Knot polynomials on braid closures

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Abstract

We explore properties of the Jones and Alexander polynomials of braid closures of braids of index 3. We analyze how the evaluations of these polynomials at $t = -1$ force implications regarding the topological structure of the braid closure, such as its number of circles or the existence of splittings. As a consequence, we construct an infinite number of pairs of distinct non-split links that have the same HOMFLYPT polynomial.

1 Introduction

A knot is an embedding of the circle S^1 in S^3, while a link is an embedding of several circles in S^3. Through the remainder of this paper, we tacitly work with tame links only.

Definition 1.1. A *splitting* of a link L is a manifold $B \subset S^3 \setminus L$ such that B is homeomorphic to S^2 and L intersects both connected components of $S^3 \setminus B$.

Informally, this occurs exactly when L can be separated into two rigid components that can be moved arbitrarily far away from each other without disturbing isotopy classes.

Example 1.2. The Hopf link features two linked circles; physical intuition implies that it exhibits no splitting. Indeed, if a split link has two circles, then it must be the unlink on two circles, which has Jones polynomial $-\frac{1}{\sqrt{t}}(1+t)$. Meanwhile, the Hopf link has different Jones polynomial $-\frac{1}{\sqrt{t}}(1+\frac{1}{t^2})$.

The Jones polynomial $V_L(t)$ is obtained upon specialization of the HOMFLYPT polynomial $X_L(q, \Lambda)$ at $q = \Lambda = t$, where we have used the capital letter Λ instead of the more standard λ in order to avoid confusion with the partitions $\lambda \vdash n$.

Proposition 1.3. *If L is a split link, then $\Delta_L(t) = 0$.*

Proof. If L splits into L_1 and L_2, then by the skein relations [1, Example 6.7] we have

$$X_L(q,\Lambda) = -\frac{(1-\Lambda q)X_{L_1}(q,\Lambda)X_{L_2}(q,\Lambda)}{\sqrt{\Lambda}(1-q)},$$

which must equal zero if $\Lambda q = 1$ and $q \neq 1$. Since the Alexander polynomial satisfies $\Delta_L(t) = X_L(t, \frac{1}{t})$, it follows that $\Delta_L(t) = 0$ for all $t \in \mathbb{C} \setminus \{1\}$. We finish by continuity. $\square$

By Proposition 1.3 that the Alexander polynomial detects all splittings. We concern ourselves with the special value of the *the knot determinant* $V_L(-1) = \Delta_L(-1)$.

Question 1.4. *Is there a link L with no splitting such that $\Delta_L(-1) = 0$?*

The answer to Question 1.4 is in the positive. In fact, given any L, there is [2] an infinite family of distinct links L' all with Jones polynomial equal to $V_{L'}(t) = -(\sqrt{t} + \frac{1}{\sqrt{t}})V_L(t)$. Thus $\Delta_{L'}(-1) = V_{L'}(-1) = X_{L'}(-1,-1) = 0$. However, the construction used to create such links are quite complicated, so we will consider simpler methods.

Theorem 1.5 (Theorems 3.8 and 3.9)**.** *There exists an infinite family of links $L_{m,k}$, indexed by pairs of integers (m,k), with the property that any link L satisfying $V_L(-1) = \Delta_L(-1) = 0$ is equivalent to some $L_{m,k}$. In addition, $L_{m,k}$ has a splitting if and only if $m = 0$.*

The construction of each $L_{m,k}$ is much simpler than that of [2]. It turns out our work also yields pairs of distinct non-split links with the same HOMFLYPT polynomial, as in our second main result.

Theorem 1.6 (Theorem 3.10)**.** *If $6 \mid k$, then $X_{\hat{\gamma}_{m,k}}(q,\Lambda) = X_{\hat{\gamma}_{m+\frac{k}{6},-k}}(q,\Lambda)$.*

The links $L_{m,k}$ all contain either two or three circles, motivating us to consider how $\Delta_L(-1)$ would behave if L had braid index at most 3 but was forced to be a knot. Surprisingly, we prove the following third main result in Section 3.2.

