firing

The Dollar Game

- 1 player game
- Played on a graph
- On each turn, select a vertex that will give a dollar to all its neighboring vertices
- We want to make all the vertices have a nonnegative amount of dollars

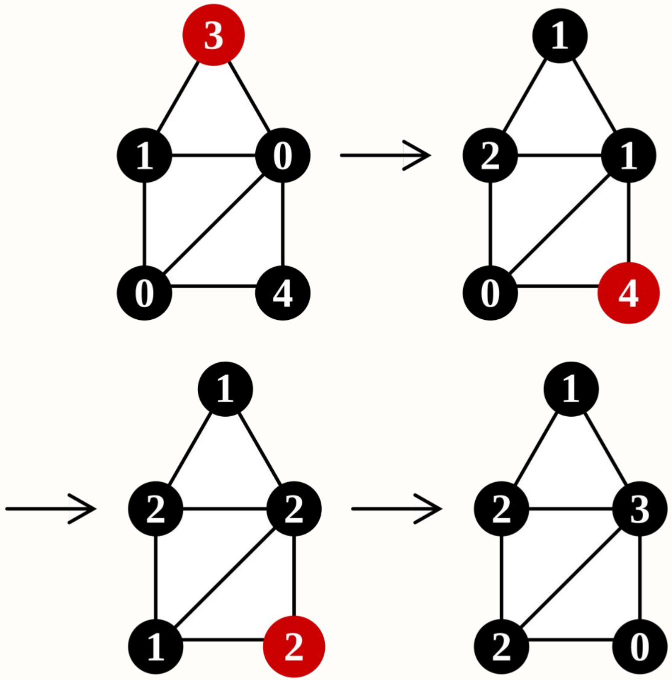
Give it a try!

Example: (Level 1)



Chip–Firing

- Similar 1 player game
- On each turn, you select a vertex to give a chip to all neighboring vertices
- Number of chips at each vertex must always be nonnegative
- **Goal of the game: Find the stable configuration, such that you can no longer fire any vertex.**



02
Past Research on Chip-Firing

Our Underlying Graph

- Perfect *k*-ary tree with a self-loop at the root
- Self-loop makes it so that every vertex has *k+1* neighbors
- Any number of chips at the root

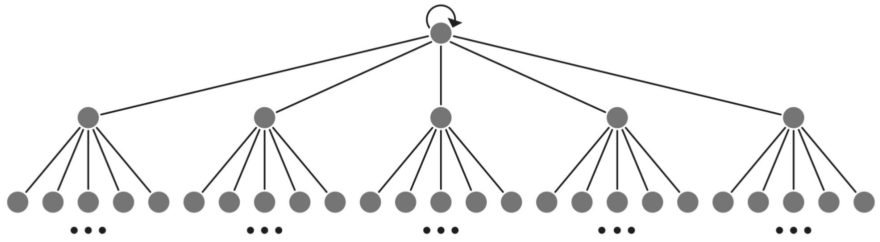


Figure 1: An infinite undirected rooted 5-ary tree with a self-loop at the root

Past Research

Past researchers used the same underlying graph with more restrictions.

- In 2023, Musiker and Nguyen found a formula to find the number of fires on an infinite **binary** tree with a self-loop at the root with $2^k - 1$ chips at the root initially.
- In 2024, Inagaki, Khovanova, and Lou generalized this and found the number of fires on an infinite **binary** tree with **any** number of chips at the root initially.

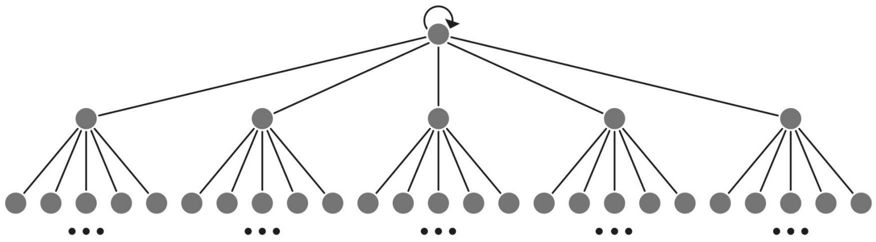


Figure 1: An infinite undirected rooted 5-ary tree with a self-loop at the root

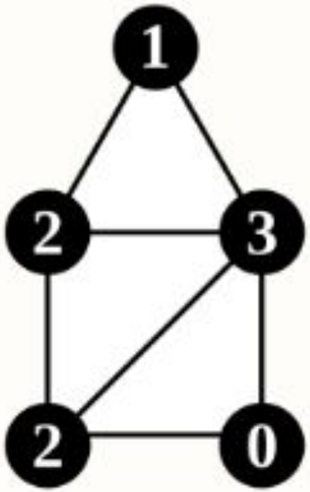
03

The Stable Configuration

Stable Configuration

- A stable configuration in the game of **Chip Firing** is a configuration of the graph such that none of the vertices can legally be fired.

- The graph to the right is in a **stable configuration** because every vertex has more connected neighbors than it has chips. Firing any vertex would result in a vertex with a negative number of chips.



