DYNAMICAL FUNCTIONALS ON ANCIENT ARF RICCI FLOWS

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Abstract. Motivated by results on the dynamical stability of Ricci-flat metrics, we introduce dynamical energy functionals for compact ancient asymptotically Ricci-flat (ARF) Ricci flows with modest decay defined via pointed solutions to conjugate heat flows. These functionals provide an upper bound for Perelman's λ-functional. Moreover, they satisfy a quantitative version of the steady Ricci breather-type rigidity satisfied by the classical λ-functional by measuring the extent to which an ARF Ricci flow coupled to a conjugate heat flow deviates from being a steady gradient soliton. Such an observation is related to how a priori asymptotics for these functionals imply asymptotics for their first variations. In addition, motivated by the work of Colding and Minicozzi, we derive local eigenvalue estimates for normalized Ricci flows coupled with conjugate heat flows.

1. Introduction

Perelman's functionals have been extensively studied on solutions to the Ricci flow equation defined on compact manifolds. A classical result in this setting asserts that if the λ-functional assumes the same value at two distinct times along a Ricci flow (that is, if the flow is a steady Ricci breather), then the solution in question is a Ricci-flat, steady gradient Ricci soliton between those two times. These functionals have found a myriad of other applications to geometric problems concerning compact 3-manifolds. Their utility has recently catalyzed efforts to relate them to the limiting behavior of the underlying Ricci flow solutions on which they are defined. One such application is the dynamical stability of Ricci-flat metrics. A metric is said to be *dynamically stable* if all sufficiently close metrics converge to it along the Ricci flow. In [Ye93], it is shown that any sufficiently pinched Einstein metric of nonzero scalar curvature is dynamically stable in the C^2-topology. [GIK02] addresses the same question for Ricci-flat metrics, concluding that any metric in a sufficiently small $C^{2,\alpha}$-neighborhood of a Ricci-flat metric converges exponentially quickly to a Ricci-flat metric. Furthermore, [HM14] and [Ses06] connect dynamical stability to *linear stability*, which is related to the stability of Perelman's energy functional at that metric, understood in the classical sense.

Another research direction related to geometric functionals on compact manifolds was initiated by Hein and Naber. In [HN13], they introduce a time-dependent, pointed entropy functional along a solution to the Ricci flow that is defined in terms of Perelman's entropy functional evaluated at a pointed conjugate heat kernel. Lower bounds for the pointed entropy may be converted into local curvature upper bounds along the Ricci flow in question. Analogues of the pointed entropy have found applications to problems concerning classifications of Ricci flows on certain classes of non-compact manifolds, as discussed in [Bam21a], [Bam21b], [CMZ23], [CMZ24], and [LO25].

The principal purpose of this paper is to liaise between the dynamical stability of Ricci-flat metrics and dynamical functionals defined on ancient Ricci flows on compact manifolds converging to Ricci-flat metrics. More precisely, we define dynamical energy functionals on certain classes of such flows in terms of conjugate heat flows. The range of ancient flows on which these functionals are defined is dictated by the Lojasiewicz-Simons inequality of Haslhofer-Müller [HM14, Theorem 1.3], which plays a salient role in proving their dynamical stability results.

1.1. **Main results.** On compact Ricci flows with bounded curvature tending to a Ricci-flat metric at a sufficiently fast rate (as formalized in the proceeding result), we derive a monotone dynamical λ-functional, denoted λ^s_{dyn}, akin to the pointed entropy functional introduced in [HN13], that serves as an upper bound for Perelman's λ-functional. This new functional further provides an alternative method of classifying Ricci breathers along such Ricci flows.

Theorem 1.1. *Suppose that $(M^n, g(t))$ is an ARF Ricci flow of order $\theta \in [\frac{2}{5}, \frac{1}{2})$ on a compact manifold in the sense of Definition 2.1.*

(a) If $f^s : M \times (-\infty, s] \to \mathbb{R}$ is the solution to the conjugate heat flow satisfying the initial condition $f^s(\cdot, s) \equiv 0$, then the sequence of functionals

$$\lambda^s_{\mathrm{dyn}}(t) := \mathcal{F}[g(t), f^s(t)]$$

defined on $(-\infty, s]$ is uniformly bounded. Furthermore, after parabolically rescaling $g(t)$, this functional is bounded from below by Perelman's λ-functional evaluated at $g(t)$.

(b) If there exists a pair of non-positive times (t_1, t_2) satisfying

$$t_1 < t_2 \quad \text{and} \quad \lambda^s_{\mathrm{dyn}}(t_1) = \lambda^s_{\mathrm{dyn}}(t_2),$$

then $(g(t))_{t\in[t_1,t_2]}$ is a Ricci-flat, steady gradient Ricci soliton.

In [HM14], it is proven that the inequality (2.1) holds true with respect to a uniform exponent $\theta \in (0, \frac{1}{2}]$ on a sufficiently small $\mathcal{C}^{2,\alpha}$-neighborhood of any Ricci-flat metric. In particular, any ancient solution to the Ricci flow converging to a Ricci-flat metric g_{RF} satisfies (2.1) on an arbitrarily large interval of the form $(-\infty, T]$ for some $T < 0$. According to [Has12], flows tending to integrable metrics (i.e. Ricci-flat metrics whose infinitesimal Ricci-flat deformations are integrable) such as Calabi-Yau and Kähler metrics satisfy (2.1) with the optimal exponent $\theta = \frac{1}{2}$, which is only true when the flow converges exponentially. In fact, every known example of a compact ancient Ricci flow satisfies (2.1) and (2.2) with respect to the optimal exponent $\theta = \frac{1}{2}$, so our assumption that $\theta \geqslant \frac{2}{5}$ offers some modest leeway.

Our second main result may be interpreted as a quantitative version of Theorem 1.1

Theorem 1.2. *Suppose that $(M^n, g(t))$ is an ancient solution to the Ricci flow equation satisfying all the properties listed in Definition 2.1. If $\lambda^s_{\mathrm{dyn}}(t) = O(|t|^{-k})$ for some $k \geqslant 1$, then the first variation of λ^s_{dyn} satisfies $\frac{d}{dt}\lambda^s_{\mathrm{dyn}}(t) = O(|t|^{\max(-1-k, 1-\beta)})$.*

Aside from our main results, we conclude the paper by establishing local upper bounds for eigenvalues of the drift Laplacian along a Ricci flow coupled with a conjugate heat flow as well as its sharpness for the first eigenvalue. Such a result is motivated by recent work of Colding and Minicozzi, who derived sharp upper bounds for eigenvalues of the drift Laplacian along a modified Ricci flow [CMI24].

Theorem 1.3. *If $(g(t), f(t))$ is a coupled system of Riemannian metrics and smooth functions on a compact manifold M^n satisfying*

$$\begin{cases} \partial_t g = -2\mathrm{Ric} - 2\nabla^2 f \\ \partial_t f = -\mathrm{R} - \Delta f, \end{cases}$$

and $\lambda_k(t)$ denotes the kth eigenvalue of the drift Laplacian for f on $(M^n, g(t), f(t))$ for some $k \geqslant 1$, then we have the local upper bound

$$\lambda_k(t) \leqslant \frac{\lambda_k(t_0)}{1 - 2(t - t_0)\lambda_k(t_0)} \quad \text{for all } t \in \left[t_0, t_0 + \frac{1}{2\lambda_k(t_0)}\right) \tag{1.1}$$

for any t_0 in the interval of definition of the coupled system.

1.2. **Organization of the article.** In Section 2, we establish some curvature estimates for ARF Ricci flows of order $\theta \in [\frac{2}{5}, \frac{1}{2})$ on compact manifolds (cf. Definition 2.1). In Section 3, we use these estimates to prove Theorems 1.1 and 1.2. In Section 4, by adapting the methods of [CMI24] to the setting of Ricci flows coupled to conjugate heat flows, we prove Theorem 1.3 and show that the inequality (1.1) is sharp for the first eigenvalue. We sequester some of the lengthier calculations belying many of the proofs in the appendix.

1.3. **Notation and conventions.** We liberally use the following notational conventions throughout the paper.

- dV_g denotes the volume form computed with respect to a given Riemannian metric g.
- g_e denotes the standard Euclidean metric, and $\nabla^e = \nabla^{g_e}$, $dV_e = dV_{g_e}$, etc.
- Given a function $f(x)$ and a non-negative function $g(x)$, we write $f = O(g)$ to mean that $|f(x)| \leqslant cg$ for some constant $c > 0$ independent of x for sufficiently large x. The variable in consideration will be clear from the context, but it will most often be time.
- Given two tensors A and B, we denote by $A * B$ a tensor derived from the tensor product $A \otimes B$ in the sense of [Top06, Section 2.1]. In particular, $|A*B| = O(|A|\cdot|B|)$.
- Given a function $f \in \mathcal{C}^\infty(M)$, we denote by Ric_f and Δ_f the weighted Ricci curvature tensor $\mathrm{Ric}_f = \mathrm{Ric} + \nabla^2 f$ and the drift Laplacian $\Delta_f = \Delta - \nabla_{\nabla f}$.

