Polyhedra and Euler Characteristic

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MIT PRIMES Circle

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Summary

1. Polyhedra and Euler's formula

2. Cellular decomposition and Euler's characteristic

3. Real projective plane $\mathbb{RP}^2$

Polyhedra

Definition

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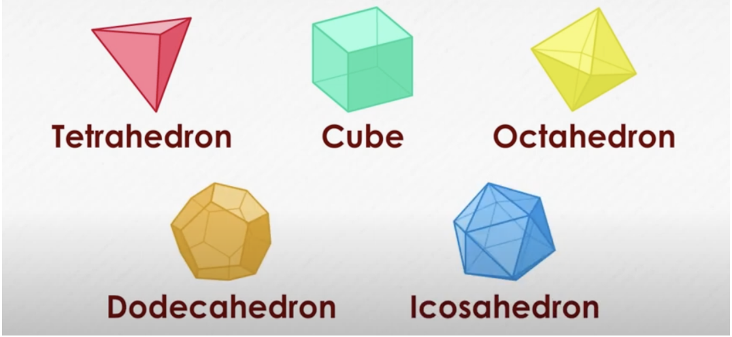

Euler's Formula

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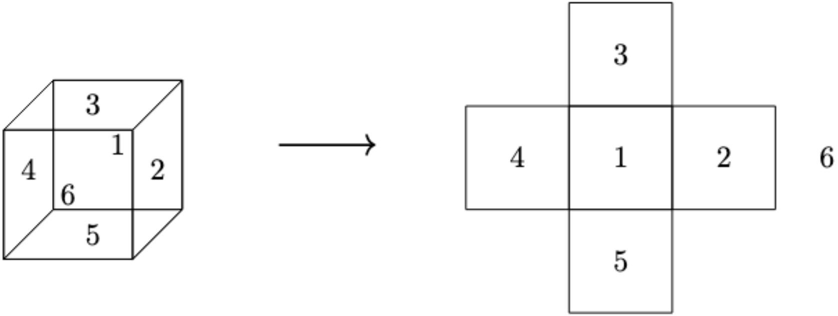
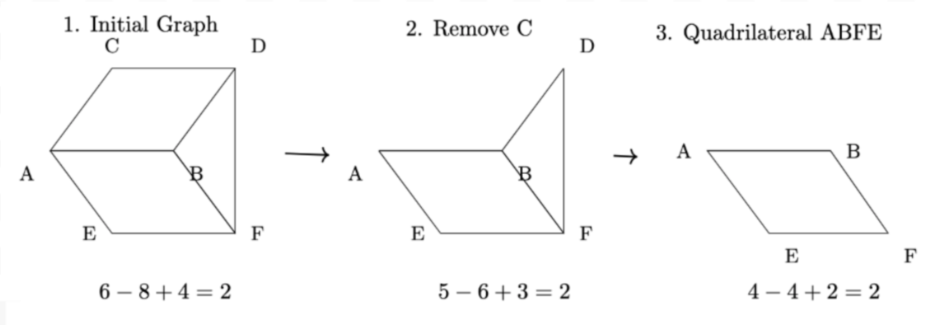


$8 - 12 + 6 = 2 \quad \longrightarrow \quad 8 - 12 + 6 = 2$

Euler's Formula: Proof by Induction



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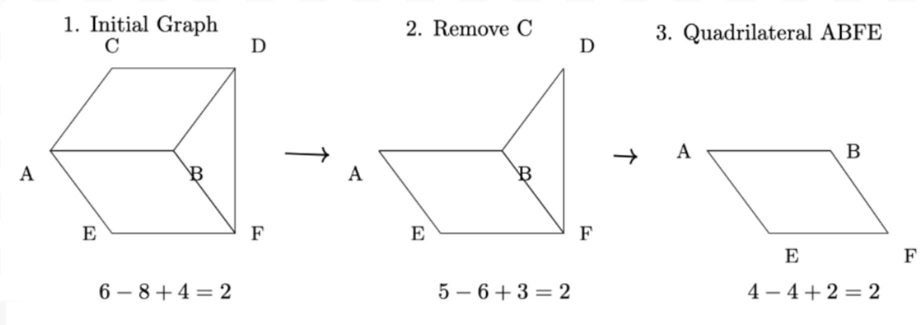

Question

Why is Euler's formula the same for all polyhedra?

