Dynamical Systems and Chaos Theory

PRIMES Circle 2025

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May 18, 2025

Introduction

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- The study of dynamical systems possesses significant applications in finance, through interest accumulation over time, and ecology through the iteration of the logistic function to model the evolution of a population over time.

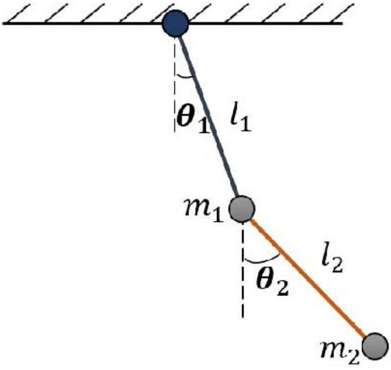

Figure: Double Pendulum is a Dynamical System

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$$
\begin{aligned}
x_0 &= 0 \\
f(x_0) &= x_1 \\
f(x_1) &= x_2 = f(f(x_0)) \\
f(x_2) &= x_3 = f(f(f(x_0)))
\end{aligned}
$$

Attracting, Repelling, and Neutral Fixed Points

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- **Neutral**: If $|f'(x)| = 1$, then the fixed point is *neutral*.

What is an Orbit?

Mathematically, for a function $f : X \to X$, the **orbit** of a point $x_0 \in X$ is the set:

$$\mathcal{A}(x_0) = \{x_0, f(x_0), f(f(x_0)), f^3(x_0), \dots\}$$

Through the iteration of the function, the behavior of the function can be analyzed to make predictions.
Let's look at these definitions in action!

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Example

We will investigate the basic function $f(x) = x^2$. The beauty in chaos showcases how very small changes in the initial conditions (x_0) drastically alters the behavior of the function.

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Starting at $x_0 = 0.99$
$0.99 \to 0.9801 \to 0.96 \to 0.92... \to 0$

Repelling Point

Starting at $x_0 = 1.01$

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Starting at $x_0 = 1.01$
$1.01 \to 1.0201 \to 1.04 \to 1.08... \to \infty$

Graphical Interpretation of Iteration

- Through the iteration of the function and our understanding of the point's behavior, we are able to make predictions about the location of the point after r iterations.

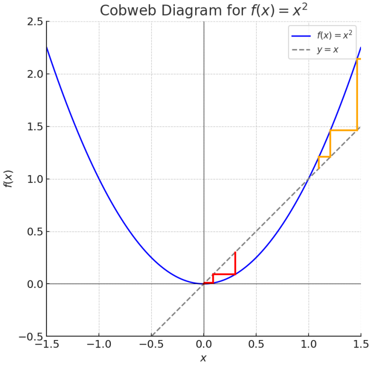

Figure: The red line showcases the behavior of the attracting point while the yellow shows the behavior of the repelling point.

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- $f'(x) = 2x$ so, if $x \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ it is attracting since then $f'(x) < 1$. However if $x \in \left(-\infty, -\frac{1}{2}\right) \cup \left(\frac{1}{2}, \infty\right)$ it is repelling since $f'(x) > 1$.

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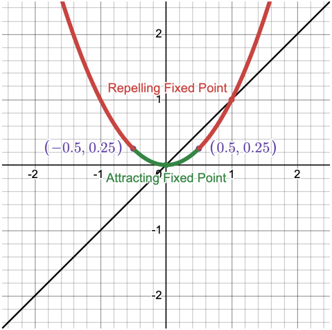