We refer the reader to [CLN06, Chapter 2] for all of the relevant evolution equations for geometric quantities along the Ricci flow, many of which we use freely throughout the paper.

1.4. **Acknowledgements.** This paper is the result of an MIT PRIMES project, in which Rio Schillmoeller was mentored by Isaac M. Lopez. Rio Schillmoeller thanks the PRIMES program for making this opportunity possible. The authors also thank Tristan Ozuch for suggesting the project, and for continuously providing invaluable insights and advice.

2. Curvature estimates for ARF Ricci flows

2.1. **Definitions.** The following definition, which should be compared with the estimates in the hypothesis of Theorem 1.1, makes precise the class of Ricci flows we shall be interested in throughout the paper.

Definition 2.1 (Asymptotically Ricci-flat flow)**.** Let $(M^n, g(t))_{t\in(-\infty,S]}$ be an ancient Ricci flow on a compact Riemannian manifold M^n. We say that $g(t)$ is *asymptotically Ricci-flat*

(ARF) of order $\theta \in (0, \frac{1}{2}]$ and regularity $m \in \mathbb{N}_0$ if there exists a Ricci-flat metric g_{RF} and constants $C_k, T_k > 0$ for each $k \in \{1, ..., m\}$ such that

(a) the Lojasiewicz-Simon inequality for the λ-functional,

$$||\mathrm{Ric}_{f_{g(t)}}||_{L^2(e^{-f_{g(t)}}dV_{g(t)})} \geqslant |\lambda[g(t)]|^{1-\theta}, \tag{2.1}$$

holds for some $\theta \in [\frac{2}{5}, \frac{1}{2})$ for all $t \leqslant -T_0$, where $f_{g(t)}$ is the minimizer of $\lambda[g(t)]$;

(b) the inequality

$$|\nabla^{k,g(t)}[g(t) - g_{\mathrm{RF}}]|_{g(t)} \leqslant C_k |t|^{-\frac{\theta}{1-2\theta}} \tag{2.2}$$

holds for all $t \leqslant -T_k$.

As in [Has12], one can interpolate between higher order derivative estimates with respect to g_{RF} and $g(t)$.

We shall often find it helpful to write the inequalities from the previous definition in terms of $\beta := \frac{\theta}{1-2\theta}$. In particular, (2.1) implies that there exist constants $C_0, T_0 > 0$ such that

$$|g(t) - g_{\mathrm{RF}}| \leqslant C_0 |t|^{-\beta}$$

for all $t \in (-\infty, 0]$ satisfying $t \leqslant -T_0$. Consequently, if $g(t)$ is ARF of order θ, then the k-th order variant of (2.2) holds true with the exponent $-\beta - k$ for each $k \in \{0, ..., m\}$. In particular, $\beta \geqslant c$ if and only if $\theta \geqslant \frac{c}{2c+1}$, and the metric decays exponentially quickly to g_{RF} when $\beta = \infty$ or $\theta = \frac{1}{2}$. For simplicity of notation, we utilize the definition in terms of β pervasively throughout the remainder of the paper and write $g(t) \in \mathcal{M}^m_\beta(g_{\mathrm{RF}})$ to mean that $g(t)$ is ARF of order $\frac{\theta}{1-2\theta}$ and regularity m.

2.2. **Ricci curvature estimates.** We devote the remainder of this section to deriving Ricci curvature estimates for ARF flows that decay polynomially in time. We remark that if $g(t) \in \mathcal{M}^{m+2}_\beta(g_{\mathrm{RF}})$, then $|\nabla^k \mathrm{Rm}[g(t)]|_{g(t)}$ is uniformly bounded in t for any $k \leqslant m$. However, such estimates for the norm of the Riemann curvature tensor and its derivatives are frivolous for our purposes, so we will not explore them further.

We begin by establishing an asymptotic estimate for the norm of the Ricci curvature tensor. As we shall make evident momentarily, the proof of this estimate remains valid if we consider immortal flows in lieu of ancient ones.

Lemma 2.1. *If $g(t) \in \mathcal{M}^2_\beta(g_{\mathrm{RF}})$, then $|\mathrm{Ric}[g(t)]|_{g(t)} = O(|t|^{-\beta})$.*

Proof. Given any $t \leqslant -T := -\max_{k=0,1,2}(T_k)$, we define the linear function $\psi : [0,1] \to S^2T^*M$ by

$$\psi(s) = (1-s)g(t) + s g_{\mathrm{RF}}$$

so that $\psi(0) = g(t)$ and $\psi(1) = g_{\mathrm{RF}}$. Since $|g(t) - g_{\mathrm{RF}}| = O(|t|^{-\beta})$ and δ_{ij} is uniformly equivalent to g_{RF} by virtue of the compactness of M, there exist positive constants c_1, c_2 such that

$$(1 - c_1 c_2^{-1}) g_{\mathrm{RF}} \leqslant g_{\mathrm{RF}} - c_1 \delta_{ij} \leqslant g(t) \leqslant g_{\mathrm{RF}} + c_1 \delta_{ij} \leqslant (1 + c_1 c_2) g_{\mathrm{RF}}$$

for all $t \leqslant -T$. Setting $C = \max(|1 - c_1 c_2^{-1}|^{-1}, |1 + c_1 c_2|)$, we obtain that $C^{-1} g_{\mathrm{RF}} \leqslant g(t) \leqslant C g_{\mathrm{RF}}$, whence the metrics $(g(t))_{t \leqslant -T}$ are uniformly equivalent to g_{RF}. Consequently, we may invoke [DO20, Lemma C.1] to obtain the upper bound

$$
\begin{aligned}
|\partial_s \mathrm{Ric}[\psi(s)]|_{g(t)} \leqslant A[&(1+|h|_{g(t)})|\nabla^{g(t),2}h|_{g(t)} + |\mathrm{Rm}[g(t)]|_{g(t)}|h|_{g(t)} \\
&+ (1+|h|_{g(t)})|\nabla^{g(t)}h|^2_{g(t)}]
\end{aligned} \tag{2.3}
$$

for some constant A independent of $s \in [0,1]$ and $t \in (-\infty, -T]$. Since $g(t)$ is assumed to be ARF of order β with regularity 2, we have that $|\nabla^{g(t),k}h|_{g(t)} = O(|t|^{-\beta})$ for $k \in \{0,1,2\}$, so (2.3) tells us that $|\partial_s \mathrm{Ric}[\psi(s)]|_{g(t)} = O(|t|^{-\beta})$. Consequently,

$$
|\mathrm{Ric}[g(t)] - \mathrm{Ric}(g_{\mathrm{RF}})|_{g(t)} \leqslant \int_0^1 |\partial_s \mathrm{Ric}[\psi(s)]|_{g(t)} ds = O(|t|^{-\beta}).
$$

Since $\mathrm{Ric}(g_{\mathrm{RF}}) = 0$, it follows that $|\mathrm{Ric}[g(t)]|_{g(t)} = O(|t|^{-\beta})$, as claimed. □

We continue by using the maximum principle to establish decay rates for the derivatives of the Ricci curvature. In the sequel, we frequently utilize cutoff functions detecting the time decay of the Ricci curvature beyond a fixed time. More precisely, if $|\mathrm{Rm}[g(t)]|_{g(t)} \leqslant c_0$ on $M \times [0,\infty)$ and $|\mathrm{Ric}[g(t)]|_{g(t)} \leqslant c_1 t^{-\beta}$ for $t \geqslant T$, we define the function $\psi : [0,\infty) \to [0,\infty)$ by

$$
\psi(t) = \begin{cases} n(n-1)c_0 & t \leqslant T \\ \chi n(n-1)c_0 + (1-\chi)c_1 t^{-\beta} & T \leqslant t \leqslant T+1 \\ c_1 t^{-\beta} & t \geqslant T+1, \end{cases} \tag{2.4}
$$

where χ is a non-negative, smooth cutoff function such that $\chi \equiv 1$ on $[0, T+\frac{1}{2}]$ and $\chi \equiv 0$ on $[T+1,\infty)$. By definition, ψ is uniformly bounded and $|\mathrm{Ric}[g(t)]|_{g(t)} \leqslant \psi(t)$ for all $t \geqslant 0$. Moreover, if we assume that $\beta > 1$, then $I(t) := \int_0^t \psi(u)du$ is uniformly bounded. We shall use similar constructions in other maximum principle arguments without explicitly defining the appropriate time cutoff functions.