Theorem 1.7 (Theorem 3.15)**.** *If L is a knot with braid index at most 3, then $V_L(-1) \equiv \Delta_L(-1) \equiv 1 \pmod 4$.*

2 Preliminaries

2.1 The braid group

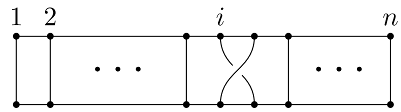
Let n be a positive integer.

Definition 2.1. The *braid group* B_n is generated by $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ with relations

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ and } \sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| \geq 2.$$

Naturally, elements of B_n are called *braids*.

We can diagrammatically depict σ_i as

$$\sigma_i =$$


.

Under this interpretation, each braid can be viewed as consisting of n strands of string, each connecting a point in the top row to a point in the bottom row. Multiplication of braids is determined by diagrammatic concatenation.

Definition 2.2. Given a braid $\alpha \in B_n$, the *braid closure* $\hat{\alpha}$ is the oriented link obtained by identifying each endpoint on the top row with the endpoint directly below it on the bottom row. The orientation of the link points away from the top row and towards the bottom row.

Example 2.3. The braid closure of $\sigma_1^3 \in B_2$ is the right-handed trefoil.

We reproduce a famous theorem of Alexander [3] below.

Theorem 2.4 ([3, Alexander's Theorem])**.** *Let L be a link. Then there exists an integer n and a braid $\alpha \in B_n$ such that L is isotopic to the braid closure $\hat{\alpha}$.*

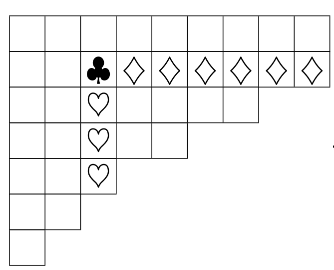
Definition 2.5. The *braid index* of a link L is the minimal satisfactory value of n in the statement of Theorem 2.4.

2.2 Young diagrams

Now we discuss partitions of n and their Young diagrams.

Definition 2.6. Let $\lambda \vdash n$ be a partition. For a box $b \in \lambda$ in its Young diagram, the *hook length* $h(b)$ is the number of boxes directly below or to the right to b, including b itself. Let $r(b)$ be the number of rows below the topmost row that b resides in, and define $c(b)$ analogously but in comparison to the leftmost column.

Example 2.7 ([4, Example A.4])**.** Consider the partition

$$(9,9,7,5,3,2,1) =$$

.

The box b marked with a club ($\clubsuit$) has 3 boxes below it, denoted with a heart ($\heartsuit$), and 6 boxes to its right, denoted with a diamond ($\diamondsuit$). Its hook length is $h(b) = 3 + 6 + 1 = 10$. Also check that $r(b) = 1$ and $c(b) = 2$.

2.3 Specht modules of the Hecke algebra

We now introduce the Hecke algebra.

Definition 2.8. For $q \in \mathbb{C}\backslash\{0\}$, the *Hecke algebra* $\mathcal{H}_n(q)$ is the generated by $g_1, g_2, \dots, g_{n-1}$ with relations

$$(g_i - q)(g_i + 1) = 0, \ \ g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}, \ \text{and} \ \ g_i g_j = g_j g_i \text{ if } |i - j| \geq 2.$$

There is a canonical epimorphism $\pi \colon \mathbb{C}[B_n] \to \mathcal{H}_n(q)$ satisfying $\pi(\sigma_i) = g_i$ for all i.

It is well-known that the irreducible representations of $\mathbb{C}[\mathfrak{S}_n]$ are the Specht modules S^λ, which are indexed by partitions $\lambda \vdash n$. As $\mathcal{H}_n(q)$ is a deformation of $\mathcal{H}_n(1) \cong \mathbb{C}[\mathfrak{S}_n]$, the q-Specht modules themselves deform as q varies. They remain irreducible for generic q.

