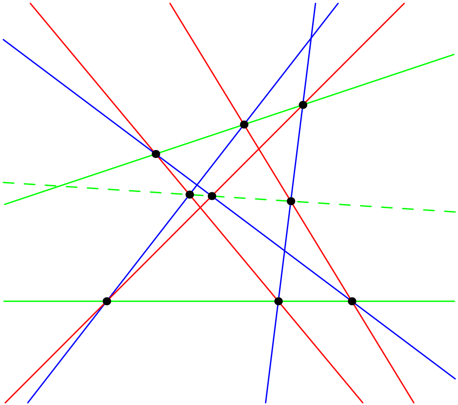
Cubics in Euclidean Geometry

Evin Liang

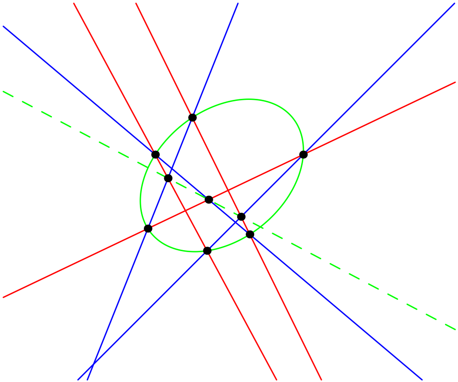
PRIMES Conference

May 17, 2025

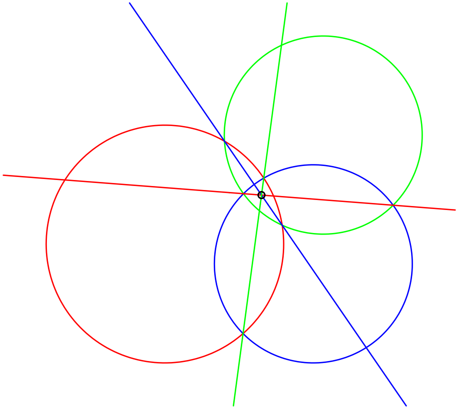
Pappus's theorem



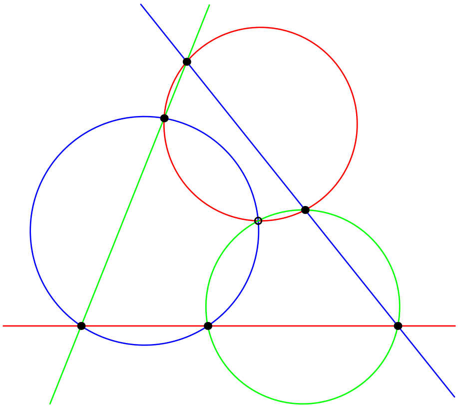
Pascal's theorem



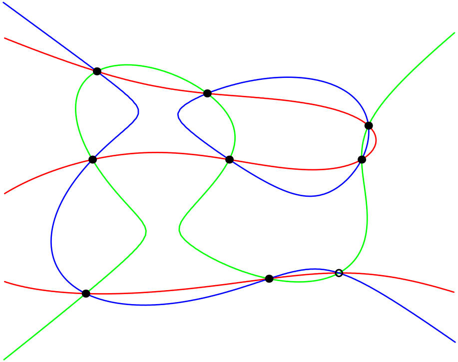
Radical axis theorem



Miquel's theorem

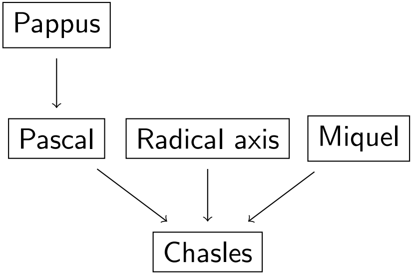


Chasles's theorem



Reflection

These theorems all generalize each other.



Cubics

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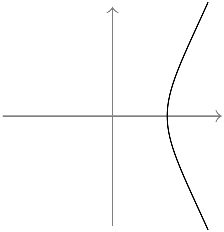
What happens if we consider cubic equations?

