FULL-TWIST PRESENTATIONS FOR FUNDAMENTAL GROUPS OF COMPLEMENTS OF COMPLEXIFIED HYPERPLANE ARRANGEMENTS

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Abstract. This paper conjectures full-twist presentations for the fundamental group of the complement of a complexified finite central real hyperplane arrangement. We first investigate the case of Coxeter arrangements, generalizing the classical Artin presentation of pure braid groups. Building on Salvetti's combinatorial model for arrangement complements, we introduce a family of generators arising from a minimal gallery of regions and describe relations produced from rank two subarrangements. We then focus on the notion of the full twist and conjecture a presentation of the fundamental group by equating all the reduced words for the full twist. We will prove this conjecture for Coxeter arrangements of types A, B, D, H_3, $I_2(m)$, and F_4.

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1. Introduction and Background

The *braid group*, introduced by Emil Artin in 1926, is a mathematical object that is fundamental to our study. It encodes the way in which n strands of braid can interlace without cutting or gluing. Braid groups arise naturally from topology, but their importance extends far beyond this field.

On the applied side, braid groups have been proposed as platforms for *public-key cryptography*, where the difficulty of reversing a braid word serves as a hard computational problem. In pure mathematics, braid groups play a foundational role in *knot theory*, where every knot can be represented as the closure of a braid. They also appear in algebraic geometry as the fundamental groups of configuration spaces. The ubiquity of braid groups demonstrates that studying their structure has implications for multiple fields.

In 1926, Artin proved the following presentation for the braid group B_n

$$B_n = \left\langle \mathbf{s_1}, \ldots, \mathbf{s_{n-1}} : \begin{array}{ll} \mathbf{s_i s_j} = \mathbf{s_j s_i} & \text{if } |i-j| > 1 \\ \mathbf{s_i s_{i+1} s_i} = \mathbf{s_{i+1} s_i s_{i+1}} & \text{for } 2 \leq i+1 \leq n-1 \end{array} \right\rangle.$$

There is a natural map π from the braid group B_n to the symmetric group S_n since one can ignore the twisting of braids to obtain solely a permutation. Therefore, we could construct the exact sequence

$$1 \to P_n \to B_n \xrightarrow{\pi} S_n \to 1$$

where $\ker(\pi) \cong P_n$. The pure braid group P_n can be visualized as the fundamental group of the space of n-tuples of distinct points in Euclidean space. It has been proven by Artin in 1947 [Art47] that the pure braid group P_n can be presented as

$$S_{rs}^{-1} S_{ij} S_{rs} = \begin{cases} S_{ij} & \text{if } r < s < i < j \\ S_{ij} & \text{if } i < r < s < j \\ S_{rj} S_{ij} S_{rj}^{-1} & \text{if } r < i = s < j \\ (S_{ij} S_{sj}) S_{ij} (S_{ij} S_{sj})^{-1} & \text{if } r = i < s < j \\ (S_{rj} S_{sj} S_{rj}^{-1} S_{sj}^{-1}) S_{ij} (S_{rj} S_{sj} S_{rj}^{-1} S_{sj}^{-1})^{-1} & \text{if } r < i < s < j \end{cases}$$

where for $1 \leq i < j \leq n+1$

$$S_{ij} \coloneqq (\mathbf{s_{j-1} s_{j-2}} \cdots \mathbf{s_{i+1}}) \mathbf{s_i}^2 (\mathbf{s_{i+1}} \cdots \mathbf{s_{j-2} s_{j-1}}).$$

We will next present a more generalized and formal definition of the braid group. Let W be a finite Coxeter group, and let $\mathcal{H}$ denote its associated real central simplicial hyperplane arrangement in $\mathbb{R}^n$. Define $V_{\mathbb{C}}^{reg} \coloneqq \mathbb{C}^n \setminus \mathcal{H}_{\mathbb{C}}$, where $\mathcal{H}_{\mathbb{C}}$ is the complexified hyperplane arrangement of $\mathcal{H}$.

Definition 1.1. The *braid group* is defined as

$$B(W) \coloneqq \pi_1(V_{\mathbb{C}}^{reg}/W).$$

In 1999, Birman-Ko-Lee, Bessis, and Brady-Watt proved an elegant result for a presentation of the braid group [BKL98, Bes03, BW01].

Theorem 1.2 (Birman-Ko-Lee, Bessis, and Brady-Watt)**.** *Let T be the set of reflections of W, and let c be a Coxeter element, which is a product of the simple reflections, in some order. Let $[\mathrm{Red}_T(c)]$ represent the set of reduced words in T for the element c. Then, we have*

$$B(W) = \langle T : [\mathrm{Red}_T(c)] \rangle.$$

Motivated by the beautiful presentation for the braid group, this project aims to establish a similar result for the pure braid groups.

Definition 1.3. The *pure braid group* of Coxeter group type W is defined as

$$P(W) \coloneqq \pi_1(V_{\mathbb{C}}^{reg}).$$

In analogy to Theorem 1.2, Nathan Williams has conjectured the following similarly elegant presentation for the pure braid group.

Definition 1.4. The *full-twist* of $\pi_1(V_{\mathbb{C}}^{\text{reg}}, x_B)$ is the element defined by

$$\mathbb{c} \coloneqq e^{2\pi i t} x_B \text{ for } 0 \leq t \leq 1.$$

Let $\mathbb{T}$ denote the set of generators of the pure braid group $P(W)$ corresponding to the reflections T of W (the precise construction will be given in Section 3).

Conjecture 1.5 ([Wil25], Page 3)**.** *Let* $[\text{Red}_{\mathbb{T}}(\mathbb{c})]$ *represent the set of minimal length words in* $\mathbb{T}$ *for the full twist* $\mathbb{c}$*. Then*

$$P(W) = \langle \mathbb{T} : [\text{Red}_{\mathbb{T}}(\mathbb{c})] \rangle.$$

In this paper, we will prove this conjecture for Coxeter groups of types A, B, D, H_3, and $I_2(m)$. The conjecture for F_4 will still be in progress. We will present Williams's novel approach using the Salvetti complex and Coxeter-Catalan theory. This approach is potentially expandable to arbitrary central real hyperplane arrangements. Refer to Conjecture 4.3 for the statement of the more general conjecture.

This conjecture has two significances. Firstly, it provides a uniform presentation for pure braid groups, simplifying Artin's relations. Moreover, it extends the combinatorial framework of the Coxeter-Catalan theory, suggesting a connection between algebraic topology and combinatorics.

This paper is structured as follows. In Section 2, we establish the mathematical prerequisites necessary for understanding the research. This section reviews posets of regions and finite central real hyperplane arrangements. In Section 3, we define and explain the Salvetti complex and how it can be applied to generate a presentation for the fundamental group of the complexified hyperplane complement. In Section 4, we state the main conjecture for this research, detailing a presentation using the full twist. In Section 5, we develop the framework for proving the conjecture for finite Coxeter arrangements. In Section 6, we provide explicit computations for Coxeter groups of types A, B, D, $I_2(m)$, H_3, and F_4 (partially).

2. Mathematical Prerequisite and Definitions

This section contains the prerequisites and definitions necessary to understand the research.

Definition 2.1. A *Coxeter group* is a group with its presentation defined by reflections. Formally, one can define a *Coxeter group* as

$$\langle r_1, r_2, \ldots, r_n | (r_i r_j)^{m_{ij}} = 1 \rangle$$

where $m_{ii} = 1$ and for $i \neq j$, $m_{ij} = m_{ji} \geq 2$ is an integer or ∞.

If we concretely represent each generator of the Coxeter group as a reflection across a linear hyperplane, we see that each element of the Coxeter group can be interpreted as a combination of reflections across the sets of linear hyperplanes.

This view of the Coxeter group indeed makes sense. Since $m_{ii} = 1$, we see that each generator of the Coxeter group has order 2. This reflects the fact that applying a reflection twice is equivalent to the identity.

Therefore, one can associate a Coxeter group W with a hyperplane arrangement $\mathcal{H}$ where each hyperplane is associated with one of the generators of W.

Example 2.2. Examples of Coxeter groups include the type A Coxeter group, or simply A_n. The group A_n, also called the symmetric group, is defined by the following presentation:

$$A_n = \langle r_1, r_2, \ldots, r_n \mid (r_i r_j)^2 = 1 \text{ for } |i-j| = 1, (r_i r_j)^3 = 1 \text{ for } |i-j| = 2, r_i^2 = 1 \rangle$$

In fact, $B(A_n)$ is equivalent to Artin's braid group on $n+1$ strands, denoted as B_{n+1}.

2.1. **Posets.** A *partially ordered set* is denoted by $\mathcal{P}$. For any subset $X \subseteq \mathcal{P}$, the *join* $\bigvee X$ is defined as the least upper bound of X in $\mathcal{P}$, provided such an element exists and is unique. Conversely, the *meet* $\bigwedge X$ is the greatest lower bound of X in $\mathcal{P}$, again assuming existence and uniqueness.

A *lattice* $\mathcal{L}$ is a poset in which every two elements possess both a greatest lower bound (meet) and a least upper bound (join). An element $j \in \mathcal{P}$ is called *join-irreducible* if it cannot be expressed as the join of strictly smaller elements; equivalently, there is no subset $X \subseteq \mathcal{P}$ with $j \notin X$ such that $j = \bigvee X$.

2.2. **Hyperplane Arrangements.** A *real hyperplane* in $\mathbb{R}^n$ is a linear subspace of codimension one. A *central real hyperplane arrangement* $\mathcal{H}$ is a finite set of such hyperplanes, each passing through the origin. The connected components of the complement $\mathbb{R}^n \setminus \mathcal{H}$ are called *regions*, and the set of all regions will be denoted by $\mathcal{R}$.

A *subarrangement* of $\mathcal{H}$ is any subset of its hyperplanes. The *rank* of a subarrangement is the codimension of the intersection of its hyperplanes. A subarrangement is called *full* if it contains every hyperplane that passes through some fixed subspace of $\mathbb{R}^n$.

2.3. **The Intersection Lattice.** For a real hyperplane arrangement $\mathcal{H}$, the *intersection lattice* $\mathcal{L}$ is the collection of all nonempty intersections of hyperplanes in $\mathcal{H}$, together with the whole space $\mathbb{R}^n$, ordered by reverse inclusion (so $X \leq_{\mathcal{L}} Y$ if and only if $Y \subseteq X$). A *face* of $\mathcal{H}$ is a set $F = C \cap X$ for $C \in \mathcal{R}$ and $X \in \mathcal{L}$, and we denote the set of faces by $\mathcal{F}$. The faces are naturally ordered by reverse inclusion: $F \geq_{\mathcal{F}} F'$ if $F \subseteq F'$. We refer the reader to [Rea03] for additional details.

For convenience, we use the notation from Williams's note [Wil25] to match the notation used for the theorems in later sections.

2.4. **Poset of Regions.** Given a hyperplane arrangement $\mathcal{H}$, its *poset of regions* is defined on its set of regions $\mathcal{R}$.

Fix a base region $B \in \mathcal{R}$. We say that a hyperplane H separates two regions B and C when every straight line connecting an interior point of B to an interior point of C necessarily crosses H. For a given region C, let $S(C)$ denote the set of all hyperplanes that separate C from B.

Let $-B$ denote the unique region for which $S(-B) = \mathcal{H}$.

The family of separating sets $\{S(C) : C \in \mathcal{R}\}$, ordered by inclusion, induces a partial order $\mathcal{P}(\mathcal{H}, B)$ on the set of regions of $\mathcal{H}$. This structure is referred to as the poset of regions of $\mathcal{H}$.

Two regions are called adjacent if they differ by exactly one separating hyperplane. In such a case, if C and D are adjacent, we write $C \xrightarrow{e} D$ when the edge e in the Hasse diagram corresponds to the relation $C < D$ in the partial order.

3. Salvetti Complex and The Fundamental Group of the Complement of the Complexified Hyperplane

Salvetti complex: We follow [Sal87, Del06]. We are only interested in the restriction of the Salvetti complex to the zero, one, and two cells, as these contain sufficient information to determine the fundamental groups. We describe this restriction of the Salvetti complex as a CW complex denoted $\mathcal{P}^*(\mathcal{H}, B)$, which can easily be built from the poset of regions $\mathcal{P}(\mathcal{H}, B)$.

- Zero-cells: The zero cells for $\mathcal{P}^*(\mathcal{H}, B)$ is equivalent to the points in the poset of regions $\mathcal{P}(\mathcal{H}, B)$.

- One-cells: Each edge e in the Hasse diagram of $\mathcal{P}(\mathcal{H}, B)$ with $C' \xrightarrow{e} C$ will be included in $\mathcal{P}^*(\mathcal{H}, B)$ as a one-cell. A second copy e^* will also be included with the opposite orientation where $C \xrightarrow{e^*} C'$. We define a *gallery* from C to C' as a sequence of edges $(e, e^*, e^{-1}, (e^*)^{-1})$ that begins at C and ends at C'. A gallery from C' to C is *positive* if it exclusively uses edges of the form e and e^*, and it is called a *loop* if $C = C'$.
- Two-cells: For each rank-two subarrangement of hyperplanes $\mathcal{A}$ and each region C, we attach one two-cell along the unique positive gallery from C to the corresponding opposite region C' with $S(C, C') = \mathcal{A}$.

The motivation for constructing the two-dimensional CW complex $\mathcal{P}^*(\mathcal{H}, B)$ is that we could generate a combinatorial model for $\mathbb{C}^n \setminus \mathcal{H}$. Therefore, the fundamental group for the hyperplane complement can be expressed as the fundamental group of the CW complex, a result shown below in Theorem 3.1.

Theorem 3.1 ([Sal87])**.** *Let $\mathcal{H}$ be a hyperplane arrangement, then*

$$\pi_1(\mathbb{C}^n \setminus \mathcal{H}, x_B) \cong \pi_1(\mathcal{P}^*(\mathcal{H}, B), B).$$

We call two galleries *homotopic* if one may be converted into the other by a sequence of operations that add or delete the boundary of a two-cell in its entirety. A gallery is termed *minimal* if, among all galleries in its homotopy class, it achieves the least possible length.