Definition 2.9. For $\sigma \in \mathfrak{S}_n$, let $\ell(\sigma) = \{(a, b) \mid 1 \leq a < b \leq n, \sigma(a) > \sigma(b)\}$ be its *length*. For $\tau = (i \ i+1) \in \mathfrak{S}_n$, define $T_\tau = g_i \in \mathcal{H}_n(q)$ for all $i < n$, and define for all σ

$$T_{\sigma\tau} := \begin{cases} T_\sigma T_\tau & \text{if } \ell(\sigma\tau) = \ell(\sigma) + 1 \\ q^{-1}(T_\sigma T_\tau + (1-q)T_\sigma) & \text{otherwise.} \end{cases}$$

Remark. One can show without much difficulty that $\{T_\sigma \mid \sigma \in \mathfrak{S}_n\}$ forms a basis for $\mathcal{H}_n(q)$, and hence it is clear that $\dim \mathcal{H}_n(q) = n!$ regardless of the choice of $q \in \mathbb{C}$. Writing out each T_σ as a product of generators g_i also expresses this basis in a "normal form" of Jones.

We can now finally demonstrate how the Specht modules deform as q changes value [5].

Definition 2.10. Let $\lambda \vdash n$ be a partition. The *canonical Young tableau* t^λ is formed by writing the first n positive integers in increasing order in the boxes of the Young diagram of λ, going row by row from left to right. Let the *row stabilizer* $P_\lambda \subset \mathfrak{S}_n$ be the subgroup of all permutations that fix the numbers in each row under the natural action of $\mathfrak{S}_n$. Define the *q-Young symmetrizer* to be

$$z_q^\lambda := \left(\sum_{\sigma \in P_\lambda} T_\sigma \right) T_{w_\lambda} \left(\sum_{\sigma \in P_{\lambda^\top}} (-q)^{-\ell(\sigma)} T_\sigma \right),$$

where $w_\lambda \in \mathfrak{S}_n$ is the unique permutation mapping t^λ to the mirror image of $t^{\lambda^\top}$. The *q-Specht module* S_q^λ is the ideal $z_q^\lambda \mathcal{H}_n(q)$, where $\mathcal{H}_n(q)$ acts from the right.

Definition 2.11. Let $\chi_\lambda \colon B_n \to \mathbb{C}$ denote the character of $\pi(\alpha)$ with respect to S_q^λ.

Remark. Regardless of the value of q, its dimension [1, Remark 3.5] is given by the *hook-length formula*

$$\dim S_q^\lambda = \frac{n!}{\prod_{b \in \lambda} h(b)}.$$

Recall the hook length $h(b)$ of a box from Definition 2.6.

Example 2.12. We first consider the q-Specht module $S_q^{(n)}$, which is of dimension 1 by the hook-length formula. Indeed, $P_{(n)} = \mathfrak{S}_n$, $P_{(n)^\top} = \{e\}$, and $w_{(n)} = e$, so

$$z_q^{(n)} = \sum_{\sigma \in \mathfrak{S}_n} T_\sigma.$$

For the transposition $\tau = (i\ i+1)$, by Definition 2.9

$$T_\sigma g_i = \begin{cases} T_{\sigma\tau} & \text{if } \ell(\sigma\tau) = \ell(\sigma) + 1 \\ qT_{\sigma\tau} + (q-1)T_\sigma & \text{otherwise.} \end{cases}$$

Define $Q_i = \{\sigma \mid \ell(\sigma\tau) = \ell(\sigma) + 1\} \subset \mathfrak{S}_n$, and note that $\sigma \in Q_i$ iff $\sigma\tau \notin Q_i$. Thus,

$$\begin{aligned} z_q^{(n)} g_i &= \sum_{\sigma \in \mathfrak{S}_n} T_\sigma g_i \\ &= \sum_{\sigma \in Q_i} T_{\sigma\tau} + \sum_{\sigma \notin Q_i} (qT_{\sigma\tau} + (q-1)T_\sigma) \\ &= \sum_{\sigma \notin Q_i} T_\sigma + q \sum_{\sigma \in Q_i} T_\sigma + (q-1) \sum_{\sigma \notin Q_i} T_\sigma \\ &= q z_q^{(n)}. \end{aligned}$$

It follows that each g_i acts by multiplication by q in $S_q^{(n)}$. This is the *trivial representation* of $\mathcal{H}_n(q)$ and generalizes the trivial representation of $\mathbb{C}[\mathfrak{S}_n]$. Indeed, choosing an eigenvalue of q makes sense as $(g_i - q)(g_i + 1) = 0$.