Cubic examples



$$y^2 = x^3 - 1$$

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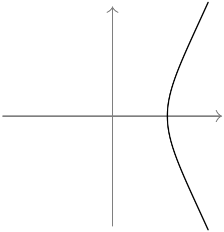

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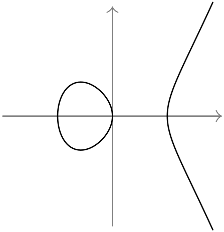

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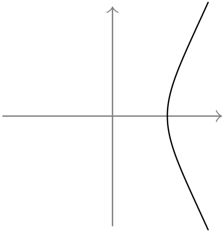

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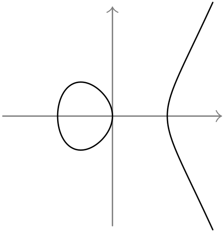

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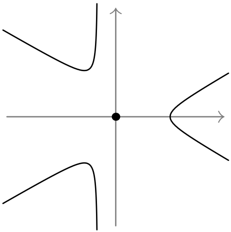
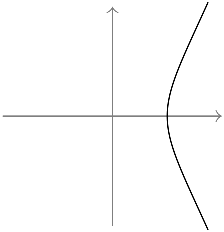


$x^3 = x^2 + y^2(3x + 1)$

Cubic examples



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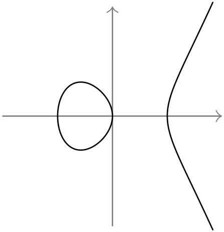

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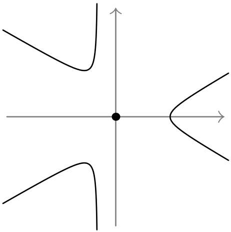

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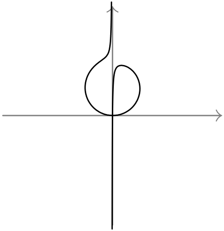
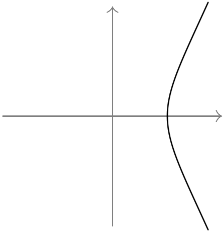


$x(x^2 + y^2) = \frac{x^2 - y^2}{100} + xy$

Cubic examples



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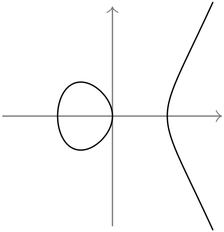

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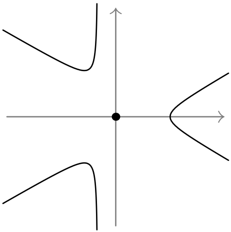

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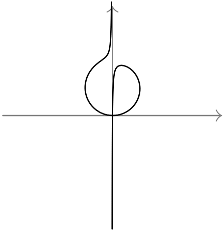

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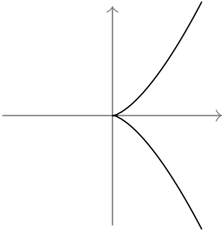

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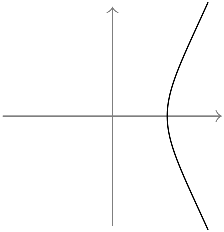

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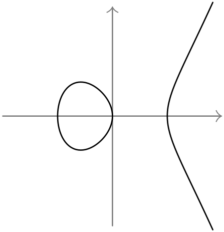

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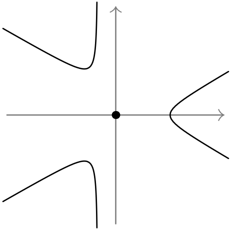

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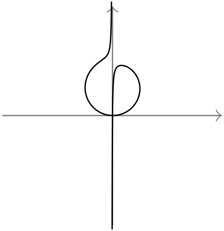

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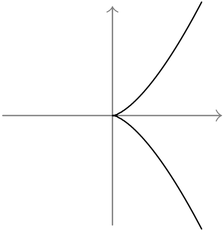

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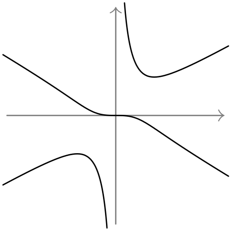


$x(3y^2 - x^2) = y$

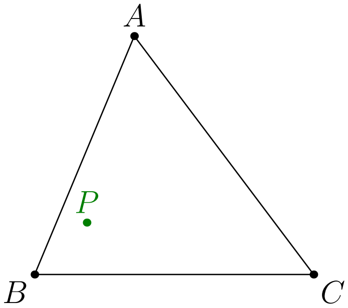
Liang-Zelich theorem

The Euler pencil is a family of important cubics in a triangle.

