Periodicity of S-Pick-Up-Bricks

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May 18, 2025

What is game theory?

Definition (Game Theory)

Game theory is the study of interactions between different "players."

Definition (Combinatorial Game Theory)

Combinatorial game theory is the study of "luckless" interactions.

Game theory is used extensively in economics and political science.

Pick-Up-Bricks

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Pick-Up-Bricks is an *impartial* game, as the set of moves available to each player is the same. Chess is not an impartial game because the players can have different possible moves (one player can move each color).

Mock Game

Solving Pick-Up-Bricks

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I did not win the previous game by luck. For one of the players, there always exists a simple strategy to win Pick-Up-Bricks.

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1. When b is a multiple of 3, the second player can match the first player such that together they remove 3 bricks every 2 turns. Then the second player will remove the last brick.
2. Otherwise, the first player can remove bricks so the number of remaining bricks is divisible by 3. Now this reduces to the first case.

The Sprague-Grundy Theorem

Theorem (Sprague-Grundy (informal))

Every position in an impartial game can be assigned a non-negative integer called a "nimber." Positions with the same nimber are strategically "equivalent." A position with no moves is equivalent to the nimber 0.

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Example

The nimber corresponding to the position with n bricks in Pick-Up-Bricks is $n \mod 3$. So every position with $n \equiv 0 \mod 3$ is equivalent to the nimber 0.

Definition (MEX of a set)

We define the MEX (Minimal EXcluded Value) of a set to be the smallest nonnegative integer not included in the set. For example, for the set $S = \{0, 1, 2, 3, 5, 10\}$, the MEX is 4, because 4 is the smallest nonnegative integer not in the set. Note that the MEX of a set can equal 0 if all numbers in the set are greater than 0.

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Theorem (MEX Principle)

The nimber of a position is equal to the MEX of the nimbers of the set of future positions that can be reached in one move.

All positions in impartial games with no moves are equivalent to the nimber 0, because $\mathrm{MEX}\{\} = 0$, i.e., the minimum excluded nonnegative integer of an empty set is just 0.

MEX on Pick-Up-Bricks

We can apply the MEX principle in a normal game of Pick-Up-Bricks:

Bricks	0	1	2	3	4	5
Nimber	0	1	2	0	1	2

As we can see, by repeatedly applying the MEX principle, every position where the number of bricks is divisible by 3 is a winning one for the second player.

Definition (S-Pick-Up-Bricks)

The game of S-Pick-Up-Bricks is a generalization of normal Pick-Up-Bricks: players can remove any amount of bricks as long as that amount is in S.

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- A normal game of Pick-Up-Bricks is just a game of S-Pick-Up bricks where $S = \{1, 2\}$.
- In a game S-Pick-Up-Bricks where $S = \{2, 3\}$, a player can remove either 2 or 3 bricks from the pile.

Example nimber sequence: $S = \{2, 3\}$

Bricks	0	1	2	3	4	5	6	7	8	9
Nimber	0									

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Nimber Sequence

Definition (Nimber Sequence)

The *nimber sequence* for S-Pick-Up-Bricks is the infinite sequence of nimbers equivalent to the positions with $0, 1, 2, \ldots, \infty$ bricks.

Here are some example nimber sequences:

- As shown in previous slides, the nimber sequence for $S = \{2, 3\}$ is $0, 0, 1, 1, 2, 0, 0, 1, 1, 2, \ldots$, corresponding to positions with $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, \ldots$ bricks.
- The nimber sequence for normal Pick-Up-Bricks is $0, 1, 2, 0, 1, 2, \ldots$

Main Result

Theorem

For any finite set S of positive integers, the nimber sequence for S-Pick-Up-Bricks is eventually periodic.

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Proof idea

Suppose that the largest number in S is a, then:

1. If there are two identical sections of the nimber sequence of length a, then the sequence will repeat, because to determine a certain nimber in the nimber sequence, only the previus a nimbers in the sequence matter.
2. The maximum value of a MEX (when computing a nimber in the nimber sequence) is $|S|$ (the size of S). This means that all nimbers from 0 to n are possible values in the nimber sequence, so there is a maximum of $(|S| + 1)^a$ different blocks of length a in the nimber sequence.

Main Result

Theorem

For any finite set S of positive integers, the sequence of nimbers for S-Pick-Up-Bricks is eventually periodic.

Proof idea

Since there are a maximum of $(|S|+1)^a$ distinct blocks of length a in the nimber sequence, after $(|S|+1)^a+1$ blocks of length a, there must be two blocks of length a that are the same. By our first claim, this implies that the nimber sequence is periodic.

Consequences of the Main Result

Definition (Immediate Periodicity)

A nimber sequence is *immediately periodic* if its repeating portion starts at the first term in the sequence.

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$S = \{2, 4, 7\}$ is an example where the nimber sequence is not immediately periodic. Its nimber sequence begins as:

$$0, 0, 1, 1, 2, 2, 0, 3, 1, 0, 2, 1, 0, 2, \ldots.$$

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Other Results

1. If $S = \{a, b\}$, the period of S-Pick-Up-Bricks is $a + b$, unless b is an odd multiple of a, in which case the period is $2a$.
2. If $S = \{p, p + 1, ..., q - 1, q\}$, then the period of the S-Pick-Up-Bricks game is $p + q$
3. If S only contains odd integers, the period is 2.

Further ideas

Some further ideas we have about this topic are:

- **Question:** Is there a better way to find the period than just repeatedly applying the MEX principle?
- **Question:** Can you determine whether a set is not immediately periodic without explicitly computing?
- **Conjecture:** If the largest number in S is l, and if S can be transformed into $\{l\}$ by increasing the smallest number in the set $a \leq \lceil \frac{l}{2} \rceil$ times, the original set has a period of $2l - a$.

For further detail and more proofs, you can check out our paper.

Questions?

Thank you!