Example 2.13. Following Example 2.12, one can analogously check that $S_q^{(n)^\top}$ is the *sign representation*, in which each g_i acts as -1.

Example 2.14. The *Burau representation* is the q-Specht module $S_q^{(n-1,1)^\top}$ [1, Note 5.7].

Let us investigate the case where $n = 3$ and $\lambda = (2,1)$. One verifies using Definition 2.10 that $P_{(2,1)} = P_{(2,1)^\top} = \langle g_1 \rangle$, so

$$z_q^{(2,1)} = (1 + g_1) g_2 \left(1 - \frac{g_1}{q}\right).$$

We check that $(q - g_1)g_1 = g_1 - q$, so $z_q^{(2,1)} g_1 = -z_q^{(2,1)}$. Since by Remark 2.3 we have that $\dim S_q^{(2,1)} = \frac{3!}{3\cdot 1\cdot 1} = 2$, it follows that $S_q^{(2,2)}$ is spanned by $z_q^{(2,1)}$ and $z_q^{(2,1)} g_2$.

Now, one can verify, say by direct computation, that $z_q^{(2,2)} g_2 g_1 = -q^2 z_q^{(2,2)} + q z_q^{(2,2)} g_2$, while by the relations of $\mathcal{H}_3(q)$ we have $z_q^{(2,2)} g_2^2 = q z_q^{(2,2)} + (q-1) z_q^{(2,2)} g_2$. It follows that $S_q^{(2,1)}$ is the two-dimensional module on which each g_i acts as H_i where

$$(H_1, H_2) = \left(\begin{pmatrix} -1 & -q^2 \\ 0 & q \end{pmatrix}, \begin{pmatrix} 0 & q \\ 1 & q-1 \end{pmatrix} \right).$$

The Burau representation for $n = 3$ is often portrayed with each g_i acting as G_i, where

$$(G_1, G_2) = \left(\begin{pmatrix} q & 0 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & q \\ 0 & q \end{pmatrix} \right).$$

This is valid as $G_i = CH_iC^{-1}$ for each i, where $C = \left[\begin{smallmatrix} 0 & q \\ 1 & q \end{smallmatrix}\right]$.

Example 2.15. The q-Specht module $S_q^{(2,2)}$ exhibits similarities to the Burau representation from Example 2.14. Again, Definition 2.10 implies that $P_{(2,2)} = \langle g_1, g_3 \rangle$, so

$$z_q^{(2,2)} = (1 + g_1)(1 + g_3)g_2 \left(1 - \frac{g_1}{q}\right)\left(1 - \frac{g_3}{q}\right).$$

Note that $(q - g_1)(q - g_3)g_i = -(q - g_1)(q - g_3)$ and thus $z_q^{(2,2)} g_i = -z_q^{(2,2)}$ for $i \in \{1, 3\}$, and by Remark 2.3 we also have $\dim S_q^{(2,2)} = \frac{4!}{3 \cdot 2 \cdot 2 \cdot 1} = 2$. Again, $S_q^{(2,2)}$ is spanned by $z_{(2,2)}$ and $z_{(2,2)}g_2$.

In fact, the same computations from Example 2.14 hold: it can be shown by hand that $z_q^{(2,2)} g_2 g_i = -q^2 z_q^{(2,2)} + q z_q^{(2,2)} g_2$ and $z_q^{(2,2)} g_2^2 = q z_q^{(2,2)} + (q-1) z_q^{(2,2)} g_2$ for $i \in \{1, 3\}$. Hence, the representation $S_q^{(2,2)}$ copies the Burau representation $S_q^{(2,1)}$ by having g_1 and g_3 act as H_1 and g_2 acts as H_2. Again using a change of basis with matrix C, equivalently we can replace each H_i with G_i in the above statement.