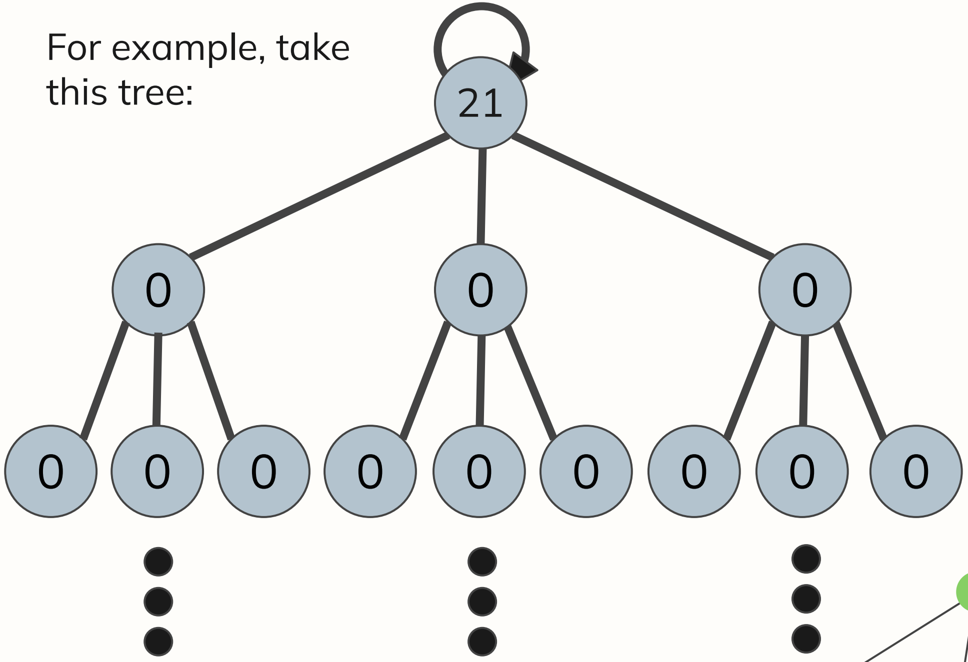
Will our graph have a stable configuration?

- In 2019, Klivans showed that either a stable configuration can be achieved after a finite number of fires or a stable configuration cannot be achieved.
- In 1991, Björner, Lovász, and Shor, proved that if the number of chips is less than the number of edges, then the game is finite and will reach a stable configuration.
- In our case, since we have an infinite *k*-ary tree, the number of edges is infinite, and we have a finite number of chips at the root. Thus, the game is finite and it reaches a stable configuration.
- The stable configuration is unique.

The Stable Configuration of our Tree

If we start with N chips at the root, where $\frac{k^n-1}{k-1} \leq N \leq \frac{k^{n+1}-k}{k-1}$ then the vertices containing chips in the stable configuration form a perfect k-ary tree with height n-1. Furthermore, every vertex on the same layer has the same number of chips.

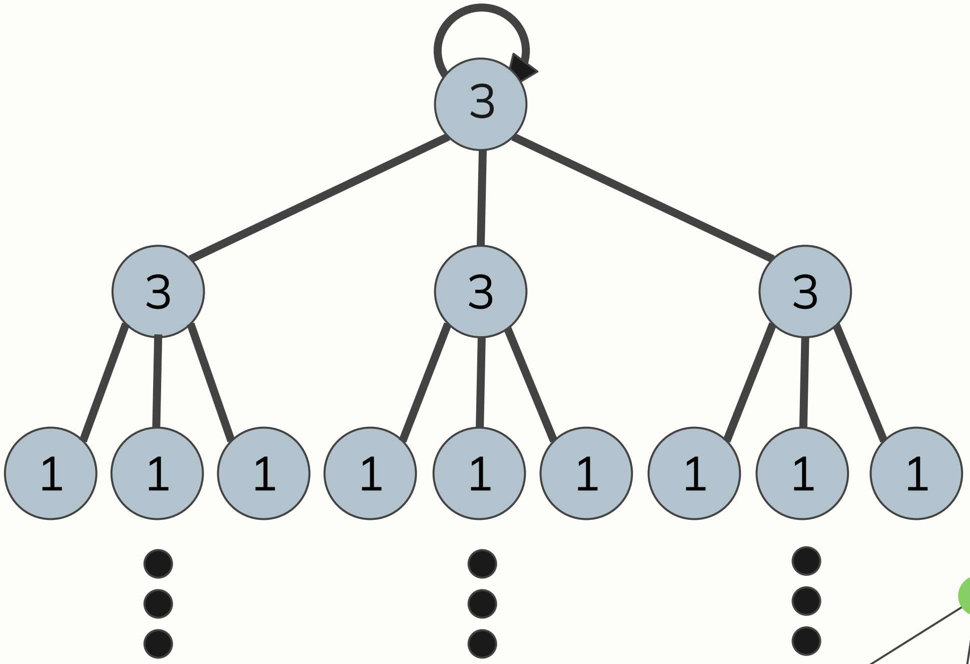
For example, take this tree:



The Stable Configuration of our Tree

The tree ends up having a stable configuration like this:

All numbers on each level are equal, forming a perfect ternary tree with 3 layers.