Given a step $C' \xrightarrow{e} C$, we write H_e for the hyperplane in the symmetric arrangement uniquely determined by e. For any region C, let $\mathrm{gal}(C)$ denote a choice of minimal gallery from B to C. By [Del72], all positive minimal galleries with the same start and endpoint are homotopic. Hence, the notion of $\mathrm{gal}(C)$ is independent of which minimal gallery is selected.

3.1. **Generators.** We adhere to the approach outlined in [Sal87] and notation similar to [Wil25]. First of all, we will construct a large generating set for $\pi_1(\mathcal{P}^*(\mathcal{H}, B), B)$. If $C' \xrightarrow{e} C$, we define the corresponding loop $\mathbb{t}_e$ by:

$$\mathbb{t}_e := \mathrm{gal}(C')ee^* \,\mathrm{gal}(C')^{-1}.$$

Let $\mathbb{T}_{\mathrm{edge}}$ represent the set of all such loops t_e. Since each $\mathbb{t}_e$ is a loop in $\mathcal{P}^*(\mathcal{H}, B)$ with basepoint at region B, the set $\mathbb{T}_{\mathrm{edge}}$ is a valid generating set for $\pi_1(\mathcal{P}^*(\mathcal{H}, B), B)$.

Next, we will construct a smaller generating set. Fix a positive minimal gallery

$$\mathbf{b} := b_1, b_2, \ldots, b_N$$

from B to $-B$ in $\mathcal{P}^*(\mathcal{H}, B)$. This defines a set of elements in $\pi_1(\mathbb{C}^n \setminus \mathcal{H}_{\mathbb{C}}, x_B) \cong \pi_1(\mathcal{P}^*(\mathcal{H}, B), B)$, denoted by:

$$\mathbb{T}_{\mathbf{b}} = \{\mathbb{b}_i\}_{i=1}^N.$$

where

$$\mathbb{b}_i := (b_1 b_2 \cdots b_{i-1}) b_i b_i^* (b_{i-1}^{-1} \cdots b_2^{-1} b_1^{-1}), \quad 1 \le i \le N,$$

and order the hyperplanes as:

$$H_1 <_b H_2 <_b \cdots <_b H_N$$

where $H_i := H_{b_i}$.

Therefore, Theorem 3.2 shows that the set $\mathbb{T}_{\mathbf{b}}$ is also a valid generating set for $\pi_1(\mathcal{P}^*(\mathcal{H}, B), B)$ as any other $\mathbb{t}_e \in \mathbb{T}_{\mathrm{edge}}$ can be expressed in terms of elements from $\mathbb{T}_{\mathbf{b}}$.

Theorem 3.2 ([Sal87], Lemma 12, Corollary 12)**.** *For any positive minimal gallery $\boldsymbol{b}$ from B to $-B$, $\mathbb{T}_b$ is a generating set of $\pi_1(\mathcal{P}^*(\mathcal{H}, B), B)$. Specifically, if $C \xrightarrow{e} C'$ with $H_e = H_{b_k}$, then*

$$\mathbb{t}_e = \left(\prod_{k > i \ge 1, H_i \notin S(C)} \mathbb{b}_i \right)^{-1} \mathbb{b}_k \left(\prod_{k > i \ge 1, H_i \notin S(C)} \mathbb{b}_i \right).$$

Next, we state a theorem that precisely generates the family of relations for the presentation of the fundamental group with the generators $\mathbb{T}_{\mathbf{b}}$. In fact, one family of relations is needed for each full rank-two subarrangement.

For a full rank-two subarrangement $\mathcal{A}$ with hyperplanes $H_{e_1} <_{\mathbf{b}} \cdots <_{\mathbf{b}} H_{e_M}$, we denote by $[\mathcal{A}]_{\mathbb{T}_{\mathbf{b}}}$ the set of relations

$$\mathbb{t}_{e_M} \cdots \mathbb{t}_{e_2}\mathbb{t}_{e_1} = \mathbb{t}_{e_1}\mathbb{t}_{e_M} \cdots \mathbb{t}_{e_2} = \cdots = \mathbb{t}_{e_{M-1}} \cdots \mathbb{t}_{e_2}\mathbb{t}_{e_1}\mathbb{t}_{e_M}.$$

Theorem 3.3 ([Sal87], Page 616)**.** *Fix a positive minimal gallery* $\mathbf{b}$ *from* B *to* $-B$ *in* $\mathcal{P}^*(\mathcal{H}, B)$*. For each full rank-two subarrangement* $\mathcal{A}$ *of* $\mathcal{H}$*, we can choose one two-cell in* $\mathcal{P}^*(\mathcal{H}, B)$ *with edges labeled*

$$e_1, e_2, \ldots, e_M, e_{M+1}, \ldots, e_{2M}$$

where $\{H_{e_i}\}_{i=1}^{M} = \mathcal{A}$ *and* $H_{e_1} <_{\mathbf{b}} \cdots <_{\mathbf{b}} H_{e_M}$*. Then*

$$\pi_1(\mathcal{P}^*(\mathcal{H}, B), B) = \langle \mathbb{T}_{\mathbf{b}} : [\mathcal{A}]_{\mathbb{T}_{\mathbf{b}}} \rangle,$$

where the relations range over all full rank-two subarrangements $\mathcal{A} \subset \mathcal{H}$*.*

Therefore, in order to use this theorem effectively, one must pick a gallery $\mathbf{b}$ such that the full rank 2 subarrangement can be directly expressed in terms of the generators $\mathbb{T}_{\mathbf{b}}$ without resorting to handling conjugations. In the case of finite Coxeter arrangements, there exists an optimal choice for the positive minimal gallery, which will be presented in Section 5.

4. Full Twist Presentations

Let $\mathcal{H}$ be a central real hyperplane arrangement with base region B. Let $\mathbf{b}$ be a positive minimal gallery from B to $-B$, and let N be the length of $\mathbf{b}$, i.e., the number of hyperplanes of $\mathcal{H}$. For a permutation $\pi : [N] \to [N]$, we say that the product $\prod_{i=1}^{N} \mathbf{b}_{\pi(i)}$ is ***b****-oriented* if for each full rank-two subarrangement $\mathcal{A}$, the restriction of the product to the $\mathbb{b}_i$ corresponding to hyperplanes in $\mathcal{A}$ respects the $\mathbf{b}$-orientation of $\mathcal{A}$, which is the order in which the gallery $\mathbf{b}$ crosses the hyperplanes of $\mathcal{A}$.

Lemma 4.1 ([Wil25], Page 8)**.** *For a fixed positive minimal gallery* ***b*** *and generators* $\mathbb{T}_{\mathbf{b}}$*,*

$$\mathbb{c} = \mathbb{b}_N \mathbb{b}_{N-1} \cdots \mathbb{b}_1.$$

Theorem 4.2 ([Wil25], Page 9)**.** *We have*

$$\prod_{i=1}^{N} \mathbb{b}_{\pi(i)} \in \mathrm{Red}_{\mathbb{T}_{\mathbf{b}}}(\mathbb{c}) \implies \prod_{i=1}^{N} \mathbf{b}_{\pi(i)} \text{ is } \mathbf{b}\text{-oriented.}$$

In [Wil25], Nathan Williams proved the forward direction described in the theorem. However, the backward direction has not yet been proven and could be approached by attempting to prove that any $\mathbf{b}$-oriented word is homotopic to $\mathbb{b}_N \mathbb{b}_{N-1} \cdots \mathbb{b}_1$, which is the full twist by Lemma 4.1.

Again, we can reiterate the generalized version of the conjecture that we desire to prove after defining the full twist. Let $\mathrm{Red}_{\mathbb{T}_{\mathbf{b}}}(\mathbb{c})$ represent the set of minimal length positive words in the generators $\mathbb{T}_{\mathbf{b}}$ for $\mathbb{c}$. Define $[\mathrm{Red}_{\mathbb{T}_{\mathbf{b}}}(\mathbb{c})]$ as setting all the elements of $\mathrm{Red}_{\mathbb{T}_{\mathbf{b}}}(\mathbb{c})$ equal.

Conjecture 4.3 ([Wil25], Page 10)**.** *We have*

$$\pi_1(V_{\mathbb{C}}^{\mathrm{reg}}, x_B) = \langle \mathbb{T}_{\mathbf{b}} : [\mathrm{Red}_{\mathbb{T}_{\mathbf{b}}}(\mathbb{c})] \rangle.$$

5. Noncrossing Subarrangements

We now specialize to the case of finite Coxeter arrangements. Throughout this section, let W be a finite Coxeter group with associated Coxeter arrangement $\mathcal{H}$, and fix a base region B of $\mathcal{H}$.

We adopt notation similar to Williams's note [Wil25] to match the notation for Theorem 5.1.

5.1. **Noncrossing Subarrangements.** The base region B specifies the identity e for the Coxeter group W. The reflections across the bounding hyperplanes of B are the simple reflections S of W, and the reflections in all the hyperplanes of $\mathcal{H}$ are the reflections T of W. If $w \in W$, let $\ell_T(w)$ represent the minimal length of a word for w using the generating set T. For two elements w, $u \in W$, we say $w \leq_T u$ if and only if $\ell_T(u) = \ell_T(w) + \ell_T(w^{-1}u)$. A Coxeter element c is obtained by multiplying together all of the simple reflections in S in some order. Given such a c, an element $w \in W$ is called *c-noncrossing* whenever $w \leq_T c$. Define

$$\mathrm{Fix}(w) = \ker(w - 1)$$

to be the points of $\mathbb{R}^n$ fixed by w. A full subarrangement $\mathcal{A} \subset \mathcal{H}$ is *c-noncrossing* if

$$\bigcap \mathcal{A} = \mathrm{Fix}(w)$$

for some c-noncrossing element w; otherwise, it is *c-crossing*. Write $\mathrm{NC}(\mathcal{H}, c)$ for the poset of noncrossing subarrangements $\mathcal{A}$, ordered by reverse inclusion of $\bigcap \mathcal{A}$.

On the other hand, index the simple reflections of W so that a reduced word for c is $\mathsf{c} = s_1 \cdots s_n$ (our definitions will not depend on the particular reduced word chosen). For any element $w \in W$, Reading defines its *c-sorting word* $\mathsf{w}(\mathsf{c})$ to be the left-most reduced word for w that is a subword of $\mathsf{c}^\infty = s_1 \cdots s_n | s_1 \cdots s_n | \cdots$ [Rea07].

Define $\mathbb{T}_c := \mathbb{T}_\mathbf{b}$ to be the generators of $P(W)$ given by the gallery specified by the c-sorting word $w_\circ(c)$ for the long element. We have the following structural theorem, which will give us control over the c-noncrossing rank-two subarrangements.

Theorem 5.1 ([Wil25], Page 13)**.** *For each c-noncrossing rank-two subarrangement $\mathcal{A}$ of $\mathcal{H}$, there is a choice of two-cells in $\mathcal{P}^*(\mathcal{H}, B)$ with edges labeled as $e_1, e_2, \ldots, e_M, e_{M+1}, \ldots, e_{2M}$, so that $\{H_{e_i}\}_{i=1}^M = \mathcal{A}$, $H_{e_1} <_\mathbf{b} \cdots <_\mathbf{b} H_{e_M}$, and $\mathbb{t}_{e_i} \cong \mathbb{b}_{H_{e_i}}$.*

A consequence of Theorem 5.1 is that for each c-noncrossing rank-two subarrangement $\mathcal{A}$, the relations $[\mathcal{A}]_{\mathbb{T}_c}$ can be easily expressed in the generators $\mathbb{T}_c$ without any conjugations from Theorem 3.2. If $\mathcal{A}$ contains the hyperplanes $H_{i_1} <_c \cdots <_c H_{i_k}$, the relation corresponding to this subarrangement would be

$$\mathbb{t}_{i_k} \cdots \mathbb{t}_{i_2} \mathbb{t}_{i_1} = \mathbb{t}_{i_1} \mathbb{t}_{i_k} \cdots \mathbb{t}_{i_1} \mathbb{t}_{i_2} = \cdots = \mathbb{t}_{i_{k-1}} \cdots \mathbb{t}_{i_1} \mathbb{t}_{i_k}.$$

Therefore, if we use the positive minimal gallery associated with the c-sorting word for the long element, the relations for the c-noncrossing rank two subarrangements are easily produced. However, the c-crossing rank two subarrangements require more computation, as one can witness in the next section.

There is one last important definition and theorem we need to include before the last section.

Definition 5.2. Write $\mathsf{w}(\mathsf{c}) = \mathsf{w}(\mathsf{c})_1 \mathsf{w}(\mathsf{c})_2 \cdots \mathsf{w}(\mathsf{c})_k$ where each $\mathsf{w}(\mathsf{c})_i$ is a subword of c. We call w *c-sortable* if $\mathsf{w}(\mathsf{c})_1 \supseteq \mathsf{w}(\mathsf{c})_2 \supseteq \cdots \supseteq \mathsf{w}(\mathsf{c})_k$.

Theorem 5.3 ([Wil25], Page 14)**.** *For each hyperplane H, there is exactly one join-irreducible $j_H \in \mathrm{Sort}(W, c)$ such that $cov(j_H) = \{H\}$ where $cov(w)$ is the set of hyperplanes covered by w.*

6. Computation for Coxeter Groups

In this section, we carry out explicit calculations for each irreducible Coxeter type that arises in our study. Our goal is to exhibit the generating relations in $P(W)$ corresponding to rank-two subarrangements, distinguishing the noncrossing cases (which follow directly from the c-sorting gallery) from the crossing ones (which require additional analysis).

6.1. **Notation.** For each finite Coxeter group W, we will decide a Coxeter element c with $\mathsf{w}_\circ(\mathsf{c})$ being the c-sorting word for the long element. According to Theorem 5.1, this denotes an order to the reflections of W as

$$t_1 <_c \cdots <_c t_N.$$

With this positive minimal gallery, we can fix the generating set for $P(W)$ as

$$\mathbb{t}_c := \{\mathbb{t}_1, \ldots, \mathbb{t}_N\}.$$

To simplify the notation, we will index our generating set by some totally ordered set I (for example, like the alphabet) and $i \in I$ denotes the element $\mathbb{t}_i$ by $\mathbb{i}$.