Liang-Zelich theorem

The Euler pencil is a family of important cubics in a triangle. The Liang-Zelich theorem gives four equivalent descriptions of these cubics.

Example



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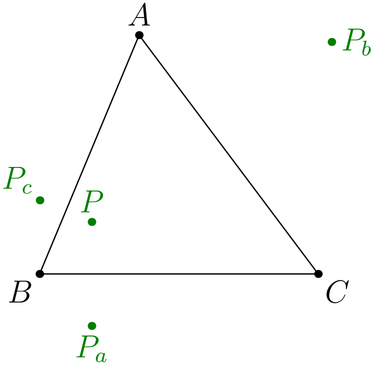

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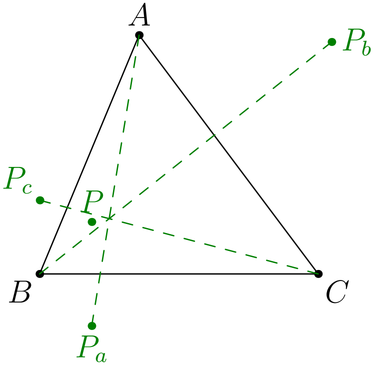

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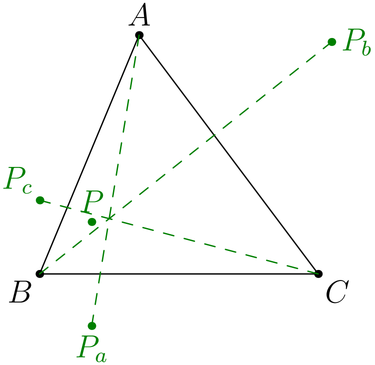

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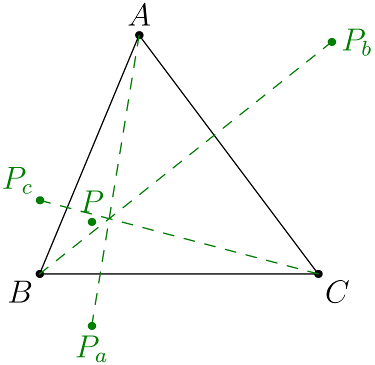

Do these lines always concur?
If not, when do they concur?

The Neuberg cubic

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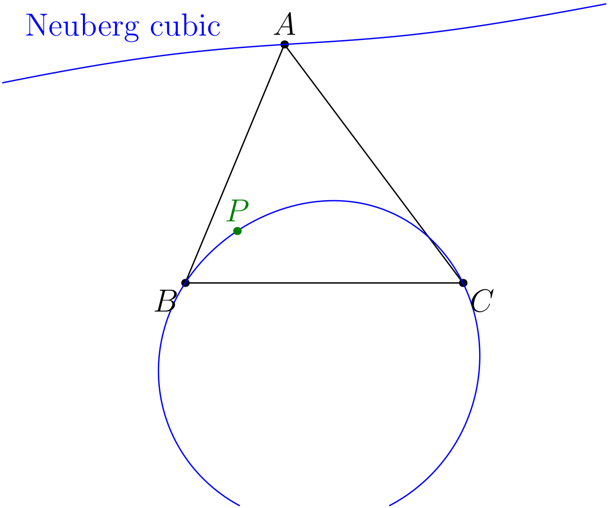
The Neuberg cubic