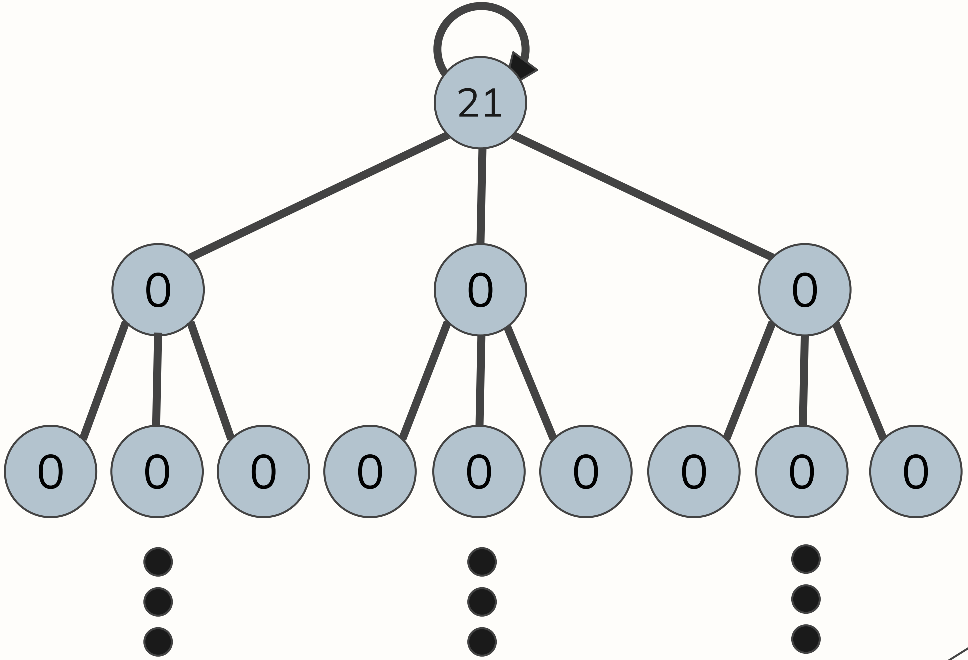
Reaching a Stable Configuration

We can use these four steps to go to the stable configuration.

1. Fire the root repeatedly until it cannot fire anymore
2. Fire the root’s children and their subtrees in parallel
3. Whenever the k children of the root fire, fire the root
4. Repeat the second and third steps on subtrees until we reach the stable configuration

Achieving the Stable Configuration

We will show the process that makes the previous tree obtain its stable configuration.



Achieving the Stable Configuration

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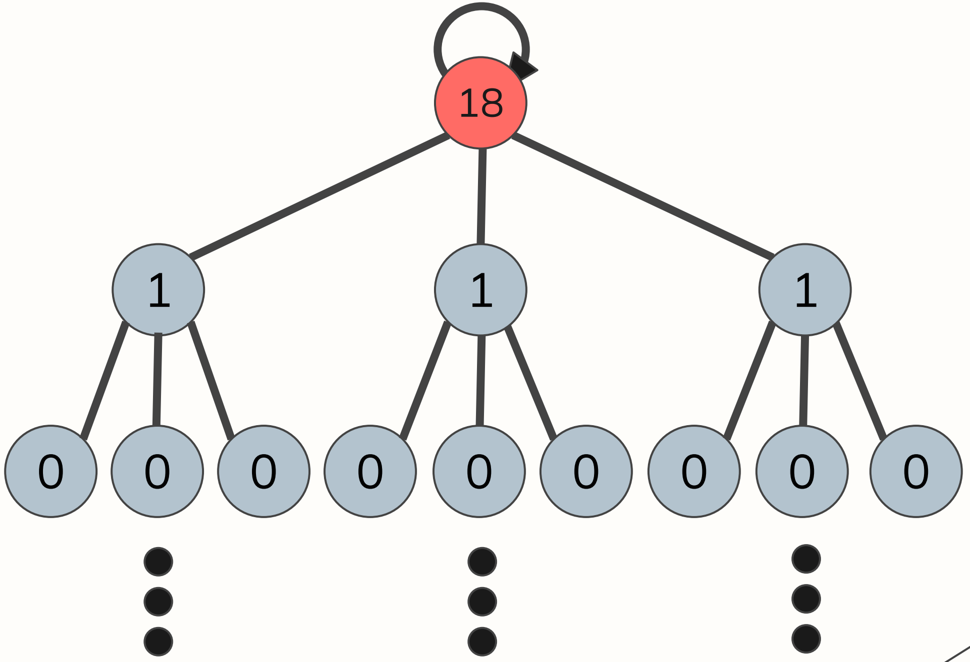

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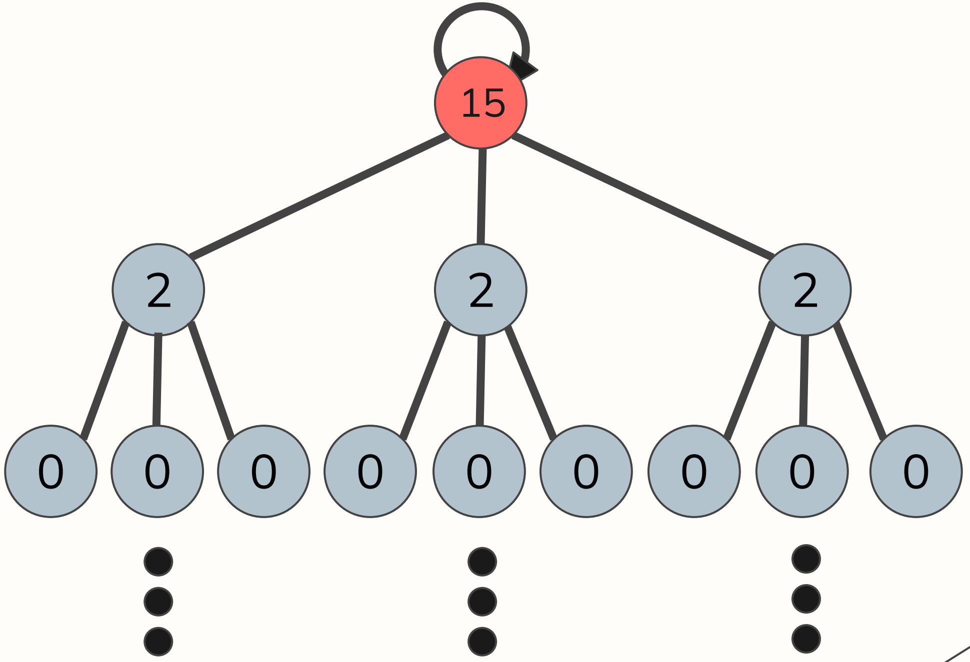