Before establishing some decay for $|\nabla \mathrm{Ric}|$, we demonstrate the utility of the cutoff function ψ by proving the following simpler result pertaining to the volume evolving along an ARF flow, which will prove invaluable in its own right in the next section.

Lemma 2.2. *If $g(t) \in \mathcal{M}^2_\beta(g_{\mathrm{RF}})$ for some $\beta > 1$, then there exists a positive constant C independent of t such that $V(t) := \mathrm{Vol}[g(t)] \leqslant C$.*

Proof. Set $\tau = S - t$ so that $\partial_\tau = -\partial_t$. After integrating the variational formula $\partial_\tau(dV) = \mathrm{R}dV$ over M, Lemma 2.1 tells us that $\partial_\tau V \leqslant \psi V$. Consequently, the maximum principle tells us that $V(\tau) \leqslant V(0)\exp\left(\int_0^\tau \psi(u)du\right) = V(0)e^{I(\tau)}$, hence the desired result follows from the observation that $I(\tau)$ is bounded provided that $\beta > 1$. □

The following result provides us with an asymptotic estimate for some of the derivatives of the Ricci curvature. The number of times we can differentiate while preserving some time decay is dictated by how much regularity we assume the ARF flow to have.

Lemma 2.3. *If $g(t) \in \mathcal{M}^{m+2}_\beta(g_{\mathrm{RF}})$ and $m \leqslant \beta$, then $|\nabla^k \mathrm{Ric}[g(t)]| = O(|t|^{k-\beta})$ for any $k \leqslant m$.*

Proof. Lemma 2.1 tells us that the desired estimate holds true when $m = 0$. Let us now suppose that the estimate holds for all $j \in \{0,...,m-1\}$ for some $m \geqslant 1$. In particular, if $g(t) \in \mathcal{M}^{m+2}_\beta(g_{\mathrm{RF}})$, then for every $j \leqslant m-1$, there exist constants $C_j > 0$ and $T_j \gg 1$ such that $|\nabla^j \mathrm{Ric}|^2 \leqslant C_j |t|^{2j-2\beta-2}$ for all $t \leqslant -T_j$. We define T_m to be the maximum of the quantities T_j for $j \leqslant m-1$ and show that there exists a constant C_m such that $|\nabla^m \mathrm{Ric}|^2 \leqslant C_m|t|^{2m-2\beta-2}$ for all $t < -T_m$.

Let us fix an arbitrary negative time $s < -T_m$ and consider the subsolution defined on $[3s, 0]$. Emulating our construction of the cutoff function (2.4), we can whip up a bounded, non-negative cutoff function $\xi_s : [3s, 0] \to [0, |2s|^{2\beta-2m+1}]$ vanishing on $[3s, 5s/2]$ such that $\xi_s(t) = |t|^{2\beta-2m+1}$ for all $t \in [2s, 0]$. We may further assume that $|\partial_t \xi| \leqslant c|s|^{2\beta-2m}$ for some constant c depending only on β and m. We define the function $F_s : M \times [3s, 0] \to [0, \infty)$ by the formula

$$F_s(x,t) = \xi_s(t)[|\nabla^m \mathrm{Ric}|^2(x,t) + A|\nabla^{m-1}\mathrm{Ric}|^2(x,t)],$$

where A is a constant to be determined momentarily. Since M is compact and ξ_s is defined independently of s on $[-T_m, 0]$, $\Box F_s$ is bounded from above on $M \times [-T_m, 0]$ by a constant independent of s.

The evolution equation (A.4) permits us to estimate $\Box|\nabla^j \mathrm{Ric}|^2$ for $j \leqslant m$ and $t \in [2s, -T_m]$ as follows. We extract a quadratic $\nabla^j \mathrm{Ric}$ term from the $i = 0$ and $i = j$ summands. To tend to the remaining summands, we bound the derivatives of the Riemann curvature tensor from above by a uniform constant and estimate $|\nabla^{j-i}\mathrm{Ric}|$ using the inductive hypothesis. In particular, we obtain linear $\nabla^j \mathrm{Ric}$ terms with coefficients of order $O(|t|^{i-\beta-1})$, each of which can be bounded from above by a coefficient of order $O(|t|^{j-\beta-1})$. We deduce from these observations that

$$\Box|\nabla^j \mathrm{Ric}|^2 \leqslant -2|\nabla^{j+1}\mathrm{Ric}|^2 + B_{1,j}|\nabla^j \mathrm{Ric}|^2 + B_{2,j}|t|^{j-\beta-1}|\nabla^j \mathrm{Ric}|.$$

We caution the reader that the constants B_i may vary from line to line. Substituting $j = m-1$ and $j = m$ variants of this estimate into the formula for the action of the heat operator on F, we obtain that

$$\begin{aligned}\Box F_s \leqslant{}& [(B_1 - 2A)\xi + (\partial_t \xi)] \cdot |\nabla^m \mathrm{Ric}|^2 + B_2 \xi |t|^{m-\beta-1}|\nabla^m \mathrm{Ric}| \\ &+ [B_3 A\xi + A(\partial_t \xi)] \cdot |\nabla^{m-1}\mathrm{Ric}|^2 + B_4 A\xi|t|^{m-\beta-2}|\nabla^{m-1}\mathrm{Ric}|.\end{aligned}$$

Recalling that $\xi = |t|^{2\beta-2m+1}$ on $[2s, -T_m]$, we can write this estimate explicitly as

$$\begin{aligned}\Box F_s \leqslant{}& [(B_1 - 2A)|t|^{2\beta-2m+1} - (2\beta - 2m + 1)|t|^{2\beta-2m}]\ |\nabla^m \mathrm{Ric}|^2 + B_2|t|^{\beta-m}|\nabla^m \mathrm{Ric}| \\ &+ [B_3 A|t|^{2\beta-2m+1} - A(2\beta - 2m + 1)|t|^{2\beta-2m}]|\nabla^{m-1}\mathrm{Ric}|^2 + B_4 A|t|^{\beta-m-1}|\nabla^{m-1}\mathrm{Ric}|.\end{aligned}$$

Since $m \leqslant \beta$, we can jettison some of the terms and apply Young's inequality twice to further simplify this upper bound to

$$\Box F_s \leqslant 2(B_1 - 2A)|t|^{2\beta-2m+1}|\nabla^m \mathrm{Ric}|^2 + 2AB_2|t|^{2\beta-2m+1}|\nabla^{m-1}\mathrm{Ric}|^2.$$

Consequently, if we set $A = B_1$, then the first term becomes non-positive. Furthermore, by induction, $|\nabla^{m-1}\mathrm{Ric}|^2 \leqslant C_{m-1}|t|^{2m-2\beta-2}$ for all $t \leqslant -T_m$, making the second term, and therefore $\Box F_s$, uniformly bounded by a constant independent of s on $M \times [2s, -T_m]$. Using the presumed uniform bounds for ξ_s and $\partial_t \xi_s$, by enlarging A if necessary, analogous computations allow one to draw the same conclusion on $M \times [3s, 2s]$. The maximum principle therefore implies that

$$\xi_s|\nabla^m \mathrm{Ric}|^2 \leqslant F_s \leqslant C_m(t - 3s)$$

for all $t \in [3s, 0]$, where C_m depends on $\max_{M\times[-T_m,0]} |\nabla^m \mathrm{Ric}|^2$, the constants C_j for each $j \leqslant m-1$, and the uniform bounds for the norms of the Riemann curvature and its derivatives.

In particular, C_m is independent of s. Since $\xi_s(s) = |s|^{2\beta-2m+1}$, it follows that $|\nabla^m \mathrm{Ric}|^2(s) \leqslant 2C_m|s|^{2m-2\beta}$. Since the choice of $s < -T_m$ is arbitrary, the desired result follows. $\square$

3. Dynamical energy functional on ancient ARF Ricci flows

3.1. **Estimates for conjugate heat flows.** Suppose that $(M^n, g(t))_{t\in(-\infty,T]}$ is an ancient solution to the Ricci flow equation on a compact manifold M^n. We also assume that $g(t) \in \mathcal{M}^3_\beta(g_{\mathrm{RF}})$ for some $\beta \geqslant 2$. Given a fixed time $s \in (-\infty, T]$, we denote by f^s the solution to the conjugate heat flow

$$\partial_t f^s = -\Delta f^s + |\nabla f^s|^2 - \mathrm{R} \tag{3.1}$$

with initial datum $f^s(\cdot, s) = 0$. While there is nothing special about this initial condition, it streamlines the proofs of the forthcoming estimates, all of which apply to any solution to (3.1) with initial datum $f^s(\cdot, s) = u$ for any smooth function u on M with bounded derivatives with respect to $g(s)$ (or equivalently, any other metric along the flow).