2.4 The HOMFLYPT polynomial

In [1, Section 6], Jones proposed a multivariable polynomial link invariant $X_L(q, \Lambda)$, which reduces to the Jones polynomial under the specialization $q = \Lambda = t$ and to the Alexander polynomial under the specialization $q = \frac{1}{\Lambda} = t$. We recall the key definitions organized in [4].

Definition 2.16 ([6, Theorem 1.1])**.** Let $\lambda \vdash n$ be a partition. The *Ocneanu weight* corresponding to λ is the product

$$\Omega_\lambda = \prod_{b \in \lambda} \frac{q^{r(b)} w - q^{c(b)} z}{1 - q^{h(b)}},$$

where $z = -\frac{1-q}{1-\Lambda q}$ and $w = 1 - q + z$.

Definition 2.17. Let $e \colon B_n \to \mathbb{Z}$ denote the *braid exponent*, the additive homomorphism where $e(\sigma_i) = 1$ for all i.

Definition 2.18 ([1, Definition 6.1])**.** For a braid $\alpha \in B_n$, the *HOMFLYPT polynomial under Jones normalization* is the product

$$X_{\hat{\alpha}}(q, \Lambda) := \left(-\frac{1 - \Lambda q}{\sqrt{\Lambda}(1 - q)}\right)^{n-1} (\sqrt{\Lambda})^{e(\alpha)} \sum_{\lambda \vdash n} \Omega_\lambda \chi_\lambda(\alpha).$$

This two-variable polynomial $X_{\hat{\alpha}}(q, \Lambda)$ is a link invariant. The sum $\sum_{\lambda \vdash n} \Omega_\lambda \chi_\lambda(\alpha)$ is sometimes referred to as the *Ocneanu trace*.

Remark. The HOMFLYPT polynomial often admits the much more common normalization $P_L(a, z)$, related to the Jones normalization by $X_L(q, \Lambda) = P_L((\Lambda q)^{-1/2}, q^{1/2} - q^{-1/2})$.

Ocneanu originally calculated the eponymous weights, but this work has been lost. A reproof of the validity of these weights was given by Geck [6].

Example 2.19 ([4, Example A.10])**.** Under the Jones specialization $q = \Lambda = t$, the weights Ω_λ simplify drastically, as shown in [4]. Specifically, $\Omega_\lambda = 0$ if λ has more than two columns, and the braid closure of any $\alpha \in B_n$ has Jones polynomial

$$V_{\hat{\alpha}}(t) = \frac{(-1)^{n-1}(\sqrt{t})^{e(\alpha)-n+1}}{t+1} \sum_{k=0}^{\lfloor n/2 \rfloor} \left(\sum_{i=k}^{n-k} t^i \right) \chi_{(n-k,k)^\top}(\alpha).$$

When $n = 3$, this simplifies to

$$V_{\hat{\alpha}}(t) = (\sqrt{t})^{e(\alpha)-2}((t^2+1)\chi_{(1,1,1)}(\alpha) + t\chi_{(2,1)}(\alpha)).$$

Example 2.20 ([1, Section 7])**.** Under the Alexander specialization $q = \frac{1}{\Lambda} = t$, the simplification of the weights is less clear as it seems like $z = -\frac{1-q}{1-\Lambda q}$ will exhibit a singularity. However, it turns out that such singularities are removable due to the factor of $(1-\Lambda q)^{n-1}$ in Definition 2.18.