Also, for the sake of simplicity and convenience, we will write the relations for the generators of $P(W)$ using the notations

$$[\mathbb{t}_{i_1}\mathbb{t}_{i_2}\cdots\mathbb{t}_{i_k}] := \left(\mathbb{t}_{i_1}\mathbb{t}_{i_2}\cdots\mathbb{t}_{i_k} = \mathbb{t}_{i_2}\cdots\mathbb{t}_{i_k}\mathbb{t}_{i_1} = \cdots = \mathbb{t}_{i_k}\mathbb{t}_{i_1}\cdots\mathbb{t}_{i_{k-1}}\right)$$

$$\mathbb{t}_i^{\mathbb{t}_{i_k}\cdot\dots\cdot\mathbb{t}_{i_1}} := \left(\mathbb{t}_{i_1}^{-1}\cdots\mathbb{t}_{i_k}^{-1}\right)\mathbb{t}_i\left(\mathbb{t}_{i_k}\cdots\mathbb{t}_{i_1}\right).$$

6.2. **Type A.** Let $W(A_{n-1})$ be the symmetric group on n letters, with simple reflections $s_i = (i, i+1)$ for $1 \leq i < n$. A standard choice of Coxeter element is $c = s_{n-1}\cdots s_2 s_1$. This construction produces the c-sorting word of the longest element:

$$\mathsf{w}_\circ(\mathsf{c}) = (s_{n-1}s_{n-2}\cdots s_1)\,(s_{n-1}s_{n-2}\cdots s_2)\ \cdots\ (s_2 s_1)\ (s_1),$$

and the resulting inversion sequence arranges the transpositions in reverse lexicographic order:

$$((n-1)n) <_c ((n-2)n) <_c \cdots <_c (1n) <_c ((n-2)(n-1)) <_c \cdots <_c (12).$$

In type A, we always list the smaller element of each transposition first. Lifting these reflections into the braid group via the successive prefixes of $\mathsf{w}_\circ(\mathsf{c})$ yields

$$\begin{aligned}
((\mathbb{n}-\mathbb{1})\mathbb{n}) &= \mathbf{s}_{n-1}^2,\\
((\mathbb{n}-\mathbb{2})\mathbb{n}) &= \mathbf{s}_{n-1}\,\mathbf{s}_{n-2}^2\,\mathbf{s}_{n-1},\\
&\vdots\\
(\mathbb{1}\mathbb{n}) &= \mathbf{s}_{n-1}\,\mathbf{s}_{n-2}\ \cdots\ \mathbf{s}_1^2\ \cdots\ \mathbf{s}_{n-2}\,\mathbf{s}_{n-1},\\
&\vdots\\
(\mathbb{1}\mathbb{2}) &= \mathbf{s}_1^2.
\end{aligned}$$

6.2.1. *Rank-Two Subarrangements.* The intersection lattice of type A_{n-1} is canonically identified with the lattice of set partitions of $\{1, 2, \ldots, n\}$. A rank-two intersection corresponds either to one block of size three or to two disjoint blocks of size two. If the vertices $1, 2, \ldots, n$ are placed uniformly around a circle in clockwise order, each block can be depicted as the convex hull of its vertices. A subarrangement is called *c-noncrossing* if its blocks' convex hulls do not intersect, and *c-crossing* otherwise (see Figure 1). Changing to a different Coxeter element c' permutes the labels around the circle according to the cycle structure of c', but the noncrossing criterion remains the same when arcs are drawn between the new labels.

We use the notation $\mathcal{N}_c(W, \text{type})$ to denote the set of c-noncrossing full rank-two subarrangements of type W whose underlying root system is of the specified type, and $\mathcal{C}_c(W, \text{type})$ to denote the corresponding set of c-crossing subarrangements.

6.2.2. *Handling c-Crossing Subarrangements.* In type A_{n-1}, any full rank-two subarrangement can be made c'-noncrossing by choosing an appropriate Coxeter element c'—in fact, one may always take c' to be a cyclic rotation of the standard $c = s_{n-1} \cdots s_1$.

To make this precise, for each $1 \le k \le n-1$ define

$$c_k \ := \ s_k \, s_{k-1} \, \cdots \, s_1 \, s_{n-1} \, \cdots \, s_{k+1},$$

so that $c_{n-1} = c$. For each reflection $(rs) \in T$, let

$$J_{rs}^{c_k} \ \in \ \mathrm{Sort}(W, c_k)$$

denote the unique c_k-sortable join-irreducible element whose cover reflection is (rs).

Lemma 6.1. *If* $\{H_{ij}, H_{rs}\} \in \mathcal{C}_c\big(A_{n-1}, A_1 \times A_1\big)$*, then in fact*

$$\{H_{ij}, H_{rs}\} \ \in \ \mathcal{N}_{c_{j-1}}\big(A_{n-1}, A_1 \times A_1\big).$$

Proof. The cycle decomposition of c_{j-1} is

$$(j, n, n-1, \ldots, j+1, j-1, \ldots, 1).$$

Arranging the labels around a circle in this order shows that the chords from i to j and from r to s no longer cross. ☐

Lemma 6.2. *For any reflection* $(rs) \in T$*,*

$$S\big(J_{rs}^{c}\big) \setminus S\big(J_{rs}^{c_{j-1}}\big) \ = \ \begin{cases} \{H_{j\,s}\}, & r < j < s, \\ \varnothing, & \textit{otherwise.} \end{cases}$$

Proof. First suppose $r < j < s$. Then the product

$$(s_{j-1} s_{j-2} \cdots s_r)\,(s_{s-1} s_{s-2} \cdots s_{r+1})$$

is c_{j-1}-sortable, with its unique descent corresponding to the covered reflection (rs); hence it is a reduced expression for $J_{rs}^{c_{j-1}}$. Its inversion set is

$$S\big(J_{rs}^{c_{j-1}}\big) = \{H_{j,j+1}, \ldots, H_{r,j+1}\} \ \cup \ \{H_{s-1,s}, \ldots, H_{j+1,s}\} \ \cup \ \{H_{j-1,s}, \ldots, H_{r,s}\}.$$

On the other hand, one checks that

$$J_{rs}^{c} = s_{s-1} s_{s-2} \cdots s_r,$$

so

$$S\big(J_{rs}^{c}\big) = \{H_{s-1,s}, \ldots, H_{r,s}\}.$$

Therefore, we see that the set difference $S(J_{rs}^{c}) \setminus S(J_{rs}^{c_{j-1}}) = H_{j,s}$.

If j does not lie strictly between r and s, then the same reduced word

$$s_{s-1} s_{s-2} \cdots s_r$$

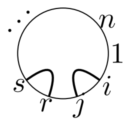
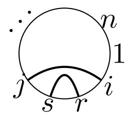
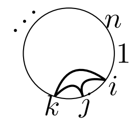
is both c_{j-1} and c-sortable, so $J_{rs}^{c_{j-1}} = J_{rs}^{c}$ and the set difference is empty. ☐

Proposition 6.3. *Fix type* A_{n-1} *and* $c = s_{n-1} s_{n-2} \cdots s_1$*. Every full rank-two subarrangement of the Coxeter arrangement of type* A_{n-1} *lies in one of the sets* $\mathcal{N}_c(A_1 \times A_1)$*,* $\mathcal{N}_c(A_2)$*, or* $\mathcal{C}_c(A_1 \times A_1)$ *given in Figure 1. Note that the relations for the* $\mathcal{C}_c(A_1 \times A_1)$ *are given by 6.1 and 6.2.*

If we recall Theorem 5.1, a direct application of this theorem for the c-noncrossing subarrangement combined with Lemma 6.1 and Lemma 6.2 yields the following presentation for $P(A_{n-1})$.

$$P(A_{n-1}) = \left\langle \ \mathbb{T}_c \ \middle| \ \begin{matrix} [(\mathbb{i}\mathbb{j})(\mathbb{r}\mathbb{s})] & \text{if } \{H_{ij}, H_{rs}\} \in \mathcal{N}_c(A_{n-1}, A_1 \times A_1) \\ [(\mathbb{i}\mathbb{j})(\mathbb{i}\mathbb{k})(\mathbb{j}\mathbb{k})] & \text{if } \{H_{ij}, H_{ik}, H_{jk}\} \in \mathcal{N}_c(A_{n-1}, A_2) \\ [(\mathbb{i}\mathbb{j})(\mathbb{r}\mathbb{s})^{(\mathbb{j}\mathbb{s})}] & \text{if } \{H_{ij}, H_{rs}\} \in \mathcal{C}_c(A_{n-1}, A_1 \times A_1) \end{matrix} \right\rangle.$$

One can easily verify that the c-crossing relation can be written positively to obtain the following presentation.

	Type A_{n-1} c-Noncrossing Full Rank-Two Subarrangements	
Label	$\mathcal{N}_c(A_{n-1}, A_1 \times A_1)$	$\mathcal{N}_c(A_{n-1}, A_2)$
Subarrangement	$\{H_{ij}, H_{rs}\}$	$\{H_{ij}, H_{ik}, H_{jk}\}$
Conditions	$i<j<r<s$ or $i<r<s<j$	$i<j<k$
Picture	 	
Relation	$[(\mathbb{ij})(\mathbb{rs})]$	$[(\mathbb{ij})(\mathbb{ik})(\mathbb{jk})]$

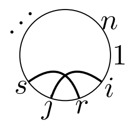
	Type A_{n-1} c-Crossing Full Rank-Two Subarrangements	
Label	$\mathcal{C}_c(A_{n-1}, A_1 \times A_1)$	
Subarrangement	$\{H_{ij}, H_{rs}\}$	
Conditions	$i<r<j<s$	
Picture		
Relation	$[(\mathbb{ij})(\mathbb{rs})^{(\mathbb{js})}]$	

FIGURE 1. The full rank-two subarrangements of type A_{n-1}.

Theorem 6.4. *The pure braid group $P(A_{n-1})$ has the positive presentation:*

$$P(A_{n-1}) = \left\langle \ \mathbb{T}_c \ \middle| \begin{array}{cr} [(\mathbb{ij})(\mathbb{rs})] & \text{if } \{H_{ij}, H_{rs}\} \in \mathcal{N}_c(A_{n-1}, A_1 \times A_1) \\ [(\mathbb{ij})(\mathbb{ik})(\mathbb{jk})] & \text{if } \{H_{ij}, H_{ik}, H_{jk}\} \in \mathcal{N}_c(A_{n-1}, A_2) \\ (\mathbb{ij})(\mathbb{is})(\mathbb{rs})(\mathbb{js}) = (\mathbb{is})(\mathbb{rs})(\mathbb{js})(\mathbb{ij}) & \text{if } \{H_{ij}, H_{rs}\} \in \mathcal{C}_c(A_{n-1}, A_1 \times A_1) \end{array} \right\rangle.$$

Proof. We only have to confirm that the positive c-crossing relations are equivalent to the original c-crossing relations. This follows easily from the computation:

$$\begin{aligned} (\mathbb{ij})\overline{(\mathbb{js})}(\mathbb{rs})(\mathbb{js}) &= \overline{(\mathbb{js})}(\mathbb{rs})(\mathbb{js})(\mathbb{ij}) \\ (\mathbb{js})(\mathbb{ij})\overline{(\mathbb{js})}(\mathbb{rs})(\mathbb{js}) &= (\mathbb{rs})(\mathbb{js})(\mathbb{ij}) \\ (\mathbb{is})(\mathbb{js})(\mathbb{ij})\overline{(\mathbb{js})}(\mathbb{rs})(\mathbb{js}) &= (\mathbb{is})(\mathbb{rs})(\mathbb{js})(\mathbb{ij}) \\ (\mathbb{ij})(\mathbb{is})(\mathbb{rs})(\mathbb{js}) &= (\mathbb{is})(\mathbb{rs})(\mathbb{js})(\mathbb{ij}). \end{aligned}$$

The last step invokes a relation of type $\mathcal{N}_c(A_{n-1}, A_2)$. □

6.3. **Type B.** We will denote the n^2 reflections of type B_n in cycle notation as

$$(\!(i,j)\!) := (i,j)(\bar{i},\bar{j}) \text{ for } 1 \leq i < j \leq n \text{ and}$$
$$(i,\bar{i}) \text{ for } 1 \leq i \leq n.$$

We write $s_i = (\!(i,i+1)\!)$ for $1 \leq i < n$ and $s_n := (n,\overline{n})$ and choose the Coxeter element $c = s_1 s_2 \cdots s_n$, which yields the c-sorting word for the long element.

$$\mathsf{w}_\circ(\mathsf{c}) = (s_1 s_2 \cdots s_n)^n$$

and the corresponding inversion sequence orders the reflections in the order

$$(\!(1,2)\!) <_c (\!(1,3)\!) <_c \cdots <_c (\!(1,n)\!) <_c (\!(1,\overline{1})\!) <_c$$

$$(\!(2,3)\!) <_c (\!(2,4)\!) <_c \cdots (\!(2,\overline{1})\!) <_c (2,\overline{2}) <_c \cdots <_c$$
$$(\!(n,\overline{1})\!) <_c (\!(n,\overline{2})\!) <_c \cdots <_c (n,\overline{n}).$$

6.3.1. *Rank-two Subarrangements.* The type B_n intersection lattice is isomorphic to the lattice of centrally symmetric set partitions on $\{1, 2, \dots, n, \overline{1}, \overline{2}, \dots, \overline{n}\}$, with at most one block B with $B = \overline{B}$ [Rei97, AR04]. Therefore, a rank-two intersection corresponds to a centrally symmetric set partition with two blocks of size three, four blocks of size two, or one block of size four.

If we space $2n$ vertices regularly around a circle, numbered clockwise from 1 up to n and then from $\overline{1}$ to $\overline{n}$, blocks can be represented by the convex hull of the vertices they contain. A c-noncrossing subarrangement is one whose blocks do not intersect, while a c-crossing subarrangement will have intersecting blocks. See Figures 2 and 3. We will characterize all full rank-two subarrangements in type B_n as presenting in the following .