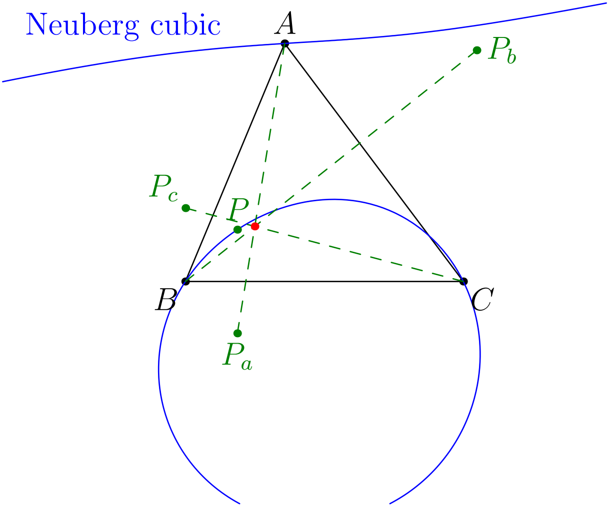
Do these lines always concur? No

If not, when do they concur? When P lies on the the Neuberg cubic



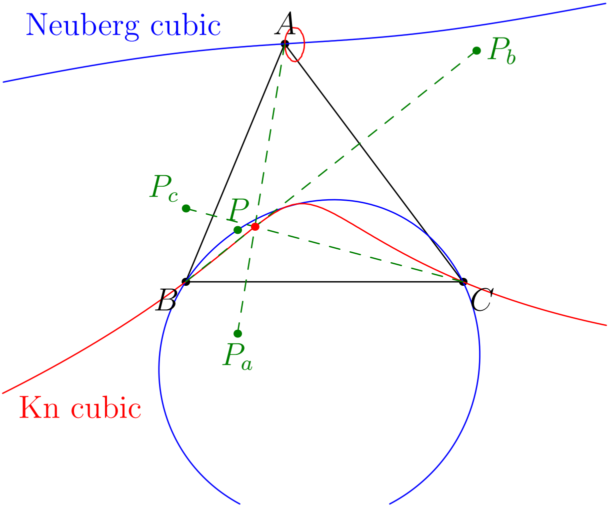
Research

What is the locus of the concurrence point?



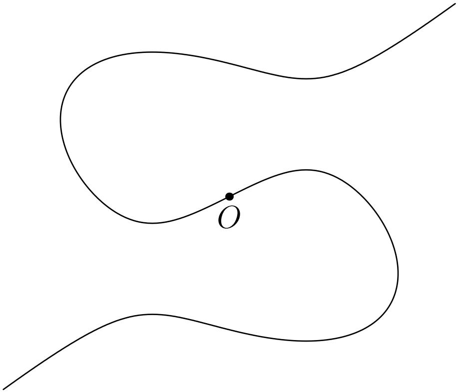
Research

What is the locus of the concurrence point?
It's also a cubic!



Adding points

We can add points on a cubic!



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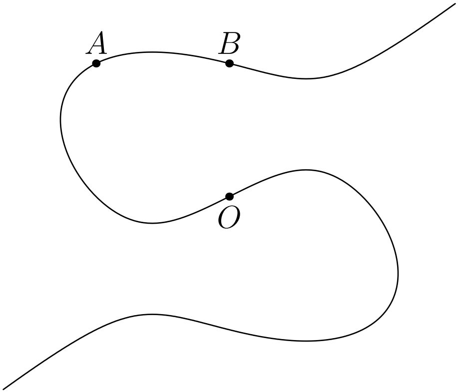

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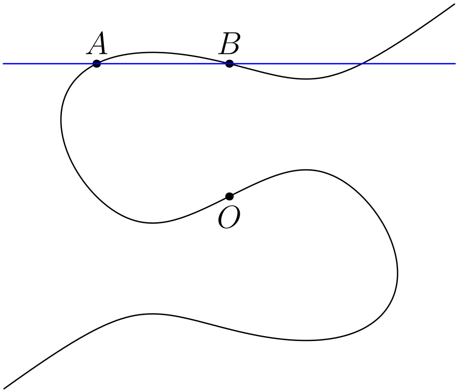

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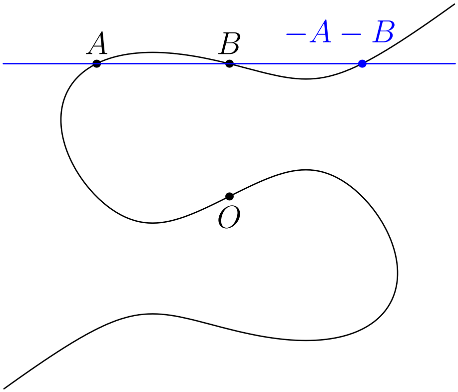