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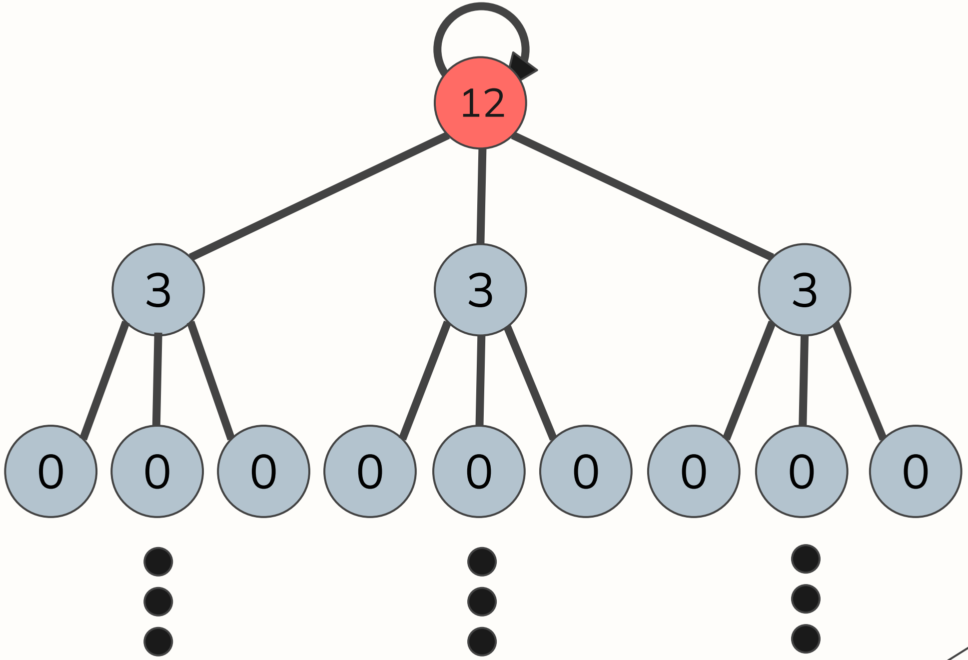

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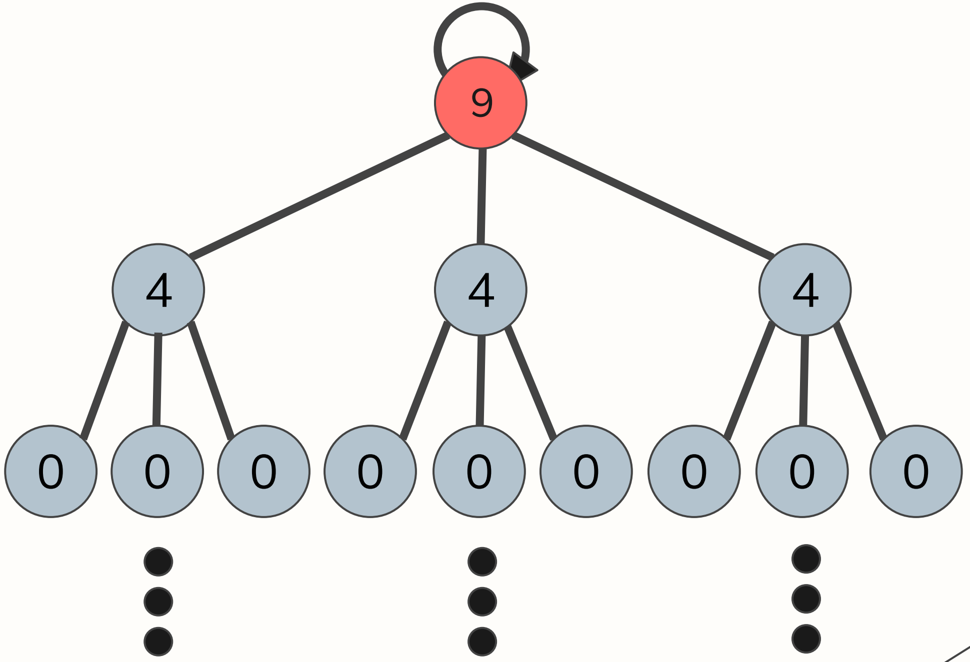