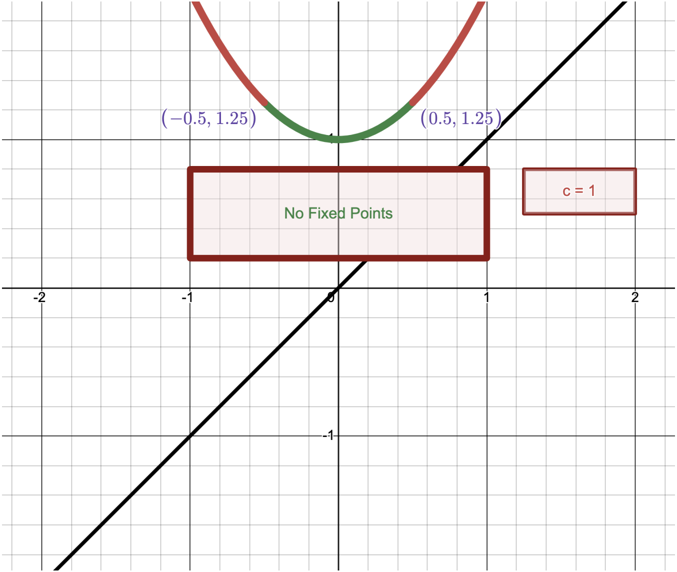
Graph of x^2

We can color our graph so that fixed points in the green are attracting and fixed points in the red are repelling.

Furthermore, as discussed previously we can graph $x = y$ to determine the fixed points.

- Fixed points in the red are repelling
- Fixed points in the green are attracting

The Quadratic Family



Fixed Points Behavior Summary

$c > \frac{1}{4}$	no fixed points
$c = \frac{1}{4}$	one neutral fixed point
$-\frac{3}{4} < c < \frac{1}{4}$	p_+ is repelling; p_- is attracting
$c = -\frac{3}{4}$	p_+ is repelling; p_- is neutral
$c < -\frac{3}{4}$	two repelling fixed points

Now that we have an understanding of what happens to the fixed points when $c > -\frac{3}{4}$, we can investigate what happens for even smaller values of c.

Period 2 Cycles

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Iterations of f(x) for x=0.5

$0.5 \rightarrow -0.75 \rightarrow -0.4375 \rightarrow -0.8086 \rightarrow -0.3452 \rightarrow \cdots \rightarrow -0.99999 \rightarrow 0.00001 \rightarrow -1 \rightarrow 0 \rightarrow -1 \rightarrow 0 \rightarrow \cdots$.

A Chaotic Function

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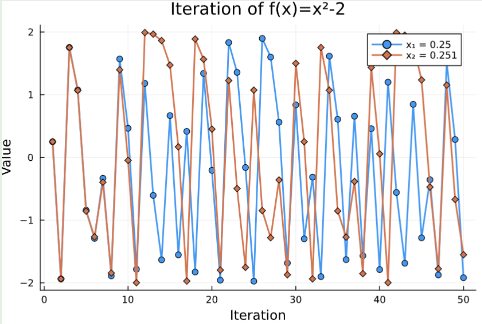
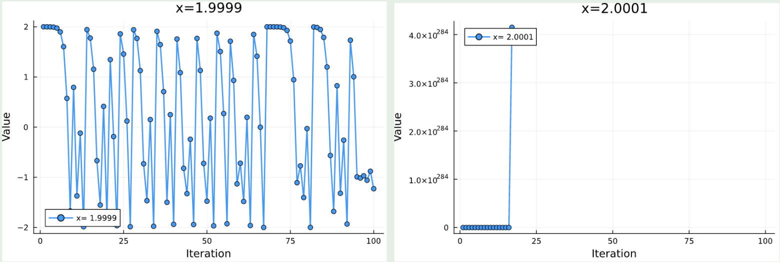
$x = 0.25$ and $x = 0.251$ for $f(x) = x^2 - 2$



A Chaotic Function

$x = 1.9999$ vs. $x = 2.0001$ for $f(x) = x^2 - 2$



Definition

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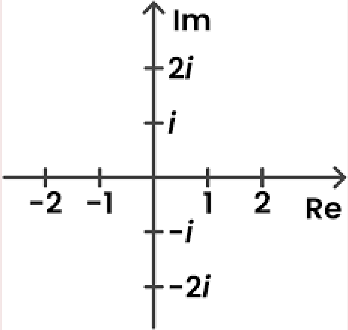
The complex plane (https://www.houseofmath.com/encyclopedia/numbers-and-quantities/numbers/complex-numbers/introduction/what-does-the-complex-plane-mean)



The Filled Julia Set

Definition

Consider some complex number w and some complex function $f(z) = z^2 + c$. We say that the orbit of w on f remains bounded if there exists some M such that $f^n(w) < M$ for all n.