Proposition 6.5. *Fix type B_{n-1} and $c = s_1 s_2 \cdots s_n$. Every full rank-two subarrangement of the Coxeter arrangement of type B_n lies in one of the sets given in Figures 2 and 3.*

6.3.2. *Dealing with c-Crossing Subarrangements.* In type B, the only rank-two subarrangements that are not c'-noncrossing for some choice of Coxeter element c' are those of the form $\mathcal{C}_c(B_n, A_1 \times A_1, 2A)$. For all other c-crossing subarrangements, we can take c' to be a cyclic rotation of $c = s_1 \cdots s_n$.

Define the Coxeter element $c_a := s_a s_{a+1} \cdots s_n s_1 \cdots s_{a-1}$ so that $c_1 = c$. For each c_a, let $J^{c_a}_{rs} \in \mathrm{Sort}(W, c_a)$ be the unique c_a-sortable join-irreducible element whose cover reflection is $(\!(rs)\!)$, and similarly for $J^{c_a}_{r\overline{s}} = J^{c_a}_{\overline{r}s}$ and $J^{c_a}_{s\overline{s}}$.

Lemma 6.6. *Each c-crossing subarrangement is noncrossing for some choice of c_a:*

- $\{H_{ij}, H_{rs}\} \in \mathcal{C}_c(B_n, A_1 \times A_1, 1A)$ *is noncrossing for* $c_{|r|}$ *and*
$$S(J^{c_{|r|}}_{ij}) \setminus S(J^c_{ij}) = \{H_{ir}, H_{r\overline{j}}\},$$
$$S(J^{c_{|r|}}_{rs}) \setminus S(J^c_{rs}) = \emptyset.$$
- $\{H_{ij}, H_{rs}\} \in \mathcal{C}_c(B_n, A_1 \times A_1, 1A')$ *is noncrossing for* $c_{|j|}$ *and*
$$S(J^{c_{|j|}}_{ij}) \setminus S(J^c_{ij}) = \emptyset,$$
$$S(J^{c_{|j|}}_{rs}) \setminus S(J^c_{rs}) = \{H_{sj}, H_{r\overline{j}}\}.$$
- $\{H_{ij}, H_{rs}\} \in \mathcal{C}_c(B_n, A_1 \times A_1, 1B)$ *is noncrossing for* $c_{|r|}$ *and*
$$S(J^{c_{|r|}}_{ij}) \setminus S(J^c_{ij}) = \{H_{ir}\},$$
$$S(J^{c_{|r|}}_{rs}) \setminus S(J^c_{rs}) = \emptyset.$$
- $\{H_{ij}, H_{rs}\} \in \mathcal{C}_c(B_n, A_1 \times A_1, 1B')$ *is noncrossing for* $c_{|r|}$ *and*
$$S(J^{c_{|r|}}_{ij}) \setminus S(J^c_{ij}) = \{H_{ir}\},$$
$$S(J^{c_{|r|}}_{rs}) \setminus S(J^c_{rs}) = \emptyset.$$
- $\{H_{ij}, H_{rs}\} \in \mathcal{C}_c(B_n, A_1 \times A_1, 1C)$ *is noncrossing for* $c_{|j|}$ *and*
$$S(J^{c_{|j|}}_{ij}) \setminus S(J^c_{ij}) = \{H_{i\overline{j}}, H_{i\overline{i}}\},$$
$$S(J^{c_{|j|}}_{rs}) \setminus S(J^c_{rs}) = \{H_{rj}\}.$$
- $\{H_{ij}, H_{rs}\} \in \mathcal{C}_c(B_n, A_1 \times A_1, 1C')$ *is noncrossing for* $c_{|r|}$ *and*
$$S(J^{c_{|r|}}_{ij}) \setminus S(J^c_{ij}) = \{H_{i\overline{r}}\},$$
$$S(J^{c_{|r|}}_{rs}) \setminus S(J^c_{rs}) = \emptyset.$$

Type B_n c-Noncrossing Full Rank-Two Subarrangements

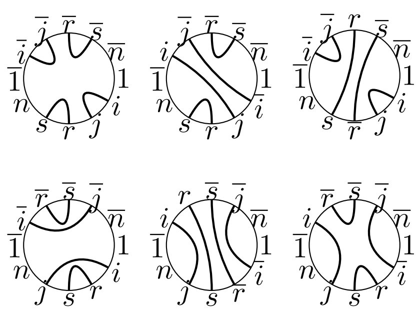
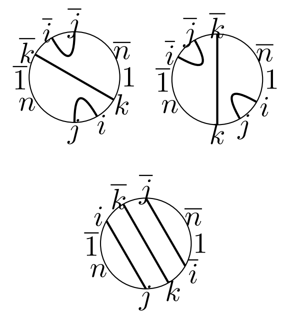
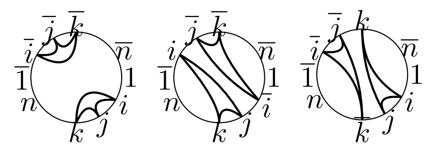
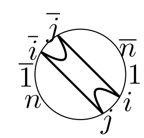
Label	$\mathcal{N}_c(B_n, A_1{\times}A_1, 1)$	$\mathcal{N}_c(B_n, A_1{\times}A_1, 2)$	$\mathcal{N}_c(B_n, A_2)$	$\mathcal{N}_c(B_n, B_2)$
Subarrangement	$\{H_{ij}, H_{rs}\}$	$\{H_{ij}, H_{k\bar{k}}\}$	$\{H_{ij}, H_{ik}, H_{jk}\}$	$\left\{\begin{matrix} H_{ij}, H_{i\bar{i}}, \\ H_{\bar{i}j}, H_{j\bar{j}} \end{matrix}\right\}$
Conditions	$\lvert i\rvert<\lvert j\rvert<\lvert r\rvert<\lvert s\rvert$ and $I=J=R=S$, $\bar{I}=J=\bar{R}=S$ or $I=J=\bar{R}=S$ $\lvert i\rvert<\lvert r\rvert<\lvert s\rvert<\lvert j\rvert$ and $I=J=R=S$, $\bar{I}=J=\bar{R}=S$ or $\bar{I}=J=R=S$	$\lvert i\rvert<\lvert j\rvert<\lvert k\rvert$, $I=J$; $\lvert k\rvert<\lvert i\rvert<\lvert j\rvert$, $I=J$ $\lvert i\rvert<\lvert k\rvert<\lvert j\rvert$, $I=\bar{J}$	$\lvert i\rvert<\lvert j\rvert<\lvert k\rvert$ and $I=J=K$, $I=\bar{J}=\bar{K}$ or $I=J=\bar{K}$	None
Picture				
Relation	$[((\mathbb{i}\mathbb{j}))((\mathbb{r}\mathbb{s}))]$	$[((\mathbb{i}\mathbb{j}))((\mathbb{k}\bar{\mathbb{k}}))]$	$[((\mathbb{j}\mathbb{k}))((\mathbb{i}\mathbb{k}))((\mathbb{i}\mathbb{j}))]$	$[((\mathbb{j}\bar{\mathbb{j}}))((\bar{\mathbb{i}}\mathbb{j}))((\mathbb{i}\bar{\mathbb{i}}))((\mathbb{i}\mathbb{j}))]$

FIGURE 2. The full rank-two c-noncrossing subarrangements of type B_n. We use capital letters to mean the sign of the corresponding letter (for example, $I = \text{sign}(i)$ and $\overline{I} = -\text{sign}(i)$). Observe that i, j, r, s, and k do not necessarily stand for positive numbers, and that $\bar{i}$, $\bar{j}$, $\bar{r}$, $\bar{s}$, and $\bar{k}$ do not necessarily stand for negative numbers.

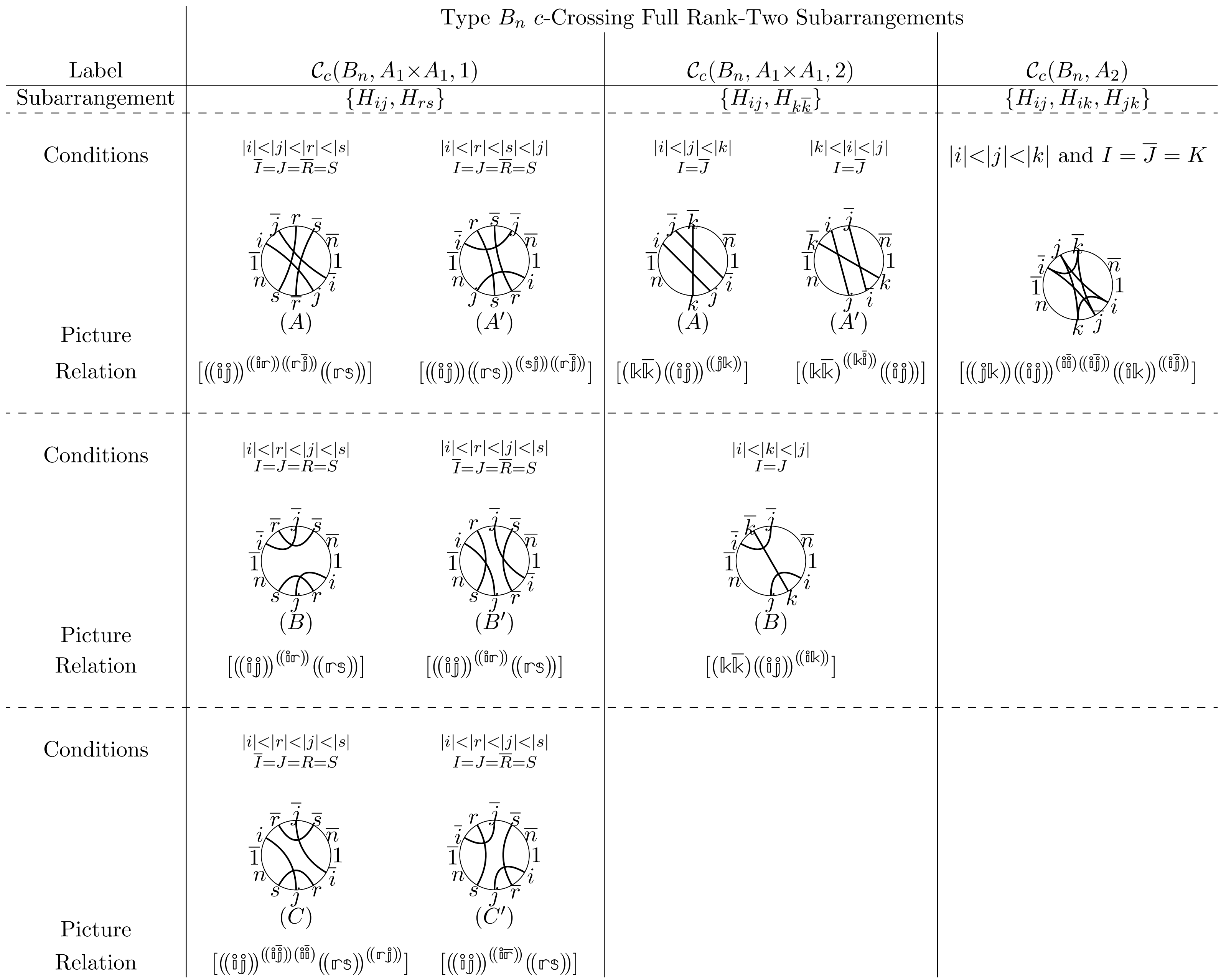


FIGURE 3. The full rank-two c-crossing subarrangements of type B_n. We use capital letters to mean the sign of the corresponding letter (for example, $I = \text{sign}(i)$ and $\bar{I} = -\text{sign}(i)$). Observe that i, j, r, s, and k do not necessarily stand for positive numbers, and that $\bar{i}$, $\bar{j}$, $\bar{r}$, $\bar{s}$, and $\bar{k}$ do not necessarily stand for negative numbers.

- $\{H_{ij}, H_{k\overline{k}}\} \in \mathcal{C}_c(B_n, A_1{\times}A_1, 2A')$ *is noncrossing for* $c_{|i|}$ *and*

$$S(J_{k\overline{k}}^{c_{|i|}}) \setminus S(J_{k\overline{k}}^{c}) = \{H_{k\overline{i}}\},$$
$$S(J_{ij}^{c_{|i|}}) \setminus S(J_{ij}^{c}) = \emptyset\}.$$

- $\{H_{ij}, H_{k\overline{k}}\} \in \mathcal{C}_c(B_n, A_1{\times}A_1, 2B)$ *is noncrossing for* $c_{|k|}$ *and*

$$S(J_{k\overline{k}}^{c_{|k|}}) \setminus S(J_{k\overline{k}}^{c}) = \emptyset,$$
$$S(J_{ij}^{c_{|k|}}) \setminus S(J_{ij}^{c}) = \{H_{ik}\}.$$

- $\{H_{ij}, H_{ik}, H_{jk}\} \in \mathcal{C}_c(B_n, A_2)$ *is noncrossing for* $c_{|j|}$ *and*

$$S(J_{jk}^{c_{|j|}}) \setminus S(J_{jk}^{c}) = \emptyset,$$
$$S(J_{ij}^{c_{|j|}}) \setminus S(J_{ij}^{c}) = \{H_{i\overline{i}}, H_{i\overline{j}}\}$$
$$S(J_{ik}^{c_{|j|}}) \setminus S(J_{ik}^{c}) = \{H_{i\overline{j}}\}.$$

Proof. The cycle notation of c_k labels the circle clockwise by the numbers $k, 1, 2, \ldots, k-1, k+1, \ldots, n, \overline{k}, \overline{1}, \overline{2}, \ldots, \overline{k-1}, \overline{k+1}, \ldots, \overline{n}$. One checks that the partitions given by the subarrangements in each case above are noncrossing with respect to this order.

To confirm the statements about the differences in inversion sets, it suffices to explicitly identify the sorting words for the c_a-sortable and c-sortable join irreducibles. This is a straightforward but tedious process. We will only present two cases where the other cases are left to interested readers.