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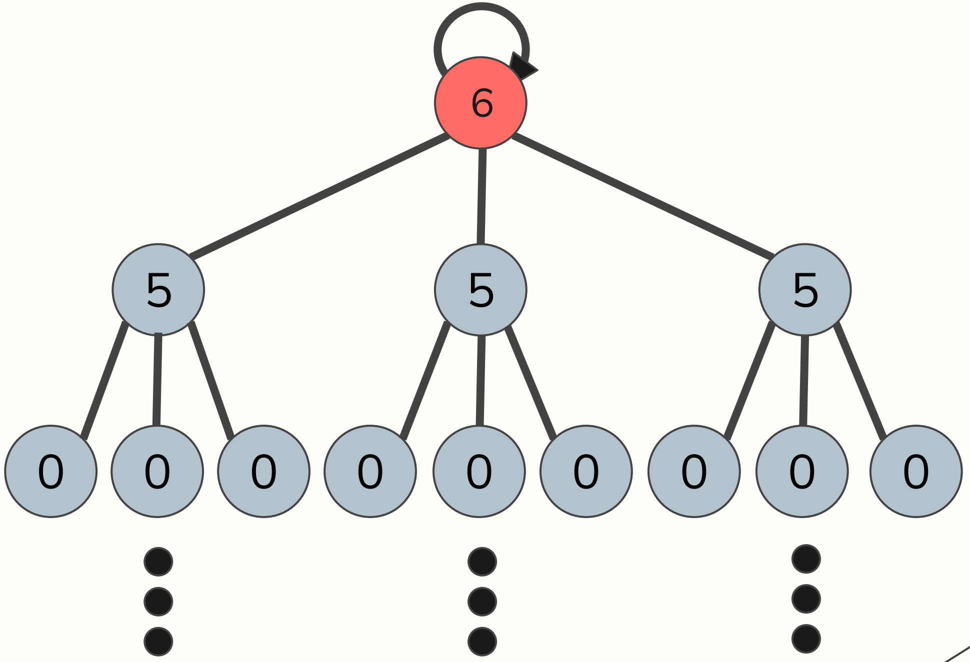

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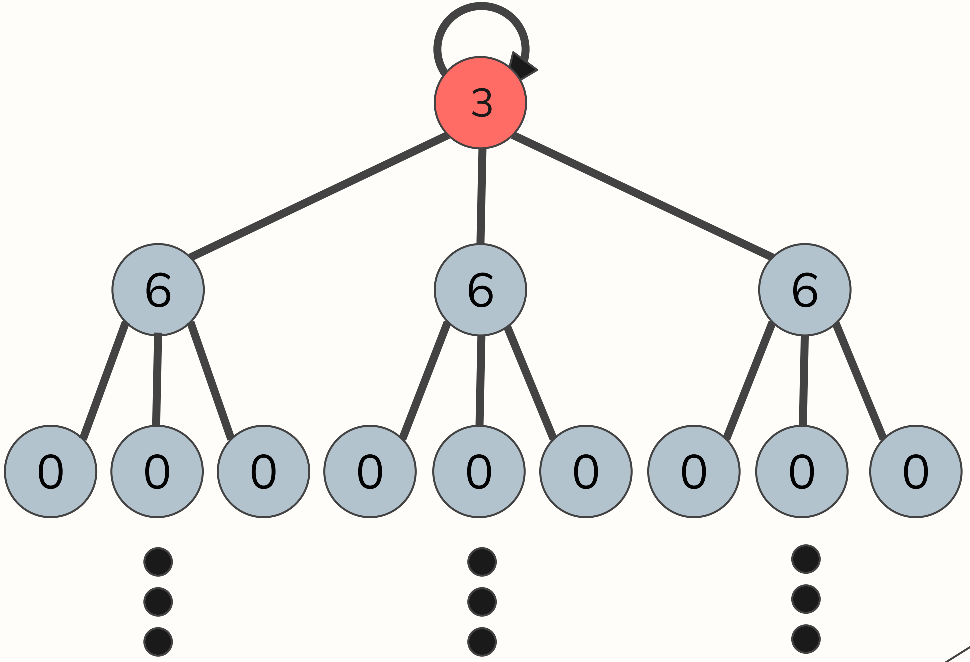

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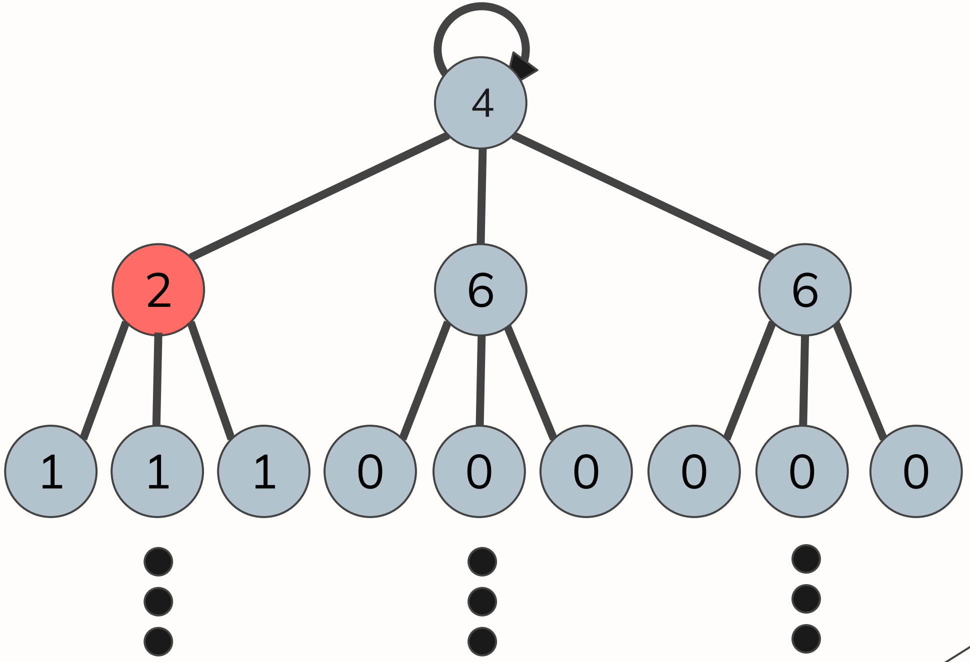