Homeomorphism

Definition: homeomorphism

Two spaces A and B are *homeomorphic* if there exists a continuous function $f : A \to B$ with a continuous inverse $f^{-1} : B \to A$. We say that f is a *homeomorphism*.

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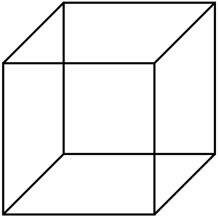
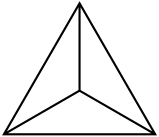
Example 1: cube and tetrahedron

A tetrahedron is homeomorphic to a cube.



Cube

$\cong$



Tetrahedron

Homeomorphism: Example

Example 2: torus

A torus is homeomorphic to a coffee mug.

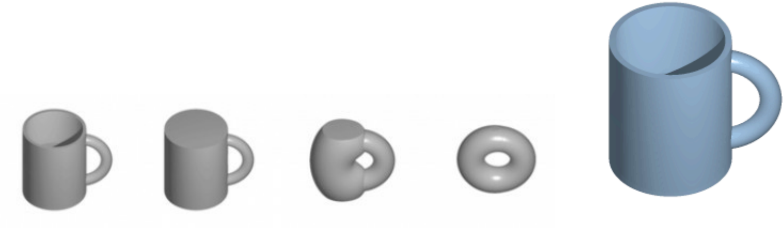


Figure 1: Wikipedia, *Mug and Torus morph*

Cellular Decomposition

Definition

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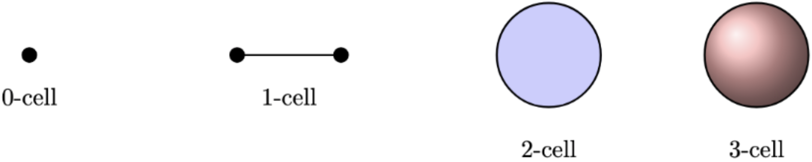

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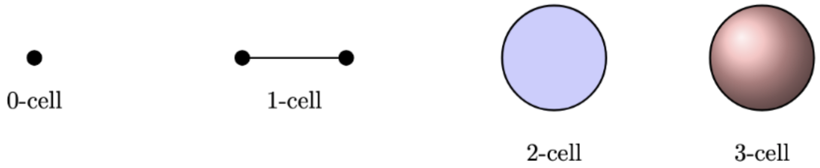

Fact

All reasonable topological spaces admit a cellular decomposition.

Cellular Decomposition: Example

Example: sphere

We can construct a cellular decomposition of the 2-sphere with 0-cells, 1-cells, and 2-cells.

Cell Type	**Quantity**	**Role in Construction**
0-cell	1	Base point (start of attachment)
1-cell	1	Connects and extends from 0-cell
2-cell	2	Forms hemispheres (e.g., upper and lower disks)

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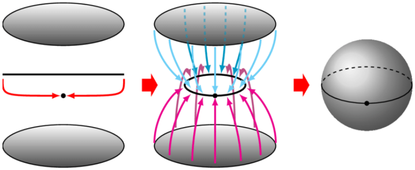

Figure 2: Picture from *Cohomology of Differential Forms and Feynman diagrams*

Euler Characteristic

Definition: Euler characteristic

Given a cellular decomposition for a space X, the *Euler characteristic of* X, denoted by $\chi(X)$, is defined as

$$\chi(X) = (\#\text{ of 0-cells}) - (\#\text{ of 1-cells}) + (\#\text{ of 2-cells}) - \cdots \pm (\#\text{ of } n\text{-cells}).$$

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Fact

- Two homeomorphic spaces have the same Euler characteristic.
- In particular, two distinct cellular decompositions for the same space X give us the same value for $\chi(X)$.