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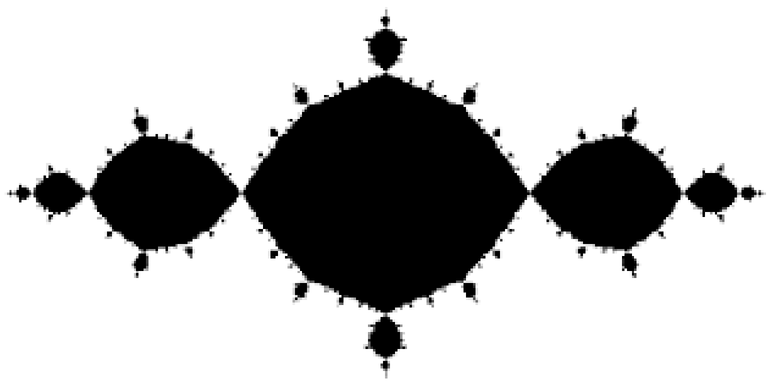
Properties of K_c

- K_c is connected if the orbit of 0 is bounded and it is totally disconnected if not.
- K_c is quasi self-similar, meaning slightly modified copies of the set are found in other parts of the set, just scaled and shifted.

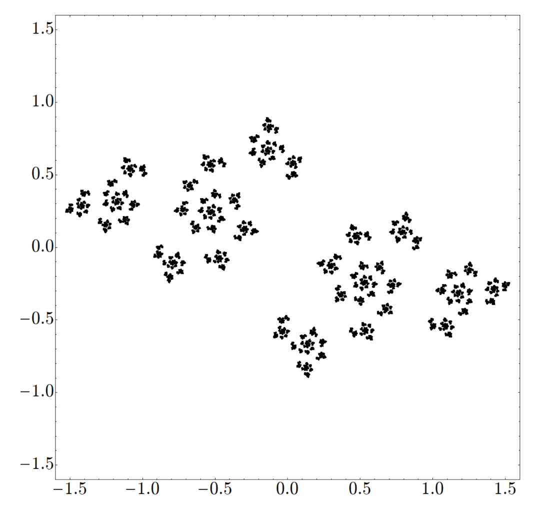
The Filled Julia Set Visualized (Connected)

The filled Julia Set for $f(z) = z^2 - 1$ (https://www.researchgate.net/figure/The-filled-Julia-set-of-f-z-z-2-1_fig1_311926148)



The Filled Julia Set Visualized (Totally disconnected)

The filled Julia Set for $f(z) = z^2 - 0.75 + 0.25i$ (https://e.math.cornell.edu/people/belk/dynamicalsystems/NotesJuliaMandelbrot.pdf)



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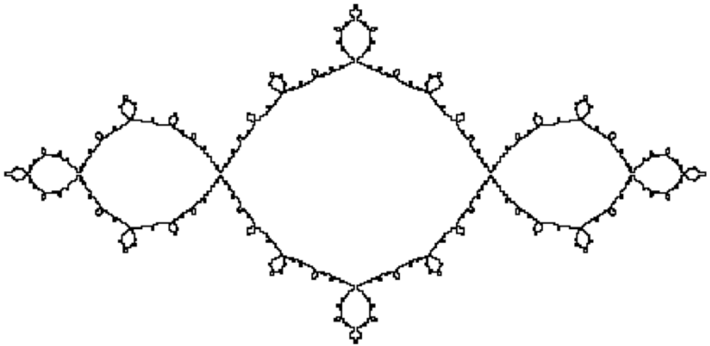