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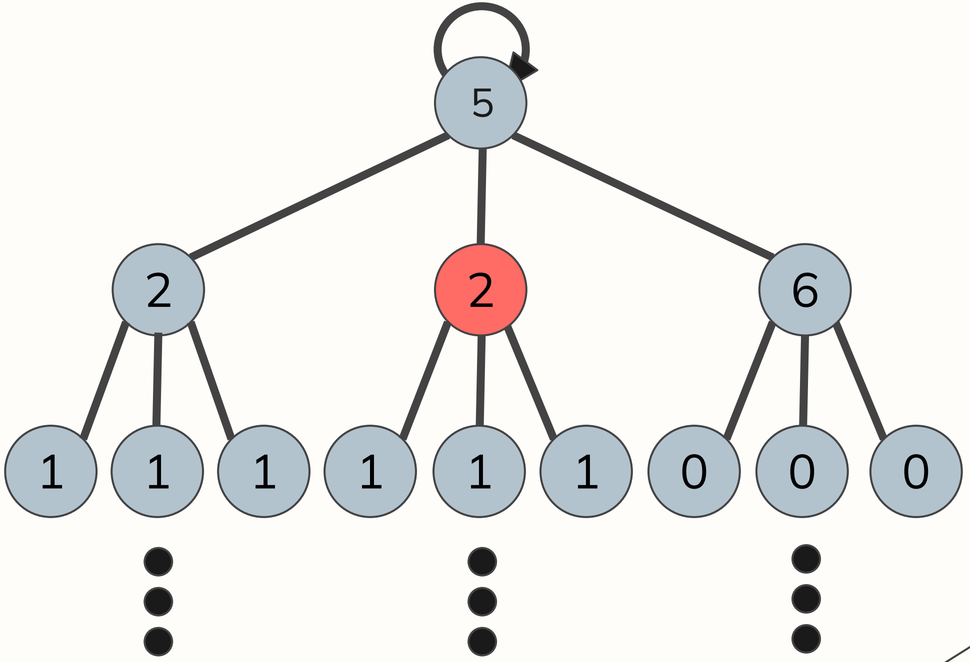

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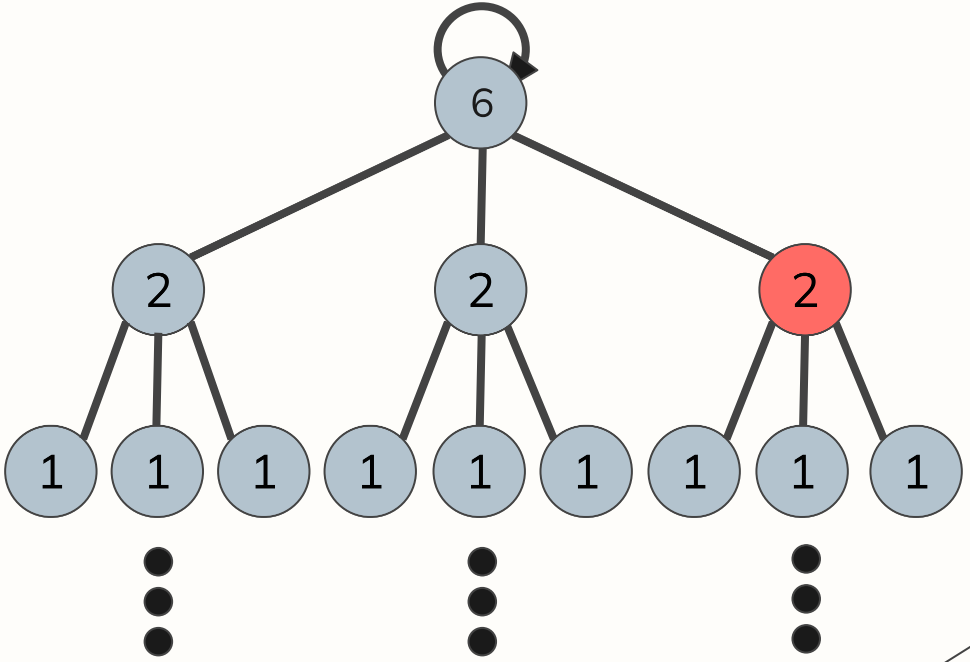