We reparameterize time by setting $\tau(t) = s - t$ so that $\partial_\tau = -\partial_t$. With respect to τ, the conjugate heat flow equation becomes

$$\partial_\tau f^s = \Delta f^s - |\nabla f^s|^2 + \mathrm{R}, \tag{3.2}$$

which is an equation readily lending itself to maximum principle applications. We henceforth denote by $\square$ the heat operator $\square = \partial_\tau - \Delta$ with respect to τ and conflate f^s with its reparameterization with respect to τ.

Our goal in this subsection is to establish uniform upper bounds independent of s for the norms of f^s, its gradient, and its Laplacian. Such bounds will prove paramount in our analysis of dynamical energy functionals defined in terms of the aforementioned one-parameter family of conjugate heat flows. Let us begin with the following elementary upper bound for $|f^s|$. For brevity of notation, we suppress the s-superscripts in some of the more cumbersome computations.

Lemma 3.1. *There exists a positive dimensional constant $a(n) > 0$ such that $0 \leqslant f^s \leqslant a(n)$ for all $s \in (-\infty, T]$.*

Proof. Since ancient flows with bounded curvature have non-negative scalar curvature, the non-negativity of f^s may be deduced from the computations presented in the proof of [LO25, Lemma 5.4].

Let us turn our attention to the stated upper bound. Since $|\mathrm{R}| \leqslant \sqrt{n}\cdot|\mathrm{Ric}|$, (3.2) implies that

$$\square f^s(x,\tau) = -|\nabla f^s|^2 + \mathrm{R} \leqslant \sqrt{n}\cdot\psi(\tau)$$

for any $x \in M$, where the function ψ is defined as in (2.4). The solution ϕ to the ODE $\partial_\tau \phi = \sqrt{n}\cdot\psi$ with initial condition $\phi(0) = 0$ satisfies

$$\phi(\tau) = \sqrt{n}\int_0^\tau \psi(u)du \leqslant (T+1)\left(n^{\frac{3}{2}}(n-1)c_0 + \frac{c_1\sqrt{n}}{(1-\beta)(T+1)^\beta}\right) =: a(n)$$

for any $\tau \geqslant 0$. Consequently, since $f^s(0) = 0$, the maximum principle tells us that

$$f^s(x,\tau) \leqslant \phi(\tau) \leqslant a(n)$$

for any $(x,\tau) \in M \times [0,\infty)$, as desired. $\square$

In a similar vein, we use cutoff functions detecting the decay rates of the first derivatives of curvature terms to establish a uniform upper bound for $|\nabla f^s|$.

Lemma 3.2. *There exists a positive dimensional constant $b(n) > 0$ such that $|\nabla f^s| \leqslant b(n)$ for all s.*

Proof. Using the identity $\partial_\tau(\nabla f) = \nabla(\partial_\tau f) + \nabla f * \mathrm{Ric}$ and the fact that $\partial_\tau g = 2\mathrm{Ric}$, we compute that

$$\partial_\tau|\nabla f|^2 = -2\mathrm{Ric}(\nabla f, \nabla f) + 2\langle\nabla(\partial_\tau f), \nabla f\rangle + (\nabla f)^{*2} * \mathrm{Ric}.$$

Substituting (3.2) and absorbing the $-2\mathrm{Ric}(\nabla f, \nabla f)$ term into the $(\nabla f)^{*2} * \mathrm{Ric}$ term, we simplify this evolution equation to

$$\partial_\tau|\nabla f|^2 = 2\langle\nabla(\Delta f - |\nabla f|^2 + \mathrm{R}), \nabla f\rangle + (\nabla f)^{*2} * \mathrm{Ric}. \tag{3.3}$$

Furthermore, by the Bochner formula, we have that

$$\Delta|\nabla f|^2 = 2\langle\nabla\Delta f, \nabla f\rangle + (\nabla f)^{*2} * \mathrm{Ric} + 2|\nabla^2 f|^2. \tag{3.4}$$

Subtracting (3.4) from (3.3) yields that

$$\Box|\nabla f|^2 = 2\langle\nabla \mathrm{R}, \nabla f\rangle - 2|\nabla^2 f|^2 + (\nabla f)^{*2} * \mathrm{Ric} - 2\langle\nabla|\nabla f|^2, \nabla f\rangle.$$

By Lemmas 2.1 and 2.3, by emulating the construction of the function defined by (2.4), we may construct cutoff functions $\psi_1, \psi_2 : [0, \infty) \to [0, \infty)$ such that

$$|\mathrm{Ric}|(x, \tau) \leqslant \psi_1(\tau) = O(\tau^{-\beta}) \quad \text{and} \quad |\nabla \mathrm{R}|(x, \tau) \leqslant \psi_2(\tau) = O(\tau^{1-\beta}).$$

Consequently, there is a positive constant C_1 such that

$$\Box|\nabla f|^2 \leqslant C_1[\psi_1(\tau)|\nabla f|^2 + \psi_2(\tau)|\nabla f|] - 2\langle\nabla|\nabla f|^2, \nabla f\rangle. \tag{3.5}$$

Furthermore, an application of Young's inequality yields that

$$2\psi_2(\tau)|\nabla f| = 2\psi_2(\tau)^{\frac{1}{2}}\psi_2(\tau)^{\frac{1}{2}}|\nabla f| \leqslant \psi_2(\tau) + \psi_2(\tau)|\nabla f|^2,$$

allowing us to absorb the $|\nabla f|$ term into the $|\nabla f|^2$ term in (3.5). In particular, defining $\psi_3 = \psi_1 + \psi_2 = O(\tau^{1-\beta})$, we obtain that

$$\Box|\nabla f|^2 \leqslant \psi_3(\tau)(|\nabla f|^2 + 1) - 2\langle\nabla|\nabla f|^2, \nabla f\rangle. \tag{3.6}$$

Since $\beta \geqslant 2$, the solution ϕ to the ODE $\partial_\tau\phi = \psi_3(\phi + 1)$ with initial condition $\phi(0) = 0$ satisfies

$$\phi(\tau) = \exp\left(\int_0^\tau \psi_3(u)du\right) - 1 \leqslant \exp\left(C_2(T+1)^{2-\beta}\right) =: b(n)^2$$

for all $\tau \geqslant 0$, where C_2 is a dimensional constant. Since $|\nabla f|(0) = 0$, the maximum principle therefore implies that $|\nabla f|^2 \leqslant b(n)^2$, as desired.

□

Availing of the uniform boundedness of $|f^s|$ and $|\nabla f^s|$, we can obtain some modest decay for λ^s_{dyn}.

Corollary 3.3. *Under all the same assumptions as before, we have that $\lambda^s_{\mathrm{dyn}}(\tau) = O(\tau^{-1})$.*

Proof. Emulating the proof of Proposition 3.10 by replacing the a priori asymptotic estimate for $|f^s|$ with an $O(1)$-bound, we obtain that $|\nabla f^s| = O(\tau^{-\frac{1}{2}})$. Since $\beta \geqslant 2$, the result follows from this asymptotic estimate and Lemma 2.1. □

While we cannot guarantee a uniform upper bound for the norm of the Hessian of f^s, we can derive a uniform bound for its Dirichlet energy, whose proof provides an appealing deviation from maximum principle arguments. As we shall observe in the proceeding section, such an estimate readily implies that the first variations of the dynamical energy functionals are uniformly bounded.

Lemma 3.4. *There exists a positive dimensional constant $c(n) > 0$ such that $\int_M |\nabla^2 f^s|^2 \leqslant c(n)$ for all s.*

Proof. The gist of the proof is to integrate the evolution (in)equalities derived in Lemma 3.2 and integrate by parts to tend to all the potentially unpleasant terms. We first establish a uniform bound for the L^2-norm of $|\Delta f^s|$ and then apply the Bochner formula to upgrade it to a uniform L^2-bound for $|\nabla^2 f^s|$. It is crucial to emphasize that all of the constants mentioned in the proof are polynomials in the uniform bounds for the curvature, its derivatives, $|f^s|$, and $|\nabla f^s|$, and are therefore independent of s.