- $\{H_{ij}, H_{rs}\} \in \mathcal{C}_c(B_n, A_1{\times}A_1, 1A)$

$$\begin{aligned} J_{ij}^{c_{|r|}} &= s_{|r|}s_{|r|+1}\cdots s_n \cdot \left(s_{|i|}s_{|i|+1}\cdots s_n \cdot s_{|j|-1}s_{|j|}\cdots s_{n-1}\right) \\ J_{ij}^{c} &= s_{|i|}s_{|i|+1}\cdots s_n \cdot s_{|j|-1}s_{|j|}\cdots s_{n-1} \\ J_{rs}^{c_{|r|}} = J_{rs}^{c} &= s_{|r|}s_{|r|+1}\cdots s_n \cdot s_{|s|-1}s_{|s|}\cdots s_{n-1}. \end{aligned}$$

  One checks that the two inversions in J_{ij}^{c} not in $J_{ij}^{c_{|r|}}$ correspond to the first copy of the simple reflection $s_{|r|-1}$ and the second copy of $s_{|r|-2}$ in the c-sorting word for J_{ij}^{c}.

- $\{H_{ij}, H_{rs}\} \in \mathcal{C}_c(B_n, A_1{\times}A_1, 1A')$

$$\begin{aligned} J_{ij}^{c_{|j|}} = J_{ij}^{c} &= s_i\cdots s_{j-1} \\ J_{rs}^{c_{|j|}} &= s_{|j|}s_{|j|+1}\cdots s_n \cdot s_{|r|}s_{|r|+1}\cdots s_n \cdot s_{|s|-1}s_{|s|}\cdots s_{n-1} \\ J_{rs}^{c} &= s_{|r|}s_{|r|+1}\cdots s_n \cdot s_{|s|-1}s_{|s|}\cdots s_{n-1}. \end{aligned}$$

  One checks that the two inversions in J_{rs}^{c} not in $J_{rs}^{c_{|j|}}$ correspond to the first copy of the simple reflection $s_{|j|-1}$ and the second copy of $s_{|j|-2}$ in the c-sorting word for J_{rs}^{c}.

□

6.3.3. *Presentations.*

Theorem 6.7. *A presentation for $P(B_n)$ is given by $\mathbb{t}_c$ subject to the four classes of c-noncrossing relations in Figure 2 and the ten classes of c-crossing relations in Figure 3.*

In fact, it is possible to show that all the c-crossing relations can be rewritten positively. We will show the derivation for positive relations for $\mathcal{C}_c(B_n, A_1{\times}A_1, 2B)$ and $\mathcal{C}_c(B_n, A_2)$, where the others are similar. For $\mathcal{C}_c(B_n, A_1{\times}A_1, 2B)$ we can compute:

$$(\mathbb{k}\overline{\mathbb{k}})\overline{((\mathbb{i}\mathbb{k}))}((\mathbb{i}\mathbb{j}))((\mathbb{i}\mathbb{k})) = \overline{((\mathbb{i}\mathbb{k}))}((\mathbb{i}\mathbb{j}))((\mathbb{i}\mathbb{k}))(\mathbb{k}\overline{\mathbb{k}})$$
$$((\mathbb{i}\overline{\mathbb{k}}))(\mathbb{i}\overline{\mathbb{i}})((\mathbb{i}\mathbb{k}))(\mathbb{k}\overline{\mathbb{k}})\overline{((\mathbb{i}\mathbb{k}))}((\mathbb{i}\mathbb{j}))((\mathbb{i}\mathbb{k})) = ((\mathbb{i}\overline{\mathbb{k}}))(\mathbb{i}\overline{\mathbb{i}})((\mathbb{i}\mathbb{j}))((\mathbb{i}\mathbb{k}))(\mathbb{k}\overline{\mathbb{k}})$$

$$(\mathbb{k}\bar{\mathbb{k}})(\!(\mathbb{i}\bar{\mathbb{k}})\!)(\mathbb{i}\bar{\mathbb{i}})(\!(\mathbb{i}\mathbb{j})\!)(\!(\mathbb{i}\mathbb{k})\!) = (\!(\mathbb{i}\bar{\mathbb{k}})\!)(\mathbb{i}\bar{\mathbb{i}})(\!(\mathbb{i}\mathbb{j})\!)(\!(\mathbb{i}\mathbb{k})\!)(\mathbb{k}\bar{\mathbb{k}}),$$

where we used a relation of type $\mathcal{N}_c(B_n, B_2)$. Then for $\mathcal{C}_c(B_n, A_2)$ we have

$$\begin{aligned}
&(\!(\mathbb{j}\mathbb{k})\!)\overline{(\!(\mathbb{i}\bar{\mathbb{j}})\!)(\mathbb{i}\bar{\mathbb{i}})}(\!(\mathbb{i}\mathbb{j})\!)(\mathbb{i}\bar{\mathbb{i}})(\!(\mathbb{i}\bar{\mathbb{j}})\!)\overline{(\!(\mathbb{i}\bar{\mathbb{j}})\!)}(\!(\mathbb{i}\mathbb{k})\!)(\!(\mathbb{i}\bar{\mathbb{j}})\!) = \overline{(\!(\mathbb{i}\bar{\mathbb{j}})\!)(\mathbb{i}\bar{\mathbb{i}})}(\!(\mathbb{i}\mathbb{j})\!)(\mathbb{i}\bar{\mathbb{i}})(\!(\mathbb{i}\bar{\mathbb{j}})\!)\overline{(\!(\mathbb{i}\bar{\mathbb{j}})\!)}(\!(\mathbb{i}\mathbb{k})\!)(\!(\mathbb{i}\bar{\mathbb{j}})\!)(\!(\mathbb{j}\mathbb{k})\!) = \overline{(\!(\mathbb{i}\bar{\mathbb{j}})\!)}(\!(\mathbb{i}\mathbb{k})\!)(\!(\mathbb{i}\bar{\mathbb{j}})\!)(\!(\mathbb{j}\mathbb{k})\!)\overline{(\!(\mathbb{i}\bar{\mathbb{j}})\!)(\mathbb{i}\bar{\mathbb{i}})}(\!(\mathbb{i}\mathbb{j})\!)(\mathbb{i}\bar{\mathbb{i}})(\!(\mathbb{i}\bar{\mathbb{j}})\!)\\
&(\!(\mathbb{i}\bar{\mathbb{j}})\!)(\!(\mathbb{j}\mathbb{k})\!)\overline{(\!(\mathbb{i}\bar{\mathbb{j}})\!)(\mathbb{i}\bar{\mathbb{i}})}(\!(\mathbb{i}\mathbb{j})\!)(\mathbb{i}\bar{\mathbb{i}})(\!(\mathbb{i}\mathbb{k})\!)(\!(\mathbb{i}\bar{\mathbb{j}})\!) = \overline{(\mathbb{i}\bar{\mathbb{i}})}(\!(\mathbb{i}\mathbb{j})\!)(\mathbb{i}\bar{\mathbb{i}})(\!(\mathbb{i}\mathbb{k})\!)(\!(\mathbb{i}\bar{\mathbb{j}})\!)(\!(\mathbb{j}\mathbb{k})\!) = (\!(\mathbb{i}\mathbb{k})\!)(\!(\mathbb{i}\bar{\mathbb{j}})\!)(\!(\mathbb{j}\mathbb{k})\!)\overline{(\!(\mathbb{i}\bar{\mathbb{j}})\!)(\mathbb{i}\bar{\mathbb{i}})}(\!(\mathbb{i}\mathbb{j})\!)(\mathbb{i}\bar{\mathbb{i}})(\!(\mathbb{i}\bar{\mathbb{j}})\!)\\
&(\!(\mathbb{j}\mathbb{k})\!)(\!(\mathbb{i}\bar{\mathbb{k}})\!)\overline{(\mathbb{i}\bar{\mathbb{i}})}(\!(\mathbb{i}\mathbb{j})\!)(\mathbb{i}\bar{\mathbb{i}})(\!(\mathbb{i}\mathbb{k})\!)(\!(\mathbb{i}\bar{\mathbb{j}})\!) = (\!(\mathbb{i}\bar{\mathbb{k}})\!)\overline{(\mathbb{i}\bar{\mathbb{i}})}(\!(\mathbb{i}\mathbb{j})\!)(\mathbb{i}\bar{\mathbb{i}})(\!(\mathbb{i}\mathbb{k})\!)(\!(\mathbb{i}\bar{\mathbb{j}})\!)(\!(\mathbb{j}\mathbb{k})\!) = (\!(\mathbb{i}\bar{\mathbb{k}})\!)(\!(\mathbb{i}\mathbb{k})\!)(\!(\mathbb{i}\bar{\mathbb{j}})\!)(\!(\mathbb{j}\mathbb{k})\!)\overline{(\!(\mathbb{i}\bar{\mathbb{j}})\!)(\mathbb{i}\bar{\mathbb{i}})}(\!(\mathbb{i}\mathbb{j})\!)(\mathbb{i}\bar{\mathbb{i}})(\!(\mathbb{i}\bar{\mathbb{j}})\!)\\
&(\!(\mathbb{j}\mathbb{k})\!)(\!(\mathbb{i}\bar{\mathbb{k}})\!)\overline{(\mathbb{i}\bar{\mathbb{i}})}(\!(\mathbb{i}\mathbb{j})\!)(\mathbb{i}\bar{\mathbb{i}})(\!(\mathbb{i}\mathbb{k})\!)(\!(\mathbb{i}\bar{\mathbb{j}})\!)(\mathbb{j}\bar{\mathbb{j}}) = (\!(\mathbb{i}\bar{\mathbb{k}})\!)\overline{(\mathbb{i}\bar{\mathbb{i}})}(\!(\mathbb{i}\mathbb{j})\!)(\mathbb{i}\bar{\mathbb{i}})(\!(\mathbb{i}\mathbb{k})\!)(\!(\mathbb{i}\bar{\mathbb{j}})\!)(\!(\mathbb{j}\mathbb{k})\!)(\mathbb{j}\bar{\mathbb{j}}) = (\!(\mathbb{i}\bar{\mathbb{k}})\!)(\!(\mathbb{i}\mathbb{k})\!)(\!(\mathbb{i}\bar{\mathbb{j}})\!)(\!(\mathbb{j}\mathbb{k})\!)\overline{(\!(\mathbb{i}\bar{\mathbb{j}})\!)(\mathbb{i}\bar{\mathbb{i}})}(\!(\mathbb{i}\mathbb{j})\!)(\mathbb{i}\bar{\mathbb{i}})(\!(\mathbb{i}\bar{\mathbb{j}})\!)(\mathbb{j}\bar{\mathbb{j}})\\
&(\!(\mathbb{j}\mathbb{k})\!)(\!(\mathbb{i}\bar{\mathbb{k}})\!)\overline{(\mathbb{i}\bar{\mathbb{i}})}(\!(\mathbb{i}\mathbb{j})\!)(\mathbb{i}\bar{\mathbb{i}})(\!(\mathbb{i}\mathbb{k})\!)(\!(\mathbb{i}\bar{\mathbb{j}})\!)(\mathbb{j}\bar{\mathbb{j}}) = (\!(\mathbb{i}\bar{\mathbb{k}})\!)\overline{(\mathbb{i}\bar{\mathbb{i}})}(\!(\mathbb{i}\mathbb{j})\!)(\mathbb{i}\bar{\mathbb{i}})(\!(\mathbb{i}\mathbb{k})\!)(\!(\mathbb{i}\bar{\mathbb{j}})\!)(\!(\mathbb{j}\mathbb{k})\!)(\mathbb{j}\bar{\mathbb{j}}) = (\!(\mathbb{i}\bar{\mathbb{k}})\!)(\!(\mathbb{i}\mathbb{k})\!)(\!(\mathbb{i}\bar{\mathbb{j}})\!)(\!(\mathbb{j}\mathbb{k})\!)(\mathbb{j}\bar{\mathbb{j}})(\!(\mathbb{i}\mathbb{j})\!)\\
&(\!(\mathbb{j}\mathbb{k})\!)(\!(\mathbb{i}\bar{\mathbb{k}})\!)\overline{(\mathbb{i}\bar{\mathbb{i}})}(\mathbb{j}\bar{\mathbb{j}})(\!(\mathbb{i}\mathbb{j})\!)(\mathbb{i}\bar{\mathbb{i}})(\!(\mathbb{i}\mathbb{k})\!)(\!(\mathbb{i}\bar{\mathbb{j}})\!)(\!(\mathbb{j}\bar{\mathbb{k}})\!)(\mathbb{k}\bar{\mathbb{k}}) = (\!(\mathbb{i}\bar{\mathbb{k}})\!)\overline{(\mathbb{i}\bar{\mathbb{i}})}(\!(\mathbb{i}\mathbb{j})\!)(\mathbb{i}\bar{\mathbb{i}})(\!(\mathbb{i}\mathbb{k})\!)(\!(\mathbb{i}\bar{\mathbb{j}})\!)(\!(\mathbb{j}\mathbb{k})\!)(\mathbb{j}\bar{\mathbb{j}})(\!(\mathbb{j}\bar{\mathbb{k}})\!)(\mathbb{k}\bar{\mathbb{k}}) = (\!(\mathbb{i}\bar{\mathbb{k}})\!)(\!(\mathbb{i}\mathbb{k})\!)(\!(\mathbb{i}\bar{\mathbb{j}})\!)(\!(\mathbb{j}\mathbb{k})\!)(\mathbb{j}\bar{\mathbb{j}})(\!(\mathbb{i}\mathbb{j})\!)(\!(\mathbb{j}\bar{\mathbb{k}})\!)(\mathbb{k}\bar{\mathbb{k}})\\
&(\!(\mathbb{j}\mathbb{k})\!)(\!(\mathbb{i}\bar{\mathbb{k}})\!)\overline{(\mathbb{i}\bar{\mathbb{i}})}(\mathbb{j}\bar{\mathbb{j}})(\!(\mathbb{i}\mathbb{j})\!)(\mathbb{i}\bar{\mathbb{i}})(\!(\mathbb{i}\bar{\mathbb{j}})\!)(\!(\mathbb{j}\bar{\mathbb{k}})\!)(\!(\mathbb{i}\mathbb{k})\!)(\mathbb{k}\bar{\mathbb{k}}) = (\!(\mathbb{i}\bar{\mathbb{k}})\!)\overline{(\mathbb{i}\bar{\mathbb{i}})}(\!(\mathbb{i}\mathbb{j})\!)(\mathbb{i}\bar{\mathbb{i}})(\!(\mathbb{i}\mathbb{k})\!)(\!(\mathbb{i}\bar{\mathbb{j}})\!)(\mathbb{j}\bar{\mathbb{j}})(\!(\mathbb{j}\bar{\mathbb{k}})\!)(\mathbb{k}\bar{\mathbb{k}})(\!(\mathbb{j}\mathbb{k})\!) = (\!(\mathbb{i}\bar{\mathbb{k}})\!)(\!(\mathbb{i}\mathbb{k})\!)(\!(\mathbb{i}\bar{\mathbb{j}})\!)(\!(\mathbb{j}\mathbb{k})\!)(\mathbb{j}\bar{\mathbb{j}})(\!(\mathbb{i}\mathbb{j})\!)(\!(\mathbb{j}\bar{\mathbb{k}})\!)(\mathbb{k}\bar{\mathbb{k}})\\
&(\!(\mathbb{j}\mathbb{k})\!)(\!(\mathbb{i}\bar{\mathbb{k}})\!)(\!(\mathbb{i}\bar{\mathbb{j}})\!)(\mathbb{j}\bar{\mathbb{j}})(\!(\mathbb{i}\mathbb{j})\!)(\!(\mathbb{j}\bar{\mathbb{k}})\!)(\!(\mathbb{i}\mathbb{k})\!)(\mathbb{k}\bar{\mathbb{k}}) = (\!(\mathbb{i}\bar{\mathbb{k}})\!)\overline{(\mathbb{i}\bar{\mathbb{i}})}(\mathbb{j}\bar{\mathbb{j}})(\!(\mathbb{i}\mathbb{j})\!)(\mathbb{i}\bar{\mathbb{i}})(\!(\mathbb{i}\mathbb{k})\!)(\!(\mathbb{i}\bar{\mathbb{j}})\!)(\!(\mathbb{j}\bar{\mathbb{k}})\!)(\mathbb{k}\bar{\mathbb{k}})(\!(\mathbb{j}\mathbb{k})\!) = (\!(\mathbb{i}\bar{\mathbb{k}})\!)(\!(\mathbb{i}\mathbb{k})\!)(\!(\mathbb{i}\bar{\mathbb{j}})\!)(\!(\mathbb{j}\mathbb{k})\!)(\mathbb{j}\bar{\mathbb{j}})(\!(\mathbb{i}\mathbb{j})\!)(\!(\mathbb{j}\bar{\mathbb{k}})\!)(\mathbb{k}\bar{\mathbb{k}})\\
&(\!(\mathbb{i}\bar{\mathbb{k}})\!)(\!(\mathbb{i}\bar{\mathbb{j}})\!)(\!(\mathbb{j}\mathbb{k})\!)(\mathbb{j}\bar{\mathbb{j}})(\!(\mathbb{i}\mathbb{j})\!)(\!(\mathbb{j}\bar{\mathbb{k}})\!)(\!(\mathbb{i}\mathbb{k})\!)(\mathbb{k}\bar{\mathbb{k}}) = (\!(\mathbb{i}\bar{\mathbb{k}})\!)\overline{(\mathbb{i}\bar{\mathbb{i}})}(\mathbb{j}\bar{\mathbb{j}})(\!(\mathbb{i}\mathbb{j})\!)(\mathbb{i}\bar{\mathbb{i}})(\!(\mathbb{i}\bar{\mathbb{j}})\!)(\!(\mathbb{j}\bar{\mathbb{k}})\!)(\!(\mathbb{i}\mathbb{k})\!)(\mathbb{k}\bar{\mathbb{k}})(\!(\mathbb{j}\mathbb{k})\!) = (\!(\mathbb{i}\bar{\mathbb{k}})\!)(\!(\mathbb{i}\mathbb{k})\!)(\!(\mathbb{i}\bar{\mathbb{j}})\!)(\!(\mathbb{j}\mathbb{k})\!)(\mathbb{j}\bar{\mathbb{j}})(\!(\mathbb{i}\mathbb{j})\!)(\!(\mathbb{j}\bar{\mathbb{k}})\!)(\mathbb{k}\bar{\mathbb{k}})\\
&(\!(\mathbb{i}\bar{\mathbb{j}})\!)(\!(\mathbb{j}\mathbb{k})\!)(\mathbb{j}\bar{\mathbb{j}})(\!(\mathbb{i}\mathbb{j})\!)(\!(\mathbb{j}\bar{\mathbb{k}})\!)(\!(\mathbb{i}\mathbb{k})\!)(\mathbb{k}\bar{\mathbb{k}}) = (\!(\mathbb{i}\bar{\mathbb{j}})\!)(\mathbb{j}\bar{\mathbb{j}})(\!(\mathbb{i}\mathbb{j})\!)(\!(\mathbb{j}\bar{\mathbb{k}})\!)(\!(\mathbb{i}\mathbb{k})\!)(\mathbb{k}\bar{\mathbb{k}})(\!(\mathbb{j}\mathbb{k})\!) = (\!(\mathbb{i}\mathbb{k})\!)(\!(\mathbb{i}\bar{\mathbb{j}})\!)(\!(\mathbb{j}\mathbb{k})\!)(\mathbb{j}\bar{\mathbb{j}})(\!(\mathbb{i}\mathbb{j})\!)(\!(\mathbb{j}\bar{\mathbb{k}})\!)(\mathbb{k}\bar{\mathbb{k}}).
\end{aligned}$$