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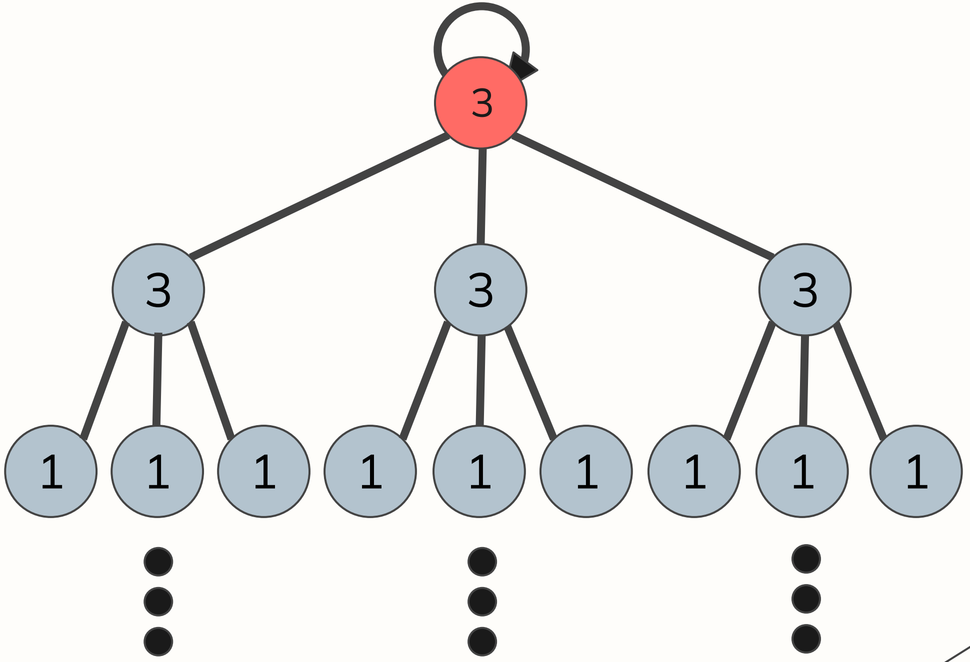
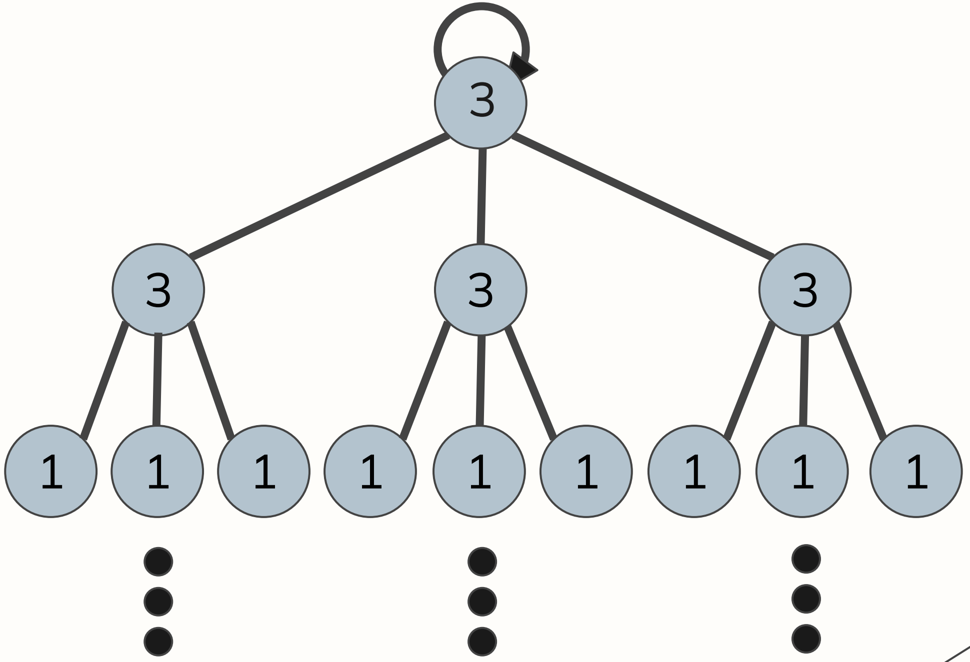


Achieving the Stable Configuration

We have now reached the stable configuration!



of chips per layer in stable configuration

Proposition 4. *If we start with N chips at the root, where $\frac{k^n-1}{k-1} \leq N \leq \frac{k^{n+1}-k}{k-1}$, then for $0 \leq i \leq n-1$, the resulting stable configuration has $a_i + 1$ chips on each vertex on layer $i+1$, where $a_{n-1} \ldots a_2 a_1 a_0$ is the base k expansion of $N - \frac{k^n-1}{k-1}$ with possible leading zeros.*

- Each vertex has between 1 and *k* chips
- Each vertex in the same layer has the same number of chips
- Very similar to base *k*
- In the example, $N - \frac{k^n-1}{k-1}$=8, which is 022 in base 3, and the number of chips in each layer was 1, 3, 3, satisfying the proposition

04

Number of times each vertex fires

of times each vertex fires

Theorem 6. *Given the total number of chips N and the index $n = \lfloor \log_k(N(k-1)+1) \rfloor$, the number of fires for each vertex on layer $i+1$ is:*

$$f_i(N,k) = \sum_{j=1}^{n-i-1} \left(\frac{k^j - 1}{k-1} \right) c_{i+j}(N,k).$$

We let $c_i(N,k)$ be the number of chips on a vertex in layer *i+1* in the stable configuration

The layer number of a vertex is one more than its distance to the root, in particular the layer number of the root is 1.

of times each vertex fires example

- In the previous example of 21 chips at the root of a ternary tree, the root fires 7 times and each of its children fire once
- The formula gives the number of fires for layer 1 should be

$$f_0(21,3) = \sum_{j=1}^{2} \left(\frac{3^j - 1}{3 - 1}\right) c_j(21,3) = 7$$

- And the number of fires for layer 2 should be

$$f_1(21,3) = \sum_{j=1}^{1} \left(\frac{3^j - 1}{3 - 1}\right) c_j(21,3) = 1$$

As expected.

Total number of fires

- We have the recursive formula: $$F(N) = f_0(N) + kF\left(\left\lceil \frac{N}{k} \right\rceil - 1\right).$$
- Total number of fires is equal to the sum of number of root fires and the sum of the fires of each of the *k* subtrees
- Each subtree behaves like the original tree with $\left\lceil \frac{N}{k} \right\rceil$-1 chips

05

Sequences from chip-firing

Total number of fires

- For N in $\{mk+1, mk+2, \ldots, (m+1)k\}$, the same vertex for the tree fires the same number of times as the root will have 1 to k chips and the number of fires will stay the same.
- The table shows the total number of fires given m and k.