Integrating (3.3) over M, we obtain that

$$\int_M \partial_\tau |\nabla f|^2 dV_g = 2\int_M \langle \nabla(\Delta f - |\nabla f|^2 + \mathrm{R}), \nabla f\rangle dV_g + \int_M [(\nabla f)^{*2} * \mathrm{Ric}] dV_g.$$

Since M is compact, we may integrate by parts to write the previous equation as

$$\int_M \partial_\tau |\nabla f|^2 dV_g = -2\int_M |\Delta f|^2 dV_g + \underbrace{\int_M [2\langle \nabla(\mathrm{R} - |\nabla f|^2), \nabla f\rangle + (\nabla f)^{*2} * \mathrm{Ric}] dV_g}_{:=J(\tau)}. \tag{3.7}$$

We avail of our previously established estimates for Ric, ∇Ric, f, and ∇f to obtain a uniform upper bound for $|J(\tau)|$. In particular, as we were made privy to in the proof of Lemma 3.2, $|\langle \nabla \mathrm{R}, \nabla f\rangle|$ and $|(\nabla f)^{*2} * \mathrm{Ric}|$ are uniformly bounded, so $\int_M [\langle \nabla \mathrm{R}, \nabla f\rangle + (\nabla f)^{*2} * \mathrm{Ric}] dV_g$ is as well by Lemma 2.2. Consequently, there exists a positive constant C_1 such that

$$|J(\tau)| \leqslant C_1 + 2\left|\int_M \langle \nabla |\nabla f|^2, \nabla f\rangle dV_g\right|$$

This upper bound, (3.7) and the triangle inequality together imply that

$$\begin{aligned} \left|\int_M \partial_\tau |\nabla f|^2 dV_g\right| &\geqslant 2\int_M |\Delta f|^2 dV_g - |J(\tau)| \\ &\geqslant 2\int_M |\Delta f|^2 dV_g - C_1 - 2\left|\int_M \langle \nabla |\nabla f|^2, \nabla f\rangle dV_g\right|. \end{aligned} \tag{3.8}$$

Furthermore, integrating the upper bound (3.6), we find that

$$\int_M \Box |\nabla f|^2 dV_g \leqslant \psi_3(\tau)\int_M (|\nabla f|^2 + 1) dV_g - 2\int_M \langle \nabla |\nabla f|^2, \nabla f\rangle dV_g.$$

Since $\psi_3(\tau) = O(\tau^{1-\beta})$ and $|\nabla f|^2$ is uniformly bounded per Lemma 3.2, it follows that there is a positive constant C_2 such that

$$\left|\int_M \Box |\nabla f|^2 dV_g\right| \leqslant C_2 + 2\left|\int_M \langle \nabla |\nabla f|^2, \nabla f\rangle dV_g\right|.$$

The compactness of M allows us to recognize the left-hand side as $|\int_M \partial_\tau |\nabla f|^2 dV_g|$. Consequently, by comparing the previous upper bound to (3.8), we obtain that

$$2\int_M |\Delta f|^2 dV_g \leqslant C_3 + 4\left|\int_M \langle \nabla|\nabla f|^2, \nabla f\rangle dV_g\right|, \tag{3.9}$$

where $C_3 = C_1 + C_2$. We handle the integral on the right-hand side using the Cauchy-Schwarz inequality and integration by parts. In particular, we compute that

$$\left|\int_M \langle \nabla|\nabla f|^2, \nabla f\rangle dV_g\right| = \left|\int_M (|\nabla f|^2 \cdot \Delta f) dV_g\right| \leqslant \left(\int_M |\Delta f|^2 dV_g\right)^{\frac{1}{2}} \left(\int_M |\nabla f|^4 dV_g\right)^{\frac{1}{2}}.$$

Combining this upper bound with (3.9), we obtain that

$$\int_M |\Delta f|^2 dV_g - 2\left(\int_M |\Delta f|^2 dV_g\right)^{\frac{1}{2}} \left(\int_M |\nabla f|^4 dV_g\right)^{\frac{1}{2}} \leqslant \frac{C_3}{2}.$$

Moreover, since $|\nabla f|^4$ is uniformly bounded, so is $\int_M |\nabla f|^4 dV_g$ per Lemma 2.2, so some algebra provides us with a positive constant C_4 arising from a uniform bound for $\int_M |\nabla f|^2 dV_g$ such that

$$\left|\left(\int_M |\Delta f|^2 dV_g\right)^{\frac{1}{2}} - \left(\int_M |\nabla f|^4 dV_g\right)^{\frac{1}{2}}\right| \leqslant C_4.$$

At long last, the triangle inequality yields the uniform upper bound

$$\left(\int_M |\Delta f|^2 dV_g\right)^{\frac{1}{2}} \leqslant \left|\left(\int_M |\Delta f|^2 dV_g\right)^{\frac{1}{2}} - \left(\int_M |\nabla f|^4 dV_g\right)^{\frac{1}{2}}\right| + \left(\int_M |\nabla f|^4 dV_g\right)^{\frac{1}{2}} \leqslant C_5.$$

To obtain the Hessian bound, we apply the Bochner formula and integrate by parts to obtain that

$$\begin{aligned}\int_M |\nabla^2 f|^2 dV_g &= \frac{1}{2}\int_M [\Delta|\nabla f|^2 - 2\langle \nabla\Delta f, \nabla f\rangle - 2\mathrm{Ric}(\nabla f, \nabla f)] dV_g \\ &= \int_M |\Delta f|^2 dV_g - \int_M \mathrm{Ric}(\nabla f, \nabla f) dV_g. \end{aligned} \tag{3.10}$$

Since $|(\nabla f)^{*2} * \mathrm{Ric}|$ is uniformly bounded, so is $\int_M \mathrm{Ric}(\nabla f, \nabla f) dV_g$, and $\int_M |\Delta f|^2 dV_g$ too is uniformly bounded. Consequently, (3.10) implies that $\int_M |\nabla^2 f|^2 dV_g$ is uniformly bounded as well. □

3.2. **Properties of the dynamical energy functional for ARF flows.** With the conjugate heat flow estimates out of the way, we are in a good position to define an analogue of [LO25, Definition 5.3] for ARF flows.

If $g(t) \in \mathcal{M}^3_\beta(g_{\mathrm{RF}})$ is an ARF flow such that $\beta \geqslant 2$ and $s \leqslant 0$, we can define the pointed functional $\lambda^s_{\mathrm{dyn}} : (-\infty, s] \to \mathbb{R}$ by

$$\lambda^s_{\mathrm{dyn}}(t) = \mathcal{F}[g(t), f^s(t)],$$

where f^s is the solution to the backward heat flow (3.1) with initial condition $f^s(\cdot, s) \equiv 0$. By [CLN06, Section 5.4], this functional satisfies the monotonicity formula

$$\frac{d}{dt}\lambda^s_{\mathrm{dyn}}(t) = 2\int_M |\mathrm{Ric}_{f^s}|^2 e^{-f^s} dV_g \geqslant 0. \tag{3.11}$$

The estimates of the previous section immediately imply the following result.

Proposition 3.5. *The sequences λ^s_{dyn} and $\frac{d}{dt}\lambda^s_{\mathrm{dyn}}$ are uniformly bounded.*

Proof. Let us recall the explicit formula

$$\lambda^s_{\mathrm{dyn}}(t) = \int_M \left[|\nabla^{g(t)} f^s|^2_{g(t)} + \mathrm{R}[g(t)]\right] e^{-f^s(t)} dV_{g(t)}.$$

Lemmas 2.1, 3.1, and 3.2 imply that $|\nabla^{g(t)} f^s|^2$, $\mathrm{R}[g(t)]$, and $\exp(-f^s(t))$ are all uniformly bounded in s and t, respectively. The volume bound given by Lemma 2.2 then allows us to deduce from these bounds a uniform bound for $\lambda^s(t)$. As for the first variation bound, recalling (3.11), Young's inequality implies that

$$\frac{d}{dt}\lambda^s_{\mathrm{dyn}}(t) \leqslant 4\int_M \left[|\mathrm{Ric}[g(t)]|^2 + |\nabla^{2,g(t)} f^s|^2\right] e^{-f^s(t)} dV_{g(t)},$$

so all the previously mentioned lemmas in tandem with Lemma 3.4 imply a uniform bound for $\frac{d}{dt}\lambda^s_{\mathrm{dyn}}(t)$. □

We have the following relationship between our newly introduced dynamical λ-functional and the ordinary λ-functional.

Proposition 3.6. *After parabolically rescaling the ARF flow, the pointed dynamical functional satisfies the inequality $\lambda^s(t) \geqslant \lambda(t)$ for any $t \in (-\infty, s]$.*

Proof. After parabolically rescaling the ARF solution, we may assume that $\mathrm{Vol}[g(s)] = 1$. Since $\partial_\tau(dV_g) = \mathrm{R}\cdot dV_g$ and $\partial_\tau f = \Delta f - |\nabla f|^2 + \mathrm{R}$, the first variation of the weighted volume with respect to f is given by

$$\partial_\tau \int_M e^{-f} dV_g = \int_M (|\nabla f|^2 - \Delta f)e^{-f} dV_g = 0.$$

Consequently, $\int_M e^{-f} dV_g$ is constant in time, so $\int_M e^{-f(t)} dV_{g(t)} = \int_M e^{-f(s)} dV_{g(s)} = 1$ for any t. In particular, $f(\cdot, t)$ is a valid test function for any $t \leqslant 0$, so the desired inequality follows from the definition of the λ-functional. □

A notable property of λ^s_{dyn} is that its critical points are precisely those coupled solutions that have constant conjugate heat flows.