The Mandelbrot Set

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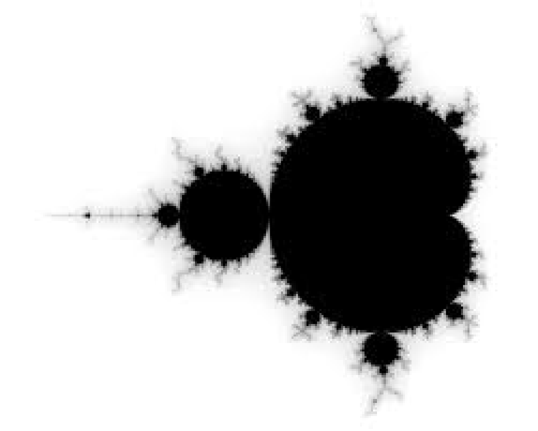
Visualization of the Mandelbrot Set (https://paulbourke.net/fractals/mandelbrot/)



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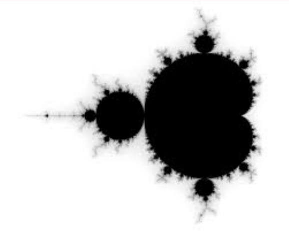


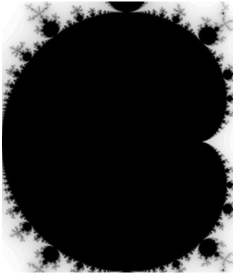
Components of the Mandelbrot Set

- The largest section of the set is the *main cardioid*
- The circle to the right of the main cardioid is the *period 2 bulb*

The Main Cardioid

Definition

The main cardioid contains all values of c that give $z^2 + c$ an attracting fixed point. Additionally, the boundary of the region are the values of c that give $z^2 + c$ a neutral fixed point.



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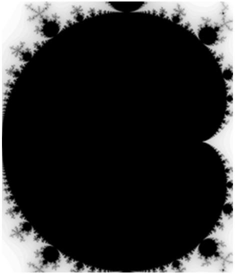


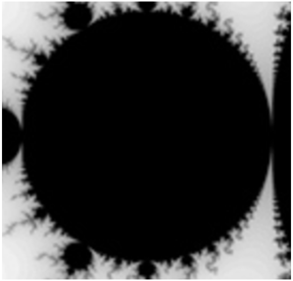
Computing the region

Our definitions of fixed points and attraction can be used to compute the region as $|1 \pm \sqrt{1-4c}| < 1$. This inequality represents the main cardioid.

The Period 2 Bulb

Definition

The period 2 bulb is the set of all values of c which give f an attracting period 2 cycle.



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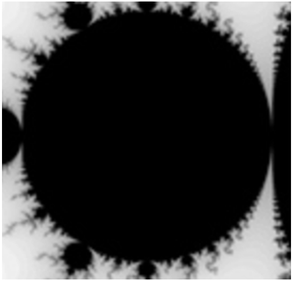
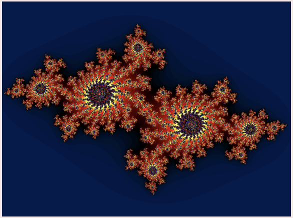


Computing the region

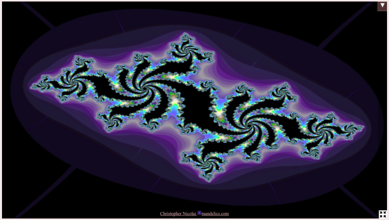
The region can be computed similarly to yield $|c+1| < \frac{1}{4}$, a circle centered at -1 with radius $\frac{1}{4}$.

Some Pretty Julia Sets

https://www.mcgoodwin.net/julia/juliajewels.html,
https://mandelics.com/photo/realtime-general-julia.html



$c=0.687 + 0.312i$



$c=-0.6078 + 0.438i$

Any questions?