$k\backslash m$	1	2	3	4	5	6	7	8	9	10
2	0	1	2	6	7	11	12	23	24	28
3	0	1	2	3	8	9	10	15	16	17
4	0	1	2	3	4	10	11	12	13	19
5	0	1	2	3	4	5	12	13	14	15
6	0	1	2	3	4	5	6	14	15	16

Difference table for total number of fires

On the right is a table of the difference between consecutive terms from the total number of fires table we showed before.

$k\backslash m$	1	2	3	4	5	6	7	8	9	10
2	1	1	4	1	4	1	11	1	4	1
3	1	1	1	5	1	1	5	1	1	5
4	1	1	1	1	6	1	1	1	6	1
5	1	1	1	1	1	7	1	1	1	1
6	1	1	1	1	1	1	8	1	1	1

Tables for the unique values

- The table on the right shows the unique values from the previous table for consecutive differences between the number of total fires.
- Each of these sequences are in the OEIS. Notably, when $k = 2$ the sequence are the Eulerian numbers.
- All these sequences have the same recursive formula:

$$a(n,k) = k \cdot a(n-1,k) + n$$

- In base 10, the numbers look like the numbers from 1 to n concatenated, but this pattern fails after the 9th term.

$k \backslash m$	1	2	3	4	5	6	7	A#
2	1	4	11	26	57	120	247	A000295
3	1	5	18	58	179	543	1636	A000340
4	1	6	27	112	453	1818	7279	A014825
5	1	7	38	194	975	4881	24412	A014827
6	1	8	51	310	1865	11196	67183	A014829
7	1	9	66	466	3267	22875	160132	A014830
8	1	10	83	668	5349	42798	342391	A014831
9	1	11	102	922	8303	74733	672604	A014832
10	1	12	123	1234	12345	123456	1234567	A014824

06

Schizophrenic Numbers

Schizophrenic Numbers

- A **schizophrenic number** is an irrational number that has properties similar to rational numbers.
- This is described in László Tóth's paper on schizophrenic numbers.
- Taking the square root of numbers in our table results in schizophrenic Numbers.
- For example, consider $a(11, 10) = 12345679011$. Then, the square root of this is:
  $111111.111105055555555390541666657673409721609556592835198056921\ldots$
- There are large sequences of repeating digits, a characteristic of schizophrenic numbers.
- A comment on OEIS by Peter Bala claimed that the inverse of schizophrenic numbers also have patterns similar to schizophrenic numbers.

More examples of schizophrenic numbers

- When we go to larger terms we can see the large patterns of schizophrenic numbers.
- For example, when taking the square root of $a(21, 10)$, the square root is:

$$11111111111.111111111000555555555555555555005541666666666666661 1\ldots$$

- We can also see that the lengths of consecutive blocks of same digits are decreasing and after a while the digits seem to be random again.
- Taking the reciprocal of the square root of $a(21, 10)$, we get the number:

$$0.00000000009000000000000000000008955000000000000000133653375000000\ldots$$

- This also has similar properties as schizophrenic numbers, which gives motivation for Peter Bala's comment.

Schizophrenic numbers in different bases

- Note when we take the square root of the terms in the other rows of our table (i.e. when $k \neq 10$), we do not get schizophrenic numbers, but when we put them in base *k*, we get schizophrenic numbers.
- László Tóth's paper showed this table of taking the square root of the numbers on the fifth row in our unique values table and putting them in base 5. Here, $f_5(n)$ is $a(n, 5)$ in our tables.

Table 1: Growing schizophrenic patterns in the base-5 expansion of the numbers $\sqrt{f_5(n)}$

n	$\sqrt{f_5(n)}$
7	$1111.1102030301340212321423323443031320022421310240_5$
9	$11111.1110103030301002441003022433343203043024414 12_5$
11	$111111.11110003030303000302132433034013044313334032_5$
13	$1.1111111111044030303030244012441021320101332242102_5 \times 5^6$
15	$1.111111111111043030303030302424102132441020133100 2_5 \times 5^7$
17	$1.1111111111111104203030303030302412124410213244033_5 \times 5^8$
19	$1.1111111111111111041030303030303030234430213244102_5 \times 5^9$
21	$1.1111111111111111110400303030303030303023310244102_5 \times 5^{10}$
23	$1.1111111111111111111103403030303030303030302311402_5 \times 5^{11}$

Acknowledgements

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Special Thanks to:

Our Friends and Family,
Especially our Parents.

Works Cited

Per Bak, Chao Tang, and Kurt Wiesenfeld, Self-Organized Criticality: An Explanation of 1/f Noise, Phys. Rev. Lett., 59(4), (1987): 381–384.

Anders Björner, László Lovász, and Peter Shor, Chip-firing games on graphs, European J. Combin. 12.4 (1991): 283–291.

Ryota Inagaki, Tanya Khovanova, and Austin Luo, On Chip-firing on Undirected Binary Tree, arXiv math.CO: 2410.00039, 2024.

Caroline J. Klivans, The Mathematics of Chip-firing. (Discrete Mathematics and its Applications), CRC Press, Boca Raton, FL, 2019.

László Tóth, On Schizophrenic Patterns in b-ary Expansions of Some Irrational Numbers, Proc. Amer. Math. Soc. 148 (2020), 461–469.

Any Questions?

Thank you!