Corollary 3.7. *The functional λ^s_{dyn} is constant if and only if f^s is constant in space and time. In particular,*

(a) if there exist two distinct times $t_1, t_2 \in (-\infty, s]$ such that $t_1 < t_2$ and $\lambda^s_{\mathrm{dyn}}(t_1) = \lambda^s_{\mathrm{dyn}}(t_2)$, then $(g(t))_{t\in[t_1,t_2]}$ is a Ricci-flat, steady gradient Ricci soliton;
(b) if there exists a decreasing sequence of times $t_n \in (-\infty, s]$ tending to $-\infty$ such that $\lambda^s_{\mathrm{dyn}}(t_n) = \lambda^s_{\mathrm{dyn}}(t_{n+1})$ for every $n \in \mathbb{N}$, then $g(t) = g_{\mathrm{RF}}$ on $(-\infty, t_1]$.

Proof. First suppose that λ^s_{dyn} is constant. Then the formula (3.11) implies that $\mathrm{Ric} = -\nabla^2 f^s$. Tracing this equation yields that $\mathrm{R} = -\Delta f^s$, whence

$$\partial_t f^s = -\Delta f^s + |\nabla f^s|^2 - \mathrm{R} = |\nabla f^s|^2. \tag{3.12}$$

Furthermore, we integrate by parts to compute that

$$\lambda^s_{\mathrm{dyn}}(t) = \int_M (|\nabla f^s|^2 + \mathrm{R})e^{-f^s} dV_g = \int_M (|\nabla f^s|^2 - \Delta f^s)e^{-f^s} dV_g = 0.$$

Since $\mathrm{R} \geqslant 0$, it follows that $\mathrm{R} = 0$ and $\nabla f^s \equiv 0$, making f^s constant in space. Since $\partial_t f^s = |\nabla f^s|^2$, the constancy of f^s in space implies that it is also constant in time.

Conversely, suppose that f^s is constant in space and time. Then $\partial_t f^s = \Delta f^s = \nabla f^s = 0$. Since f^s satisfies the first equality of (3.12), it follows that $\mathrm{R} \equiv 0$. Consequently, $\lambda^s_{\mathrm{dyn}} \equiv 0$.

Turning our attention to part (a), since λ^s_{dyn} is non-decreasing, the previous argument tells us that f^s is constant on $[t_1, t_2]$, hence $\mathrm{Ric}[g(t)] = -\nabla^{2,g(t)} f^s = 0$ for all $t \in [t_1, t_2]$. As for part (b), select any time $t \leqslant t_1$. Since $t \in [t_{n_0+1}, t_{n_0}]$ for some $n_0 \in \mathbb{N}$, part (a) implies that $g(t) = g(t_n)$ for all $n \in \mathbb{N}$. It then follows from the ARF assumption that

$$|g(t) - g_{\mathrm{RF}}| = |g(t_n) - g_{\mathrm{RF}}| \leqslant C_0 |t_n|^{-\beta}$$

for every $n \in \mathbb{N}$ for some constant C_0 independent of n, readily implying the result. □

The preceding three results prove our first main theorem.

Proof of Theorem 1.1. Part (a) follows directly from Propositions 3.5 and 3.6, whereas part (b) is tantamount to part (a) of Corollary 3.7. □

Part (b) of Corollary 3.7 is rigid in the sense that ARF flows with vanishing dynamical energy functionals correspond to trivial solutions. We turn our attention to generalizing this result by showing that prescribing a decay rate for λ^s_{dyn} is equivalent to prescribing one for Ric_f, thereby proving Theorem 1.2. This observation quantifies the notion that these energy functionals measure the deviation of a solution from being a Ricci-flat, steady gradient soliton.

By definition, if $|\nabla f^s|^2 = O(\tau^{-k})$ for some $k > 1$, then $\lambda^s_{\mathrm{dyn}}(\tau) = O(\tau^{-\min(k,\beta)})$. We will show that a slightly weaker version of the converse also holds. As a first step, we prove that a decay rate for λ^s_{dyn} transmutes into a decay rate for f, which we only know to be bounded otherwise (cf. Lemma 3.1). The proof hinges on adapting Li's method of converting L^2-bounds into pointwise ones to our ARF situation. To accomplish this, it is necessary to have uniform equivalence of the diameters of the metrics along the flow. The subsequent lemma allays this potential concern. We henceforth denote by d_τ the diameter with respect to $g(\tau)$.

Lemma 3.8. *For τ sufficiently large, there exists a constant c_0 independent of τ such that*

$$c_0^{-1} d_1 \leqslant d_\tau \leqslant c_0 d_1.$$

Proof. By Lemma 2.1, for τ sufficiently large, $-\overline{c}_0 \tau^{1-\beta} \leqslant \mathrm{Ric}[g(\tau)] \leqslant \overline{c}_0 \tau^{1-\beta}$ for some constant $\overline{c}_0$ independent of τ. Modestly tweaking the classical distance distortion estimates yields that

$$e^{-\frac{\overline{c}_0}{\beta-1}} d_1 \leqslant d_\tau \leqslant e^{\frac{\overline{c}_0}{\beta-1}} d_1$$

for τ sufficiently large. Indeed, one need only take into account the change in the sign of the first variation of Ric when computing with respect to τ. □

Lemma 3.9. *If $\lambda^s_{\mathrm{dyn}} = O(\tau^{-k})$, then $|f^s| = O(\tau^{\max(\frac{1-k}{2}, \frac{3}{2}-\beta)})$.*

Proof. By Lemma 3.8, there exists a uniform constant c such that $c^{-1} d_0 \leqslant d_\tau \leqslant c d_0$ for all τ, where d_τ is the diameter of M. Combined with Lemma 3.1 and [LS84, Corollary 1.1] for $\alpha = 2$ and $R = d_\tau(M)/2$, this bound yields a decay rate of $O(\tau^{-k/2})$ for the L^2-norm of f. Note that if $w = f + c\tau^{1-\beta}$ for some sufficiently large constant c independent of τ, then

$$\partial_\tau w \leqslant \Delta w + \mathrm{R} + c(1-\beta)\tau^{-\beta} \leqslant \Delta w$$

for sufficiently large τ. Since the diameter and volume along the flow are uniformly bounded from below and above, respectively, [Ye07, Theorems D, 5.5] and [LS84, Corollary 1.1] provide us with a uniform *normalized* Sobolev constant, as defined in [Li12, Chapter 19]. Consequently, we can apply [Li12, Lemma 19.1] to obtain that

$$\sup_{[\tau,2\tau]} f \leqslant \sup_{[\tau,2\tau]} w \leqslant a(n)\left[\int_{\tau/2}^{2\tau}\int_M (f^2+u^{2-2\beta})dV_g du\right]^{1/2} \leqslant b(n)\tau^{\max(\frac{1-k}{2},\frac{3}{2}-\beta)}$$

for some dimensional constants $a(n)$ and $b(n)$. □

Now, we are ready to prove that the gradient and Hessian of f satisfy similar decay rates.

Lemma 3.10. *If $\lambda^s_{\text{dyn}} = O(\tau^{-k})$ and $\beta > 2$, then $|\nabla f|^2 = O(\tau^{\max(\frac{1-k}{2},\frac{3}{2}-\beta)-1})$.*

Proof. Set $m = -\max(\frac{1-k}{2}, \frac{3}{2}-\beta)$ and $C = \frac{m+1}{3}$. Using Lemma 3.2, we obtain the bound

$$\Box(\tau^{m+1}|\nabla f|^2 - C\tau^m f) \leqslant C\tau^{m-\beta} - 2\langle\nabla(\tau^{m+1}|\nabla f|^2 - C\tau^m f), \nabla f\rangle.$$

Since $|\nabla f|^2(x,0) = 0$, the maximum principle implies that there is a uniform constant C_1 such that $\tau^{m+1}|\nabla f|^2 - C\tau^m f \leqslant C_1$, or equivalently, $|\nabla f|^2 \leqslant C_1\tau^{-(m+1)}$. □

As a final step, we derive a decaying upper bound for the norm of the Hessian of f.