In this computation, we have used relations of types $\mathcal{N}_c(B_n, A_2)$, $\mathcal{N}_c(B_n, B_2)$, and $\mathcal{C}_c(B_n, A_1\times A_1, 2B)$.

6.4. **Type D.** We denote the $n(n-1)$ reflections of type D_n in cycle notation by

$$(\!(i,j)\!) \;=\; (i,j)(\bar{i},\bar{j}), \qquad (\!(i,\bar{j})\!) \;=\; (i,\bar{j})(\bar{i},j), \quad 1 \le i < j \le n.$$

We write $s_i = (\!(i, i+1)\!)$ for $1 \le i < n$ and $s_{n-1} = (\!(n-1, \bar{n})\!)$. We choose the Coxeter element $c = s_1 s_2 s_3 \cdots s_n$, which yields the c-sorting word for the long element:

$$w_\circ(c) = (s_1 s_2 \cdots s_n)^{n-1}.$$

The corresponding inversion sequence orders the reflections as

$$\begin{aligned}
(\!(1,2)\!) <_c (\!(1,3)\!) <_c \cdots <_c (\!(1,n)\!) <_c (\!(2,3)\!) <_c (\!(2,4)\!) <_c \cdots <_c (\!(n-1,n)\!)\\
<_c (\!(1,\bar{2})\!) <_c (\!(1,\bar{3})\!) <_c \cdots <_c (\!(n-1,\bar{n})\!).
\end{aligned}$$

6.4.1. *Rank-two Subarrangements.* The type D_n intersection lattice is isomorphic to the lattice of centrally symmetric set partitions on $\{1, 2, \ldots, n, \overline{1}, \overline{2}, \ldots, \overline{n}\}$, with at most one block B with $B = \overline{B}$ such that $|B| \ge 4$. In other words, the block B cannot consist of only one pair $\{i, \bar{i}\}$ [AR04, Rei97]. Therefore, a rank-two intersection corresponds to a centrally symmetric set partition with two blocks of size three and four blocks of size two. Note that a block of size four is impossible because there exist no reflections of the form $(i, \bar{i})$ in the hyperplane arrangement for D_n.

If we space $2n$ vertices regularly around a circle, numbered clockwise from 1 up to n and then from $\overline{1}$ to $\overline{n}$, blocks can be represented by the convex hull of the vertices they contain. A c-noncrossing subarrangement is one whose blocks do not intersect, while a c-crossing subarrangement will have intersecting blocks; see Figures 4 and 5. We characterize all full rank-two subarrangements in type D_n in the following proposition.

Proposition 6.8. *Fix type D_n and $c = s_1 s_2 \cdots s_n$. Every full rank-two subarrangement of the Coxeter arrangement of type D_n lies in one of the sets given in Figures 4 and 5.*

6.4.2. *Dealing with c-Crossing Subarrangements.* In type D, every full rank-two subarrangement is actually c'-noncrossing for some choice of Coxeter element c', which we can take as a cyclic rotation of $c = s_1 \cdots s_n$.

Define the Coxeter element $c_a := s_a s_{a+1} \cdots s_n s_1 \cdots s_{a-1}$ so that $c_1 = c$. For each c_a, let $J^{c_a}_{rs} \in \mathrm{Sort}(W, c_a)$ be the unique c_a-sortable join-irreducible element whose cover reflection is $(\!(rs)\!)$, and similarly for $J^{c_a}_{r\bar{s}} = J^{c_a}_{\bar{r}s}$.

Lemma 6.9. *Each c-crossing subarrangement is noncrossing for some choice of c_a:*

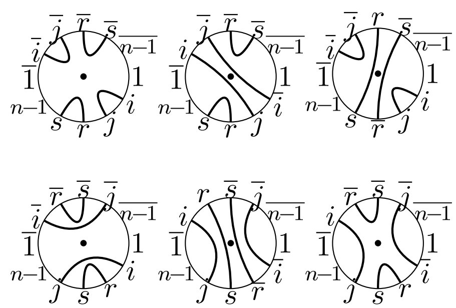
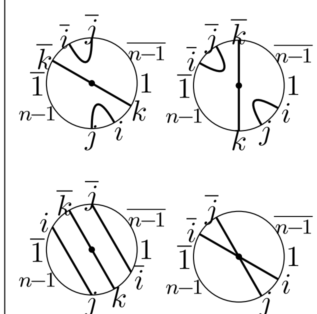
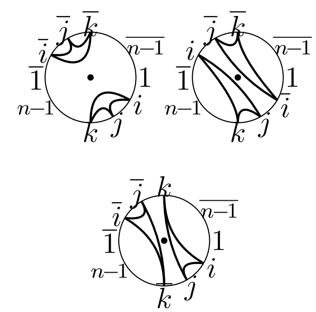
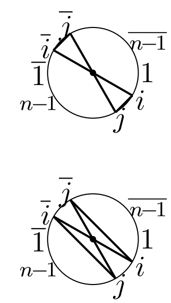
Type D_n c-Noncrossing Full Rank-Two Subarrangements				
Label	$\mathcal{N}_c(D_n, A_1{\times}A_1, 1)$	$\mathcal{N}_c(D_n, A_1{\times}A_1, 2)$	$\mathcal{N}_c(D_n, A_2, 1)$	$\mathcal{N}_c(D_n, A_2, 2)$
Subarrangement	$\{H_{ij}, H_{rs}\}$	$\{H_{ij}, H_{kn}\}, \{H_{in}, H_{j\overline{n}}\}$	$\{H_{ij}, H_{ik}, H_{jk}\}$	$\{H_{in}, H_{ij}, H_{jn}\}$
Conditions	$\lvert i\rvert<\lvert j\rvert<\lvert r\rvert<\lvert s\rvert$ and $I{=}J{=}R{=}S$, $\overline{I}{=}J{=}\underline{R}{=}S$ or $I{=}J{=}\overline{R}{=}S$ $\lvert i\rvert<\lvert r\rvert<\lvert s\rvert<\lvert j\rvert$ and $I{=}J{=}\underline{R}{=}S$, $\overline{I}{=}J{=}\overline{R}{=}S$ or $\overline{I}{=}J{=}R{=}S$	$\lvert i\rvert<\lvert j\rvert<\lvert k\rvert$, $I{=}J$; $\lvert k\rvert<\lvert i\rvert<\lvert j\rvert$, $I{=}J$ $\lvert i\rvert<\lvert k\rvert<\lvert j\rvert$, $I{=}\overline{J}$; $\lvert i\rvert<\lvert j\rvert$, $I{=}J$	$\lvert i\rvert<\lvert j\rvert<\lvert k\rvert$ and $I{=}J{=}\underline{K}$, $I{=}\overline{J}{=}\overline{K}$ or $I{=}J{=}\overline{K}$	$\lvert i\rvert < \lvert j\rvert$ and $I{=}J$ or $\overline{I}{=}J$
Picture				
Relation	[((ij))((rs))]	[((ij))((kn))], [((ij))((jn̄))]	[((jk))((ik))((ij))]	[((in))((ij))((jn))]

FIGURE 4. The full rank-two c-noncrossing subarrangements of type D_n. The point in the center corresponds to n and $\overline{n}$. We use capital letters to mean the sign of the corresponding letter (for example, $I = \text{sign}(i)$ and $\overline{I} = -\text{sign}(i)$). Observe that i, j, r, s, and k do not necessarily stand for positive numbers, and that $\overline{i}$, $\overline{j}$, $\overline{r}$, $\overline{s}$, and $\overline{k}$ do not necessarily stand for negative numbers. Also i, j, r, s, and k are never equal to n or $\overline{n}$. The letter n is denoted explicitly if they are used.