Euler Characteristic: Example

Example: 2-sphere S

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Euler characteristic of S

$\chi(S) = (\text{\# of 0-cells}) - (\text{\# of 1-cells}) + (\text{\# of 2-cells}) = 1 - 1 + 2 = 2.$

Revisiting Euler's Formula

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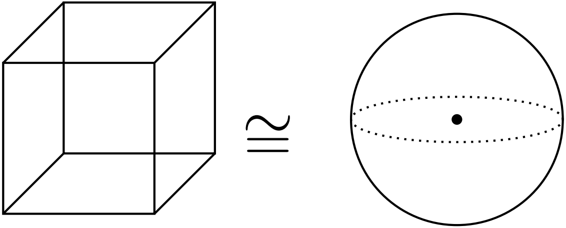
Answer

All polyhedra are homeomorphic to a sphere, which has Euler characteristic 2:

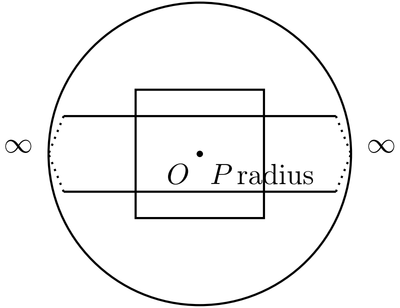
$$V - E + F = (\#\text{ of 0-cells}) - (\#\text{ of 1-cells}) + (\#\text{ of 2-cells}) = \chi(S) = 2$$



polyhedron $\cong$ sphere S

Real Projective Plane $\mathbb{RP}^2$

1. To create $\mathbb{RP}^2$, we start with a set of parallel lines. We can claim that at ∞, they converge.
2. Then, we can make many more sets of parallel lines until their convergences create a circle with radius ∞



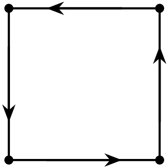
3. Finally, we can connect or glue all opposite convergences to finally create $\mathbb{RP}^2$, which unfortunately cannot be visualized in 3 dimensions. It is a 2-dimensional object living in at least 4 dimensions.

Euler's Characteristic of $\mathbb{RP}^2$

To find the Euler characteristic, we can use cellular decomposition. We will not be using anything above 2-cells to decompose $\mathbb{RP}^2$, so we will disregard all after 2-cells. We get

$$\chi(\mathbb{RP}^2) = (\text{\# of 0-cells}) - (\text{\# of 1-cells}) + (\text{\# of 2-cells}).$$

As we stated in the creation of our $\mathbb{RP}^2$ plane, we had a circle with radius infinity at one point. Because Euler's characteristics are consistent across all homeomorphic objects, we can reduce that circle to a limited square that we will modify. We will fold it, but opposite sides will be oppositely aligned.



Euler's Characteristic of $\mathbb{RP}^2$

When we fold them with the correct alignments, we get two 0-cells (vertices), two 1-cells (edges), and one 2-cell (the face). Therefore, from our prior equation, we get the following.

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Euler characteristic of $\mathbb{RP}^2$

$\chi(\mathbb{RP}^2) = (\#\text{ of 0-cells}) - (\#\text{ of 1-cells}) + (\#\text{ of 2-cells}) = 2 - 2 + 1 = 1.$

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Remarks

- We have computed $\chi(S) = 2$ and $\chi(\mathbb{RP}^2) = 1$, where S is the sphere.
- As an exercise, you can compute that $\chi(T) = 0$, where T is the torus.
- Hence, these three 2-dimensional surfaces are not homeomorphic.

Conclusion

- Covered Euler's formula and showed why it applies to all polyhedra
- Defined and explored homeomorphisms
- Used cellular decomposition to define and compute the Euler characteristic
- Constructed $\mathbb{RP}^2$ and computed its Euler characteristic

References

- *Thinking Geometrically: A Survey of Geometries* (Mathematical Association of America Textbooks) by Thomas Q. Sibley
- Wikipedia, *Mug and Torus morph*
- *Cohomology of Differential Forms and Feynman diagrams*, by Sergio L. Cacciatori, Maria Conti, and Simone Trevisan