Lemma 3.11. *If $\lambda^s_{\text{dyn}} = O(\tau^{-k})$, then $\int_M |\nabla^2 f|^2 = O(\tau^{\max(-1-k,1-\beta)})$.*

Proof. Emulate the proof of Lemma 3.4 by substituting each universal constant with an appropriate decaying upper bound derived from either the curvature estimates or our newly established decay rate for $|\nabla f|$ (see Lemma 3.10). □

Theorem 1.2 follows readily from the preceding results.

Proof of Theorem 1.2. Recalling that $|\mathrm{Ric}[g(\tau)]| = O(\tau^{-\beta})$ and that $|f|$ is uniformly bounded, we obtain from the first variation formula (3.11) and the previous lemma the asymptotic estimate $\partial_\tau\lambda^s_{\text{dyn}} = O(\tau^{\max(-1-k,1-\beta,-2\beta)}) = O(\tau^{\max(-1-k,1-\beta)})$, since $k \geqslant 1$ and $\beta \geqslant 2$. □

In tandem with Corollary 3.3, this result guarantees a decay rate for the first variation of λ^s_{dyn} for any ARF flow of order β.

Corollary 3.12. *If $g(t) \in \mathcal{M}^3_\beta(g(t))$ for some $\beta \geqslant 2$, then $\partial_\tau\lambda^s_{\text{dyn}} = O(\tau^{\max(-2,1-\beta)})$. In particular, if $\beta \geqslant 3$, then $\partial_\tau\lambda^s_{\text{dyn}} = O(\tau^{-2})$.*

4. Eigenvalue estimates for conjugate heat flows

In [CMI24], Colding and Minicozzi derive sharp upper bounds for eigenvalues of the drift Laplacian for a modified Ricci flow. Staying on the topic of backward heat flows, we dedicate this section to proving analogues of some of their results for eigenvalues of the drift Laplacian for a modified backward heat flow.

4.1. **Local upper bounds for the eigenvalues of conjugate heat flows.** Suppose we have a coupled system $(g(t), f(t))$ on a compact manifold M^n satisfying

$$\begin{cases}\partial_t g = -2\mathrm{Ric}_f \\ \partial_t f = -\mathrm{R} - \Delta f,\end{cases} \tag{4.1}$$

Observe that the weighted volume form $e^{-f}dV_g$ is constant along this coupled flow, hence why the variational formula for g is referred to as the normalized Ricci flow. Furthermore, this system is, up to diffeomorphism, equivalent to a coupling of the Ricci flow with a conjugate heat flow, which was the system we studied in the previous sections.

We recall that the drift Laplacian $\Delta_f = \Delta - \nabla_{\nabla f}$ has discrete spectrum with $\lambda_k \to \infty$ as $k \to \infty$, although the eigenvalues can have multiplicity.

Let us now derive the local upper bound for the kth eigenvalue of Δ_f along (4.1) stated in Theorem 1.3.

Proof of Theorem 1.3. Emulating the computations in [CMI24, Section 2] using the variational formulas provided in Lemma A.4 and Corollary A.5, we obtain that $\lambda_k'(t) \leqslant 2\lambda_k^2(t)$, where the derivative λ_k' is understood in the sense of [CMI24, (1.1)]. Consequently, $(-\frac{1}{2\lambda_k})' \leqslant 1$, hence

$$\frac{1}{2\lambda_k(t_0)} - \frac{1}{2\lambda_k(t)} \leqslant t - t_0$$

for any $t < t_0 + \frac{1}{2\lambda_k(t_0)}$. Algebraically manipulating this inequality yields (1.1). $\square$

4.2. **Sharpness.** We dedicate this subsection to establishing the rigidity of the eigenvalue upper bound (1.1) for $k = 1$. We define the function f and the metric g on $\mathbb{R}^n$ by

$$f(x,t) = \frac{|x|^2}{4} + \frac{n}{2}\log\left(\frac{u(t)}{u(t_0)}\right) \quad \text{and} \quad g(t) = u(t)\delta_{ij},$$

where $u(t)$ is defined on $[t_0, t_0 + u(t_0))$ by

$$u(t) = u(t_0) - (t - t_0).$$

One computes that

$$(\partial_t f, \partial_t g) = (-\Delta f, -2\nabla^2 f),$$

which is readily seen to satisfy (4.1) since $\mathrm{Ric}[g(t)] = 0$. By [CMI24, Lemma 2.41], the first eigenvalue of $g(t)$ is $\frac{1}{2u(t)}$, so $\lambda_1(t_0) = \frac{1}{2u(t_0)}$. Consequently,

$$\lambda_1(t) = \frac{1}{2u(t)} = \frac{1}{2[u(t_0) - (t - t_0)]} = \frac{\lambda_1(t_0)}{1 - 2(t - t_0)\lambda_1(t_0)},$$

thereby establishing the rigidity of the local estimate (1.1) for the first eigenvalue.

Appendix A. Evolution equations and inequalities

Here, we record several computations that are utilized throughout the paper.

A.1. **First variations of norms of some curvature tensors.** There are ample classical estimates that provide inductive local upper bounds for the first variations of the derivatives of the Riemann curvature tensor. Peruse [CLN06, Section 6.1] for a succinct treatment of such estimates. Nevertheless, as is evident from the proofs in §2 and §3, such upper bounds are insufficient for our purposes considering how we require some modest *global* time decay for the derivatives of the Ricci curvature tensor. Fortunately, when one is merely concerned about Ricci curvature estimates, it is possible to tweak the classical variational formulas enough to replace several Riemann curvature quantities with Ricci curvature quantities.

Lemma A.1. *Let $(M^n, g(t))_{t\in I}$ be a solution to the Ricci flow equation. Then*

$$\frac{\partial}{\partial t}|\mathrm{Ric}|^2 \leqslant \Delta|\mathrm{Ric}|^2 - 2|\nabla\mathrm{Ric}|^2 + c(n)|\mathrm{Rm}|\cdot|\mathrm{Ric}|^2$$

for some $c(n) > 0$.

Proof. We claim that

$$\partial_t(g^{ri}g^{sj})R_{rs}R_{ij} = 2g^{ri}g^{sj}R_{rs}(R_i^aR_{aj} + R_j^aR_{ia}). \tag{A.1}$$

To derive this formula, we first use geodesic normal coordinates in tandem with the Ricci flow equation to compute that

$$\partial_t(g^{ri})g^{sj}R_{rs}R_{ij} = 2R_i^r g^{sj}R_{rs}R_{ij} = 2R_i^aR_{aj}R_{ij},$$

which coincides with

$$2g^{ri}g^{sj}R_{rs}R_i^aR_{aj} = 2R_{ij}R_i^aR_{aj}.$$

An analogous computation for the second term arising from the product rule tells us that $g^{ri}(\partial_t g^{sj})R_{rs}R_{ij} = 2g^{ri}g^{sj}R_{rs}R_j^aR_{ia}$, proving (A.1). Similarly, to compute $g^{ri}g^{sj}\partial_t(R_{rs}R_{ij})$, we use the evolution equation for the Ricci tensor, namely

$$\partial_t R_{ij} = \Delta R_{ij} + 2R_{kij\ell}R_{k\ell} - 2R_{ik}R_{jk},$$

to obtain that

$$g^{ri}g^{sj}\partial_t(R_{rs}R_{ij}) = 2g^{ri}g^{sj}R_{rs}(\partial_t R_{ij}) = 2g^{ri}g^{sj}R_{rs}(\Delta R_{ij} + 2R_{kij\ell}R_{k\ell} - 2R_{ik}R_{jk}). \tag{A.2}$$

Furthermore, we observe that

$$\Delta|\mathrm{Ric}|^2 = 2\langle\Delta\mathrm{Ric},\mathrm{Ric}\rangle + 2|\nabla\mathrm{Ric}|^2 = 2g^{ri}g^{sj}R_{rs}\Delta R_{ij} + 2|\nabla\mathrm{Ric}|^2,$$

so we may rewrite (A.2) as

$$g^{ri}g^{sj}\partial_t(R_{rs}R_{ij}) = \Delta|\mathrm{Ric}|^2 - 2|\nabla\mathrm{Ric}|^2 + 2g^{ri}g^{sj}R_{rs}(2R_{kij\ell}R_{k\ell} - 2R_{ik}R_{jk}).$$

Adding this equation to (A.1), we find that

$$\partial_t|\mathrm{Ric}|^2 = \partial_t(g^{ri}g^{sj}R_{rs}R_{ij}) = \Delta|\mathrm{Ric}|^2 - 2|\nabla\mathrm{Ric}|^2 + B,$$

where $B = B_{aijk\ell rs}$ is the tensor

$$2g^{ri}g^{sj}R_{rs}(R_i^aR_{aj} + R_j^aR_{ia} + 2R_{kij\ell}R_{k\ell} - 2R_{ik}R_{jk}) = \mathrm{Ric}^{*3} + \mathrm{Rm} * (\mathrm{Ric})^{*2}.$$