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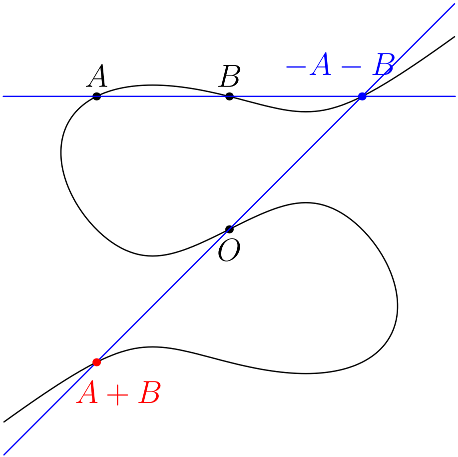
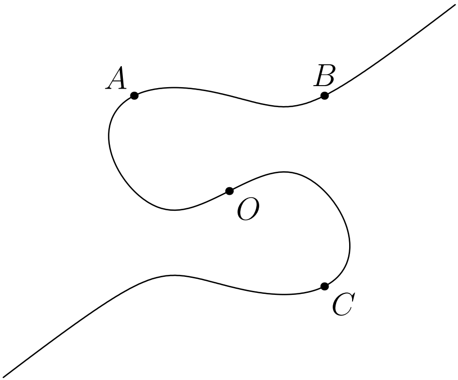


Associativity

Is it associative? Does $A + (B + C) = (A + B) + C$ hold?



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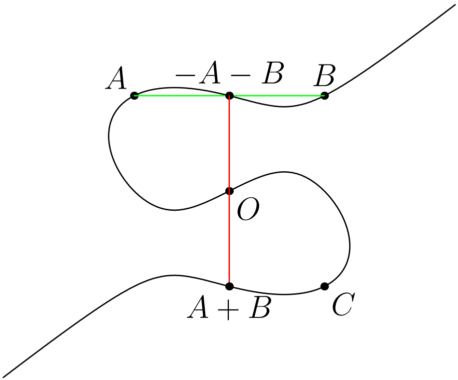

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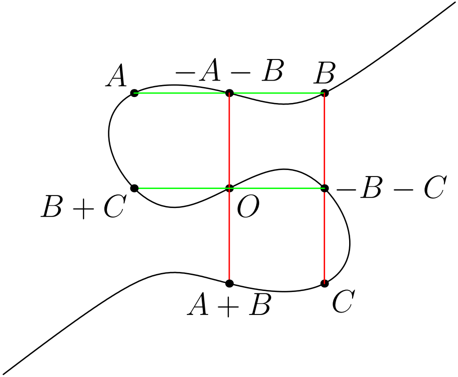

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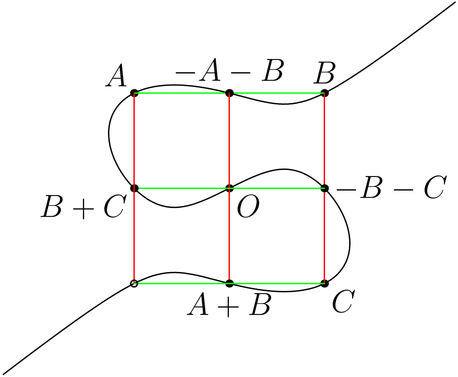

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This theorem goes beyond geometry; it belongs to algebraic geometry.

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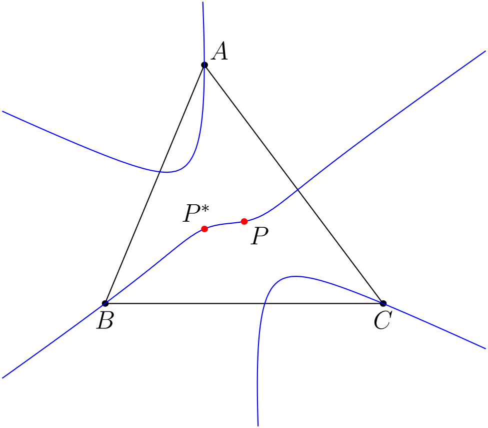
Determined by two points: pivot P, isopivot P^*



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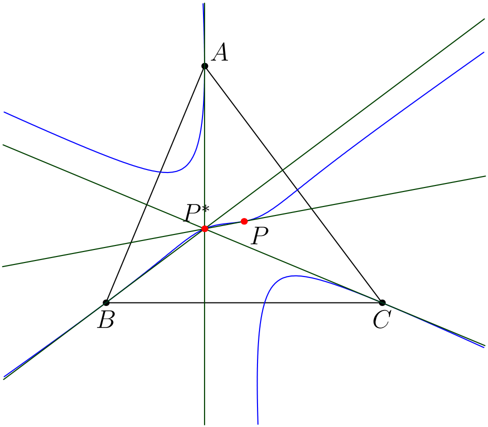