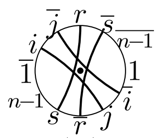
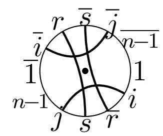
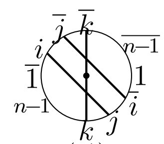
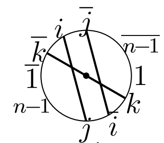
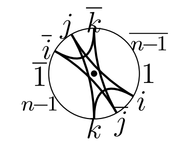
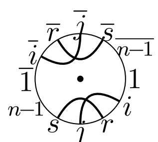
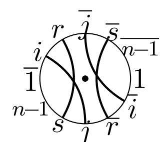
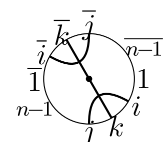
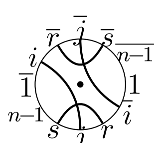
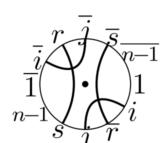
Type D_n c-Crossing Full Rank-Two Subarrangements					
Label	$\mathcal{C}_c(D_n, A_1{\times}A_1, 1)$		$\mathcal{C}_c(D_n, A_1{\times}A_1, 2)$		$\mathcal{C}_c(D_n, A_2)$
Subarrangement	$\{H_{ij}, H_{rs}\}$		$\{H_{ij}, H_{kn}\}$		$\{H_{ij}, H_{ik}, H_{jk}\}$
Conditions	$\lvert i\rvert<\lvert j\rvert<\lvert r\rvert<\lvert s\rvert$ $\overline{I}=J=\overline{R}=S$	$\lvert i\rvert<\lvert r\rvert<\lvert s\rvert<\lvert j\rvert$ $I=J=\overline{R}=S$	$\lvert i\rvert<\lvert j\rvert<\lvert k\rvert$ $I=\overline{J}$	$\lvert k\rvert<\lvert i\rvert<\lvert j\rvert$ $I=\overline{J}$	$\lvert i\rvert<\lvert j\rvert<\lvert k\rvert$ and $I=\overline{J}=K$
Picture	 (A)	 (A')	 (A)	 (A')	
Relation	$[(\!(ij)\!)^{(\!(ir)\!)(\!(r\bar{j})\!)}(\!(rs)\!)]$	$[(\!(ij)\!)(\!(rs)\!)^{(\!(sj)\!)(\!(r\bar{j})\!)}]$	$[(\!(kn)\!)(\!(ij)\!)^{(\!(ik)\!)}]$	$[(\!(kn)\!)(\!(ij)\!)^{(\!(ik)\!)}]$	$[(\!(jk)\!)(\!(ij)\!)^{(\!(i\bar{j})\!)}(\!(ik)\!)^{(\!(i\bar{j})\!)}]$
Conditions	$\lvert i\rvert<\lvert r\rvert<\lvert j\rvert<\lvert s\rvert$ $I=J=R=S$	$\lvert i\rvert<\lvert r\rvert<\lvert j\rvert<\lvert s\rvert$ $\overline{I}=J=\overline{R}=S$	$\lvert i\rvert<\lvert k\rvert<\lvert j\rvert$ $I=J$		
Picture	 (B)	 (B')	 (B)		
Relation	$[(\!(ij)\!)^{(\!(ir)\!)}(\!(rs)\!)]$	$[(\!(ij)\!)^{(\!(ir)\!)}(\!(rs)\!)]$	$[(\!(kn)\!)(\!(ij)\!)^{(\!(ik)\!)}]$		
Conditions	$\lvert i\rvert<\lvert r\rvert<\lvert j\rvert<\lvert s\rvert$ $\overline{I}=J=R=S$	$\lvert i\rvert<\lvert r\rvert<\lvert j\rvert<\lvert s\rvert$ $I=J=\overline{R}=S$			
Picture	 (C)	 (C')			
Relation	$[(\!(ij)\!)^{(\!(i\bar{j})\!)}(\!(rs)\!)^{(\!(rj)\!)}]$	$[(\!(ij)\!)^{(\!(i\bar{r})\!)}(\!(rs)\!)]$			

FIGURE 5. The full rank-two c-crossing subarrangements of type D_n. The center point for each circle represents n and $\overline{n}$. We use capital letters to mean the sign of the corresponding letter (for example, $I = \text{sign}(i)$ and $\overline{I} = -\text{sign}(i)$). Observe that i, j, r, s, and k do not necessarily stand for positive numbers, and that $\bar{i}$, $\bar{j}$, $\bar{r}$, $\bar{s}$, and $\bar{k}$ do not necessarily stand for negative numbers. Also i, j, r, s, and k are never equal to n or $\overline{n}$. The letter n is denoted explicitly if they are used.

- $\{H_{ij}, H_{rs}\} \in \mathcal{C}_c(D_n, A_1{\times}A_1, 1A)$ *is noncrossing for* $c_{|r|}$ *and*
$$S(J_{ij}^{c_{|r|}}) \setminus S(J_{ij}^{c}) = \{H_{ir}, H_{r\bar{j}}\},$$
$$S(J_{rs}^{c_{|r|}}) \setminus S(J_{rs}^{c}) = \emptyset.$$
- $\{H_{ij}, H_{rs}\} \in \mathcal{C}_c(D_n, A_1{\times}A_1, 1A')$ *is noncrossing for* $c_{|j|}$ *and*
$$S(J_{ij}^{c_{|j|}}) \setminus S(J_{ij}^{c}) = \emptyset,$$
$$S(J_{rs}^{c_{|j|}}) \setminus S(J_{rs}^{c}) = \{H_{sj}, H_{r\bar{j}}\}.$$
- $\{H_{ij}, H_{rs}\} \in \mathcal{C}_c(D_n, A_1{\times}A_1, 1B)$ *is noncrossing for* $c_{|r|}$ *and*
$$S(J_{ij}^{c_{|r|}}) \setminus S(J_{ij}^{c}) = \{H_{ir}\},$$
$$S(J_{rs}^{c_{|r|}}) \setminus S(J_{rs}^{c}) = \emptyset.$$
- $\{H_{ij}, H_{rs}\} \in \mathcal{C}_c(D_n, A_1{\times}A_1, 1B')$ *is noncrossing for* $c_{|r|}$ *and*
$$S(J_{ij}^{c_{|r|}}) \setminus S(J_{ij}^{c}) = \{H_{ir}\},$$
$$S(J_{rs}^{c_{|r|}}) \setminus S(J_{rs}^{c}) = \emptyset.$$
- $\{H_{ij}, H_{rs}\} \in \mathcal{C}_c(D_n, A_1{\times}A_1, 1C)$ *is noncrossing for* $c_{|j|}$ *and*
$$S(J_{ij}^{c_{|j|}}) \setminus S(J_{ij}^{c}) = \{H_{i\bar{j}}\},$$
$$S(J_{rs}^{c_{|j|}}) \setminus S(J_{rs}^{c}) = \{H_{rj}\}.$$
- $\{H_{ij}, H_{rs}\} \in \mathcal{C}_c(D_n, A_1{\times}A_1, 1C')$ *is noncrossing for* $c_{|r|}$ *and*
$$S(J_{ij}^{c_{|r|}}) \setminus S(J_{ij}^{c}) = \{H_{i\bar{r}}\},$$
$$S(J_{rs}^{c_{|r|}}) \setminus S(J_{rs}^{c}) = \emptyset.$$
- $\{H_{ij}, H_{ik}, H_{jk}\} \in \mathcal{C}_c(D_n, A_2)$ *is noncrossing for* $c_{|j|}$ *and*
$$S(J_{jk}^{c_{|j|}}) \setminus S(J_{jk}^{c}) = \emptyset,$$
$$S(J_{ij}^{c_{|j|}}) \setminus S(J_{ij}^{c}) = \{H_{i\bar{j}}\}$$
$$S(J_{ik}^{c_{|j|}}) \setminus S(J_{ik}^{c}) = \{H_{i\bar{j}}\}.$$

Proof. The cycle notation of c_k labels the circle clockwise by the numbers $k, 1, 2, \ldots, k-1, k+1, \ldots, n-1, \overline{k}, \overline{1}, \overline{2}, \ldots, \overline{k-1}, \overline{k+1}, \ldots, \overline{n-1}$. The point in the center essentially swaps n and $-n$ because of the transposition $(n, \overline{n})$. One checks that the partitions given by the subarrangements in each case above are noncrossing with respect to this order.

To confirm the statements about the differences in inversion sets, it suffices to explicitly identify the sorting words for the c_a-sortable and c-sortable join irreducibles. This is a straightforward but tedious process. We will only present two cases where the other cases are left to interested readers.

- $\{H_{ij}, H_{rs}\} \in \mathcal{C}_c(D_n, A_1{\times}A_1, 1A)$
$$J_{ij}^{c_{|r|}} = s_{|r|}s_{|r|+1}\cdots s_n \cdot (s_{|i|}s_{|i|+1}\cdots s_n \cdot s_{|j|-1}s_{|j|}\cdots s_{n-1})$$
$$J_{ij}^{c} = s_{|i|}s_{|i|+1}\cdots s_n \cdot s_{|j|-1}s_{|j|}\cdots s_{n-1}$$
$$J_{rs}^{c_{|r|}} = J_{rs}^{c} = s_{|r|}s_{|r|+1}\cdots s_n \cdot s_{|s|-1}s_{|s|}\cdots s_{n-1}.$$

  One checks that the two inversions in J_{ij}^{c} not in $J_{ij}^{c_{|r|}}$ correspond to the first copy of the simple reflection $s_{|r|-1}$ and the second copy of $s_{|r|-2}$ in the c-sorting word for J_{ij}^{c}.

- $\{H_{ij}, H_{rs}\} \in \mathcal{C}_c(D_n, A_1 \times A_1, 1A')$

$$J_{ij}^{c|j|} = J_{ij}^{c} = s_i \cdots s_{j-1}$$
$$J_{rs}^{c|j|} = s_{|j|}s_{|j|+1} \cdots s_n \cdot s_{|r|}s_{|r|+1} \cdots s_n \cdot s_{|s|-1}s_{|s|} \cdots s_{n-1}$$
$$J_{rs}^{c} = s_{|r|}s_{|r|+1} \cdots s_n \cdot s_{|s|-1}s_{|s|} \cdots s_{n-1}.$$

One checks that the two inversions in J_{rs}^c not in $J_{rs}^{c|j|}$ correspond to the first copy of the simple reflection $s_{|j|-1}$ and the second copy of $s_{|j|-2}$ in the c-sorting word for J_{rs}^c.

∎

6.4.3. *Presentations.* Applying Lemma 6.9, one can obtain the relations for the different types of the subarrangement similar to the process in types A and B.

Theorem 6.10. *A presentation for $P(D_n)$ is given by $\mathbb{t}_c$ subject to the two classes of c-noncrossing relations in Figure 4 and the seven classes of c-crossing relations in Figure 5.*

Similarly to type B_n, it is possible to show, through extensive computation, that all the c-crossing relations can be rewritten positively. Here, we will show the derivation for positive relations for $\mathcal{C}_c(D_n, A_1 \times A_1, 1D)$, where we compute

$$(\!(\mathbb{rs})\!)\overline{(\!(\mathbb{ir})\!)}(\!(\mathbb{ij})\!)(\!(\mathbb{ir})\!) = \overline{(\!(\mathbb{ir})\!)}(\!(\mathbb{ij})\!)(\!(\mathbb{ir})\!)(\mathbb{rs})$$
$$(\!(\mathbb{is})\!)(\!(\mathbb{ir})\!)(\!(\mathbb{rs})\!)\overline{(\!(\mathbb{ir})\!)}(\!(\mathbb{ij})\!)(\!(\mathbb{ir})\!) = (\!(\mathbb{is})\!)(\!(\mathbb{ir})\!)\overline{(\!(\mathbb{ir})\!)}(\!(\mathbb{ij})\!)(\!(\mathbb{ir})\!)(\mathbb{rs})$$
$$(\!(\mathbb{rs})\!)(\!(\mathbb{is})\!)(\!(\mathbb{ij})\!)(\!(\mathbb{ir})\!) = (\!(\mathbb{is})\!)(\!(\mathbb{ij})\!)(\!(\mathbb{ir})\!)(\mathbb{rs}),$$

where we used a relation of type $\mathcal{N}_c(B_n, A_2)$.

6.5. **Dihedral Groups.** The dihedral group $I_2(m)$ only has two simple reflections. The choice of any Coxeter element indexes the m generators of $P(I_2(m))$ as

$$\mathbb{t}_c := \{\mathbb{1} <_c \mathbb{2} <_c \cdots <_c \mathbb{m}\}.$$

One can easily see the dihedral has a single rank-two subspace which is c-noncrossing. Hence, we can deduce the following positive presentation.

Theorem 6.11. *The pure braid group $P(I_2(m))$ has the positive presentation:*

$$P(I_2(m)) = \left\langle\ \mathbb{t}_c \;\middle|\; [\mathbb{m} \cdots \mathbb{1}]\ \right\rangle = \left\langle\ \mathbb{t}_c \;\middle|\; \mathrm{Red}_{\mathbb{t}_c}(\mathbb{c})\ \right\rangle.$$

Proof. The single rank-two subspace is c-noncrossing. ∎

6.6. **Type H_3.** The Coxeter diagram is $1 \xrightarrow{5} 2 \to 3$. We choose the Coxeter element $c = s_1s_2s_3$ and label the fifteen generators for $P(H_3)$ using the first fifteen letters of the alphabet.

$$\mathbb{t}_c := \{\mathbb{a} <_c \mathbb{b} <_c \cdots <_c \mathbb{o}\}.$$

6.6.1. *Presentations.* There are 15 c-noncrossing rank-two subspaces where the noncrossing partition is self-dual with four ranks by the Kreweras complement. Therefore, there are the same number of c-noncrossing rank-two subspaces as there are reflections. One can verify that there are five subspaces each of types $A_1 \times A_1$, A_2, and $I_2(5)$. The 15 relations for these subspaces are given in Figure 6.

On the other hand, the lattice of subspaces of the type H_3 Coxeter arrangement has 31 rank-two subspaces. Of the 16 remaining c-crossing rank-two subspaces, 10 are of type $A_1 \times A_1$, 5 are of type A_2, and 1 is of type $I_2(5)$. The sixteen remaining relations are also shown in Figure 6.

It follows from an unenlightening computation that the sixteen relations above are equivalent to the sixteen positive relations given in Figure 7. We will not include the details of this computation.

Label	c-Noncrossing $\mathcal{N}_c(H_3, A_1 \times A_1)$	$\mathcal{N}_c(H_3, A_2)$	$\mathcal{N}_c(H_3, I_2(5))$
Relation	[oa]	[onl]	[omkjf]
	[ml]	[ocb]	[oiged]
	[ji]	[lki]	[ljhgc]
	[gf]	[ihf]	[lfdba]
	[dc]	[fec]	[nmica]

Label	c-Crossing $\mathcal{C}_c(H_3, A_1 \times A_1)$	$\mathcal{C}_c(H_3, A_2)$	$\mathcal{C}_c(H_3, I_2(5))$
Relation	$[oh^{f}]$ $[ng^{ca}]$	$[n^{l}j^{f}d]$	
	$[n^{l}f]$ $[m^{igcba}e^{cba}]$	$[m^{ica}g^{ca}b^{a}]$	
	$[le^{c}]$ $[k^{jf}d]$	$[k^{ic}g^{c}a]$	$[n^{l}k^{jfd}h^{fd}e^{d}b]$
	$[k^{i}c]$ $[j^{fd}b]$	$[j^{fdb}e^{db}a]$	
	$[i^{ca}b^{a}]$ $[h^{gc}a]$	$[m^{kjf}h^{f}d]$	

FIGURE 6. The 15 relations arising from the c-non-crossing rank-two subspaces and the 16 relations arising from the c-crossing rank-two subspaces. The c-crossing relations are rewritten in a positive form in Figure 7.