Consequently, $|B| \leqslant c(n)|\mathrm{Rm}|\cdot|\mathrm{Ric}|^2$ since $|\mathrm{Ric}| \leqslant \sqrt{n(n-1)}|\mathrm{Rm}|$. □

Lemma A.2. *Let $(g(t))_{t\in I}$ be a solution to the Ricci flow equation. Then*

$$\frac{\partial}{\partial t}|\nabla\mathrm{Ric}|^2 = \Delta|\nabla\mathrm{Ric}|^2 - 2|\nabla\mathrm{Ric}|^2 + \mathrm{Rm} * (\nabla\mathrm{Ric})^{*2} + (\nabla\mathrm{Rm}) * \mathrm{Ric} * (\nabla\mathrm{Ric}).$$

Proof. Since $\partial_t g = -2\mathrm{Ric}$, we have that

$$\begin{aligned}\partial_t(\nabla\mathrm{Ric}) = \nabla(\partial_t\mathrm{Ric}) + \nabla(\partial_t g) * \mathrm{Ric} &= \nabla(\Delta\mathrm{Ric} + \mathrm{Rm} * \mathrm{Ric} + (\mathrm{Ric})^{*2}) + (\nabla\mathrm{Ric} * \mathrm{Ric})\\ &= \nabla\Delta\mathrm{Ric} + (\nabla\mathrm{Rm}) * \mathrm{Ric} + (\nabla\mathrm{Ric}) * \mathrm{Rm}.\end{aligned}$$

Furthermore, the Bochner formula tells us that

$$2\langle\nabla\Delta\mathrm{Ric},\nabla\mathrm{Ric}\rangle = \Delta|\nabla\mathrm{Ric}|^2 - 2|\nabla\mathrm{Ric}|^2 + \mathrm{Ric} * (\nabla\mathrm{Ric})^{*2},$$

and consequently,

$$\begin{aligned}\partial_t|\nabla\mathrm{Ric}|^2 &= 2\langle\partial_t(\nabla\mathrm{Ric}),\nabla\mathrm{Ric}\rangle + \mathrm{Ric}*(\nabla\mathrm{Ric})^{*2}\\ &= \Delta|\nabla\mathrm{Ric}|^2 - 2|\nabla\mathrm{Ric}|^2 + \mathrm{Rm}*(\nabla\mathrm{Ric})^{*2} + (\nabla\mathrm{Rm})*\mathrm{Ric}*(\nabla\mathrm{Ric}).\end{aligned}$$

□

Lemma A.3. *Let $(g(t))_{t\in I}$ be a solution to the Ricci flow equation. Then*

$$\Box(\nabla^m\mathrm{Ric}) = \sum_{i=0}^{m}\left[\nabla^i\mathrm{Rm}*\nabla^{m-i}\mathrm{Ric}\right], \tag{A.3}$$

and consequently,

$$\Box|\nabla^m\mathrm{Ric}|^2 = -2|\nabla^{m+1}\mathrm{Ric}|^2 + \sum_{i=0}^{m}[\nabla^i\mathrm{Rm}*\nabla^{m-i}\mathrm{Ric}*\nabla^m\mathrm{Ric}]. \tag{A.4}$$

Proof. The proof of the previous lemma tells us that (A.3) holds when $m = 1$. Let us now suppose that (A.3) holds up to some $m \in \mathbb{N}$. Then

$$\begin{aligned}\partial_t(\nabla^{m+1}\mathrm{Ric}) &= \nabla\partial_t(\nabla^m\mathrm{Ric}) + \nabla^m\mathrm{Ric}*\nabla\mathrm{Ric}\\ &= \nabla\left[\Delta(\nabla^m\mathrm{Ric}) + \sum_{i=0}^{m}\left[\nabla^i\mathrm{Rm}*\nabla^{m-i}\mathrm{Ric}\right]\right] + \nabla^m\mathrm{Ric}*\nabla\mathrm{Ric}.\end{aligned} \tag{A.5}$$

Using the standard commutation formula for exchanging a covariant derivative and a Laplacian, we obtain that

$$\nabla\Delta(\nabla^m\mathrm{Ric}) = \Delta(\nabla^{m+1}\mathrm{Ric}) + \mathrm{Rm}*\nabla^{m+1}\mathrm{Ric} + \nabla\mathrm{Rm}*\nabla^m\mathrm{Ric}. \tag{A.6}$$

Furthermore, since

$$\nabla(\nabla^i\,\mathrm{Rm}*\nabla^{m-i}\mathrm{Ric}) = \nabla^{i+1}\,\mathrm{Rm}*\nabla^{(m+1)-(i+1)}\mathrm{Ric} + \nabla^i\mathrm{Rm}*\nabla^{(m+1)-i}\mathrm{Ric},$$

it follows that

$$\nabla\left(\sum_{i=0}^{m}\left[\nabla^i\mathrm{Rm}*\nabla^{m-i}\mathrm{Ric}\right]\right) = \sum_{i=0}^{m+1}\nabla^i\,\mathrm{Rm}*\nabla^{(m+1)-i}\mathrm{Ric}. \tag{A.7}$$

Substituting (A.6) and (A.7) into (A.5), we obtain that

$$\begin{aligned}\Box(\nabla^{m+1}\mathrm{Ric}) = &\sum_{i=0}^{m+1}\nabla^i\,\mathrm{Rm}*\nabla^{(m+1)-i}\mathrm{Ric}\\ &+ \mathrm{Rm}*\nabla^{m+1}\mathrm{Ric} + \nabla\,\mathrm{Rm}*\nabla^m\mathrm{Ric} + \nabla^m\mathrm{Ric}*\nabla\mathrm{Ric}.\end{aligned}$$

Observing that the final three terms may be absorbed into the first completes the inductive step and in turn the proof of the lemma. □

A.2. **Evolution equations for quantities evolving along a modified backward heat flow.**

Lemma A.4. *Suppose we have a coupled system $(g(t), f(t))$ of Riemannian metrics and smooth functions on a compact manifold M^n satisfying*

$$\begin{cases} \partial_t g = -2\mathrm{Ric} - \mathrm{Hess}_f \\ \partial_t f = -\mathrm{R} - \Delta f. \end{cases} \tag{A.8}$$

If $\partial_t u = \Delta_f u$ and $\partial_t v = \Delta_f v$, then the weighted L^2-inner product of u and v and the weighted L^2-norm of $|\nabla u|$ satisfy

$$\partial_t \langle u, v\rangle_f = -2\int_M \langle \nabla u, \nabla v\rangle e^{-f} dV_g \quad \text{and} \quad \partial_t ||\nabla u||^2_{L^2(e^{-f}dV_g)} = -2\int_M |\nabla^2 u|^2 e^{-f} dV_g.$$

Proof. Using the product rule, we compute that

$$\partial_t(uv) = (\partial_t u)v + u(\partial_t v) = v\Delta_f u + u\Delta_f v = \Delta_f(uv) - 2\langle \nabla u, \nabla v\rangle.$$

Since $\Delta_f(uv)e^{-f}$ integrates to zero and $\partial_t(e^{-f}dV_g) = 0$, multiplying this series of equations by e^{-f} and integrating over M gives the first formula. As for the second formula, the chain rule and the weighted Bochner formula imply that

$$\begin{aligned} \partial_t |\nabla u|^2 &= 2\langle \nabla(\partial_t u), \nabla u\rangle - (\partial_t g)(\nabla u, \nabla u) \\ &= 2\langle \nabla(\Delta_f u), \nabla u\rangle + 2\mathrm{Ric}_f(\nabla u, \nabla u) \\ &= \Delta_f |\nabla u|^2 - 2|\nabla^2 u|^2, \end{aligned}$$

so multiplying this series of equations by e^{-f} and integrating over M yields the second formula. ☐

Corollary A.5. *Consider the quantities*

$$J_{uv}(t) = \int (uv)e^{-f}, \quad D_{uv}(t) = \int \langle \nabla u, \nabla v\rangle e^{-f}, \quad I_u(t) = J_{uu}(t), \quad E_u(t) = D_{uu}(t).$$

If $\partial_t u = \Delta_f u$ and $\partial_t v = \Delta_f v$, then along (A.8)*, we have that*

$$J'_{uv}(t) = -2D_{uv}(t), \quad I'_u(t) = -2E_u(t), \quad E'_u(t) = -2\int_M |\nabla^2 u|^2 e^{-f} \leqslant 0.$$

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