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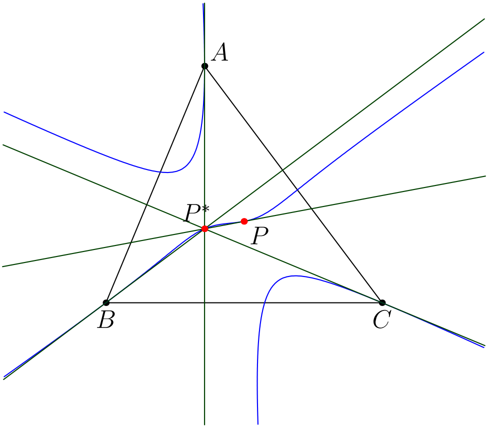
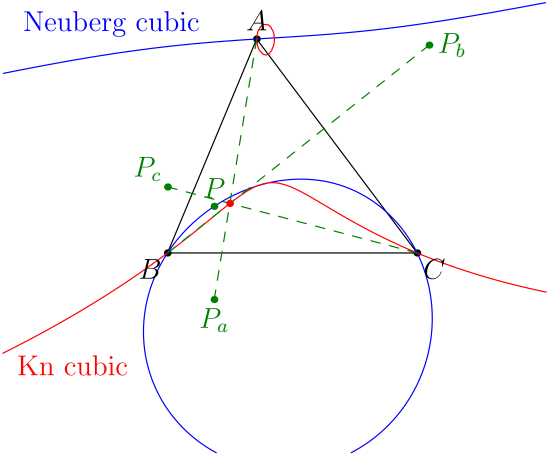
$$2A = 2B = 2C = 2P = -P^*$$



The locus

Recall that we have a map from a cubic to another curve.



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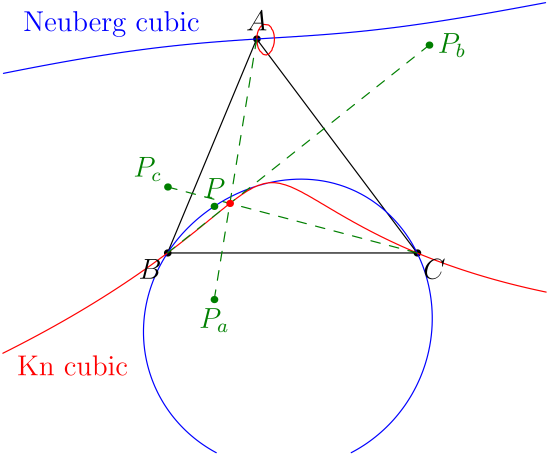
We can show that it is bijective and the image is a cubic.



The locus

Recall that we have a map from a cubic to another curve.
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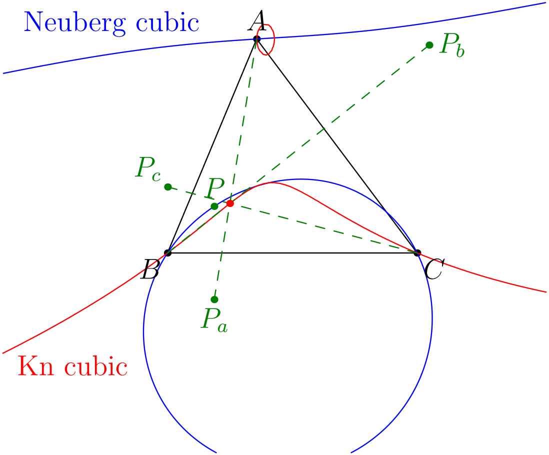
Now the fun begins!



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Theorem

The image of an isopivotal cubic in $\triangle ABC$ with pivot P is an isopivotal cubic in $\triangle f(A)f(B)f(C)$ with pivot $f(P)$.

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Lemma

If X, Y, and Z are collinear points on the image, then $f^{-1}(X) + f^{-1}(Y) + f^{-1}(Z)$ is constant.

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Lemma

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Theorem (L., 2025)

The image of the Euler cubic with pivot P is the isopivotal cubic in $\triangle ABC$ with pivot $f(P)$ and the orthocenter as isopivot.

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Acknowledgments

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I would like to thank my mentor Tanya Khovanova for her support.
Finally, thanks to all of you for listening to my presentation!

References

- Jean-Pierre Ehrmann and Bernard Gibert. *Special Isocubics in the Triangle Plane*. 2015.
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