Theorem 6.12. *The generating set* $\mathbb{t}_c$ *with relations given by the 15 c-noncrossing relations in Figure 6 and 16 positive c-crossing relations in Figure 7 is a positive presentation for* $P(H_3)$*.*

Corollary 6.13.

$$P(H_3) = \langle \mathbb{t}_c : \mathrm{Red}_{\mathbb{t}_c}(\mathbb{c}) \rangle.$$

Proof. Each relation in Figures 6 and 7 can be completed to a word for the full twist. □

$A_1 \times A_1$	A_2	$I_2(5)$
foih = hfoi	nljfdomk = ljfdonmk	jfdonmkihgecb
nmigca = migcan	= lfdonmkj	= kjhfedbonmigc
nlfo = lfon	omigcban = oigcbanm	= jhfedbonmkigc
domige = edomig	= bomigcan	= jfedbonmkihgc
clfe = eclf	nmkigcal = nmigcalk	= jfdbonmkihgec
kjfdom = jfdomk	= anmkigcl	
kicl = iclk	cljfedba = ecaljfdb	
aljfdb = baljfd	= caljfedb	
oicb = boic	mkjhfdoican = kjhfdomican	
hgcalj = gcaljh	= kjfdoihcanm	

FIGURE 7. Positive relations equivalent to the c-crossing relations in Figure 6.

6.7. **Type F_4.**

6.7.1. *Root System and Intersection Lattice.* A model for the root system $\Phi \subset \mathcal{R}^4$ of type F_4 is

$$\Phi = \{\pm e_i \pm e_j : 1 \leq i < j \leq 4\} \cup \{\pm e_i : 1 \leq i \leq 4\} \cup \left\{\tfrac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)\right\},$$

with 48 roots in total (24 long, 24 short). One convenient choice of simple roots is

$$\alpha_1 = e_2 - e_3, \quad \alpha_2 = e_3 - e_4, \quad \alpha_3 = e_4, \quad \alpha_4 = -\tfrac{1}{2}(e_1 + e_2 + e_3 + e_4).$$

Let $W(F_4)$ denote the reflection group generated by $s_1, \dots, s_4$ acting on $\mathcal{R}^4$. For each positive root $\alpha \in \Phi^+$, the reflecting hyperplane $H_\alpha = \{x \in \mathcal{R}^4 : \langle x, \alpha\rangle = 0\}$ belongs to the Coxeter arrangement $\mathcal{A}(F_4)$, which consists of $|\Phi^+| = 24$ hyperplanes. The intersection lattice

$$L(\mathcal{A}(F_4)) = \{\bigcap_{H \in S} H : S \subseteq \mathcal{A}(F_4)\},$$

ordered by reverse inclusion, is a rank-4 geometric lattice and is in bijection with the nonstandard parabolic subgroups of $W(F_4)$.

6.7.2. *Presentations.* The Coxeter diagram is $1 \to 2 \xrightarrow{4} 3 \to 4$. We pick the Coxeter element $c = s_1 s_2 s_3 s_4$ whose order (the Coxeter number) is $h = 12$ and label the 24 generators for $P(F_4)$ by the first 24 letters of the alphabet.

$$\mathbbm{t}_c := \{\mathbbm{a} <_c \mathbbm{b} <_c \cdots <_c \mathbbm{x}\}.$$

The noncrossing partition lattice $\mathrm{NC}(F_4, c)$ is a graded, self-dual poset of rank 4. One computes that there are exactly 55 such c-noncrossing subarrangements, where 24 are of type $A_1 \times A_1$, 16 are of type A_2, and 15 are of type B_2. Directly referencing Theorem 5.1, we see that these gave rise to the 55 non-crossing relations listed in Figure 8.

On the other hand, the full intersection lattice $L(\mathcal{A}(F_4))$ has 122 rank-2 subarrangements, of which the remaining 67 are c-crossing (decomposing into 48 of type $A_1 \times A_1$, 16 of type A_2, and 3 of type B_2). Choosing one standard generating face per crossing flat yields the additional 67 relations in Figure 9.

The 67 c-crossing relations can be rewritten into 67 positive relations similar to the process for H_3. The positive form of these relations is given in Figure 10. We will include an example of a computation for a c-crossing A_2 relation.

$$\begin{gathered}
\mathbbm{g}^{-1}\mathbbm{m}^{-1}\mathbbm{umg}\mathbbm{g}^{-1}\mathbbm{jgf} = \mathbbm{f}\mathbbm{g}^{-1}\mathbbm{m}^{-1}\mathbbm{umg}\mathbbm{g}^{-1}\mathbbm{jg} = \mathbbm{g}^{-1}\mathbbm{jgf}\mathbbm{g}^{-1}\mathbbm{m}^{-1}\mathbbm{umg} \\
\mathbbm{g}^{-1}\mathbbm{m}^{-1}\mathbbm{umjgf} = \mathbbm{f}\mathbbm{g}^{-1}\mathbbm{m}^{-1}\mathbbm{umjg} = \mathbbm{g}^{-1}\mathbbm{jgf}\mathbbm{g}^{-1}\mathbbm{m}^{-1}\mathbbm{umg} \\
\mathbbm{g}^{-1}\mathbbm{m}^{-1}\mathbbm{umjgf} = \mathbbm{f}\mathbbm{g}^{-1}\mathbbm{m}^{-1}\mathbbm{umjg} = \mathbbm{g}^{-1}\mathbbm{jgfeu}\mathbbm{e}^{-1} \\
\mathbbm{xmg}\mathbbm{g}^{-1}\mathbbm{m}^{-1}\mathbbm{umjgf} = \mathbbm{fxmg}\mathbbm{f}^{-1}\mathbbm{f}\mathbbm{g}^{-1}\mathbbm{m}^{-1}\mathbbm{umjg} = \mathbbm{xmg}\mathbbm{g}^{-1}\mathbbm{jgfeu}\mathbbm{e}^{-1} \\
\mathbbm{xumjgf} = \mathbbm{fxumjg} = \mathbbm{xmjgfeu}\mathbbm{e}^{-1} \\
\mathbbm{xumjgfe} = \mathbbm{fxumjge} = \mathbbm{xmjgfeu}
\end{gathered}$$

where the relations used are

$$\mathbbm{xmgf} = \mathbbm{fxmg} \quad \text{and} \quad \mathbbm{umge} = \mathbbm{mgeu}.$$

We will not include the details of the rest of the computation. Formally, we can express our two conjectures as

Conjecture 6.14. *The generating set $\mathbbm{t}_c$ with relations given by the* 55 *c-noncrossing relations in Figure 8 and* 67 *positive c-crossing relations (after rewriting completely) in Figure 10 is a positive presentation for $P(F_4)$.*

Conjecture 6.15.

$$P(F_4) = \langle \mathbb{t}_c : \mathrm{Red}_{\mathbb{t}_c}(\mathbb{c}) \rangle.$$

Since all remaining (c)-crossing relations have now been rewritten as positive relations, we can directly test whether they occur as prefixes of words for the full twist. We can use Theorem 4.2 and its conjectured backward direction to determine, for each relation, a partial selection of edges corresponding to a parabolic subsystem, possibly with additional reflections interspersed. We then complete this partial ordering arbitrarily to a full (c)-oriented DAG and verify using GAP3 that the resulting product equals the full twist [CHE16]. While it is not currently complete, we can use this approach to prove 6.15.

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	c-Noncrossing		
Label	$\mathcal{N}_c(F_4, A_1 \times A_1)$	$\mathcal{N}_c(F_4, A_2)$	$\mathcal{N}_c(F_4, B_2)$
Relation	[xe]	[xwt]	[xvsq]
	[xb]	[xph]	[xuon]
	[xa]	[xdc]	[xmgf]
	[wa]	[vua]	[wvpe]
	[vt]	[urq]	[wudb]
	[ut]	[ume]	[uonh]
	[us]	[tsp]	[trom]
	[up]	[tld]	[tqhe]
	[ta]	[qnm]	[srla]
	[rp]	[qia]	[qkjd]
	[qp]	[pol]	[pnki]
	[qo]	[mji]	[pmda]
	[ql]	[lkh]	[ljge]
	[nl]	[ife]	[hfca]
	[ml]	[hgd]	
	[mk]		
	[mh]		
	[jh]		
	[ih]		
	[ig]		
	[id]		
	[fd]		
	[ed]		
	[ec]		

FIGURE 8. The 55 relations arising from the c-non-crossing rank-two subspaces.

	c-Crossing		
Label	$\mathcal{C}_c(F_4, A_1 \times A_1)$	$\mathcal{C}_c(F_4, A_2)$	$\mathcal{C}_c(F_4, B_2)$
	$[ku^{l}]$	$[fj^{g}u^{mg}]$	$[bg^{e}n^{mhe}s^{pmhe}]$
	$[fo^{mjg}]$	$[bj^{d}r^{omld}]$	$[cj^{d}o^{mld}v^{umld}]$
	$[bp^{e}]$	$[hs^{p}w^{vp}]$	$[fk^{i}r^{qli}w^{tqli}]$
	$[jx^{m}]$	$[bf^{e}q^{he}]$	
	$[jw^{ul}]$	$[cl^{d}w^{ud}]$	
	$[ak^{i}]$	$[bm^{d}v^{ud}]$	
	$[cq^{i}]$	$[fn^{h}v^{sqph}]$	
	$[cr^{li}]$	$[aj^{d}n^{md}]$	
	$[mw^{u}]$	$[cg^{ed}p^{med}]$	
	$[dn^{m}]$	$[ck^{i}s^{rqli}]$	
	$[hv^{p}]$	$[ko^{nm}x^{pnm}]$	
	$[io^{l}]$	$[en^{h}r^{qph}]$	
	$[ek^{h}]$	$[ir^{l}v^{ul}]$	
	$[nx^{p}]$	$[go^{m}w^{vupm}]$	
	$[iw^{t}]$	$[do^{l}s^{rl}]$	
	$[bh^{e}]$	$[gk^{ji}t^{lji}]$	
	$[nw^{vsp}]$		
	$[jt^{l}]$		
	$[qw^{t}]$		
	$[jp^{m}]$		
	$[cm^{d}]$		
	$[js^{rol}]$		
	$[ao^{l}]$		
Relation	$[bk^{jgd}]$		
	$[ft^{i}]$		
	$[fp^{h}]$		
	$[es^{p}]$		
	$[fs^{qh}]$		
	$[gq^{h}]$		
	$[cn^{hgf}]$		
	$[ms^{p}]$		
	$[gv^{pm}]$		
	$[bl^{d}]$		
	$[cu^{d}]$		
	$[eo^{m}]$		
	$[hr^{q}]$		
	$[ag^{d}]$		
	$[kv^{pon}]$		
	$[nt^{q}]$		
	$[fl^{i}]$		
	$[gr^{ponm}]$		
	$[is^{q}]$		
	$[dv^{u}]$		
	$[dr^{l}]$		
	$[gu^{m}]$		
	$[lv^{u}]$		
	$[bo^{md}]$		
	$[rx^{u}]$		

FIGURE 9. The 67 relations arising from the c-crossing rank-two subspaces. The c-crossing relations are rewritten in a positive form in Figure 10.

$A_1 \times A_1$		A_2	B_2
khol = holka	taud = adut	xwtekga = tekgaxw = kgaxwte	fkrqlit = krqlitf = rqlitfk = qlitfkr
badu = adub	jagn = agnj	hgnicox = nicoxhg = icoxhgn	bgnmhep = gnmhepb = nmhepbg = mhepbgn
jaxn = axnj	nxxn = xxon	jkrguse = rgusejk = gusejkr	cjomldu = jomlduc = omlducj = mlducjo
nxnx = xnxn	rkcrm = kcrmr	xmspeth = spethxm = pethxms	
rkcrm = kcrmr	okfq = kfqeo	jldaqpe = ldaqpej = daqpejl	
okfq = kfqeo	omis = miso	kldotrx = ldotrxk = dotrxkl	
omis = miso	outb = utbou	ufdaoxm = fdaoxmu = daoxmuf	
outb = utbou	kxpu = xpuk	jrlncvx = rlncvxj = lncvxjr	
kxpu = xpuk	gulm = ulmga	judqowx = udqowxj = dqowxju	
gulm = ulmga	fugi = ugifb	ehcbaqo = hcbaqoe = cbaqoeh	
fugi = ugifb	flmq = lmqfs	vjuifbs = juifbsv = uifbsvj	
flmq = lmqfs	cgvi = vcigf	hnmfsel = nmfselh = mfselhn	
cgvi = vcigf	icxa = cxai	bromld = romldb = omldbr	
icxa = cxai	dcho = chod	fvsqph = vsqphf = sqphfv	
dcho = chod	dbfu = bfud	csrqli = srqlic = rqlics	
dbfu = bfud	dsqr = srqd	gwvupm = wvupmg = vupmgw	
dsqr = srqd	asqp = sqpa		
asqp = sqpa	vspo = spov		
vspo = spov	vqrt = qrtv		
vqrt = qrtv	vegi = egiv		
vegi = egiv	pebi = ebip		
pebi = ebip	hegs = egsh		
hegs = egsh	hwic = wich		
hwic = wich	bsoc = socbx		
fxuomjg = xuomjgf	bwvumd = wvumdb		
ronjdas = sronjda	ultdcbw = wultdcb		
vspnhew = wvspnhe	kixvpon = ixvponk		
ldcbtasrqi = dbtrlicasq	qifcaxvsph = icasqhfxvp		
phgfxewvum = hfxvpmgewu			

FIGURE 10. Positive relations are equivalent to the c-crossing relations in Figure 8.