Let μ_i be the hook-type partition made up of one row of i squares and one column of $n - i$ squares beneath. As computed in [1, Equation 7.2], we find that

$$\Delta_\alpha(t) = \frac{(-1)^n(t-1)}{(\sqrt{t})^{e(\alpha)-n+1}(t^n-1)} \sum_{i=1}^{n} (-1)^i \chi_{\mu_i}(\alpha).$$

When $n = 3$, this implies that

$$\Delta_{\hat{\alpha}}(t) = \frac{\chi_{(1,1,1)}(\alpha) - \chi_{(2,1)}(\alpha) + \chi_{(3)}(\alpha)}{(\sqrt{t})^{e(\alpha)-2}(t^2+t+1)}.$$

Remark. If we substitute $t = \zeta_n = e^{2\pi i/n}$ into Example 2.20 above, the denominator vanishes, implying that the numerator must vanish too. In particular, we must have

$$\sum_{i=1}^{n} (-1)^i \chi_{\mu_i}(\alpha) = 0$$

at $t = \zeta_n$. If we apply the KZ functor to the exact sequence proven due to the BGG resolution in [7], we expect an exact sequence of ζ_n-Specht modules on hook-shape partitions.

3 Closures of 3-braids

We investigate the Jones and Alexander polynomials of 3-braid closures at -1.

Recall from Examples 2.12 and 2.13 that $S_q^{(1,1,1)}$ is the sign representation, where each g_i acts as -1, while $S_q^{(3)}$ is the trivial representation, where each g_i acts as q. When $q=-1$, the representations become isomorphic, so $\chi_{(1,1,1)}(\alpha)=\chi_{(3)}(\alpha)$. Hence

$$V_{\hat{\alpha}}(-1)=(\sqrt{-1})^{e(\alpha)-2}(2\chi_{(1,1,1)}(\alpha)-\chi_{(2,1)}(\alpha))$$

and

$$\begin{aligned}\Delta_{\hat{\alpha}}(-1)&=(\sqrt{-1})^{-e(\alpha)+2}(\chi_{(1,1,1)}(\alpha)-\chi_{(2,1)}(\alpha)+\chi_{(3)}(\alpha))\\&=(\sqrt{-1})^{-e(\alpha)+2}(2\chi_{(1,1,1)}(\alpha)-\chi_{(2,1)}(\alpha)).\end{aligned}$$

Note the use of $\sqrt{-1}$ rather than i or $-i$ due to the lack of a continuous, single-valued square root over $\mathbb{C}$. If we define $\sqrt{-1}$ to be $\{i,-i\}$, then clearly $V_{\hat{\alpha}}(-1)=\Delta_{\hat{\alpha}}(-1)=X_{\hat{\alpha}}(-1,-1)$. Otherwise, the usual convention where $i=\sqrt{-1}$ yields that $V_{\hat{\alpha}}(-1)=(-1)^{e(\alpha)}\Delta_{\hat{\alpha}}(-1)$.

From now on, we will adopt the latter convention where $V_{\hat{\alpha}}(-1)=(-1)^{e(\alpha)}\Delta_{\hat{\alpha}}(-1)$.

Definition 3.1. Let $\phi\colon B_3\to \mathrm{SL}_2(\mathbb{Z})$ be the homomorphism satisfying $\phi(\sigma_i)=M_i$, where

$$(M_1,M_2)=\left(\begin{pmatrix}1&0\\-1&1\end{pmatrix},\begin{pmatrix}1&1\\0&1\end{pmatrix}\right).$$

Proposition 3.2. *For a braid $\alpha\in B_3$, the Jones and Alexander polynomials of $\hat{\alpha}$ satisfy $V_{\hat{\alpha}}(-1)=(-i)^{e(\alpha)}(\mathrm{tr}(\phi(\alpha))-2)$ and $\Delta_{\hat{\alpha}}(-1)=i^{e(\alpha)}(\mathrm{tr}(\phi(\alpha))-2)$.*

Proof. From Example 2.14 we note that g_i acts as $-M_i$ in $S^{-1}_{(2,1)}$, while g_i acts as -1 in $S^{-1}_{(1,1,1)}$. Thus, as found above, we have

$$\begin{aligned}V_{\hat{\alpha}}(-1)&=i^{e(\alpha)-2}(2\chi_{(1,1,1)}(\alpha)-\chi_{(2,1)}(\alpha))\\&=i^{e(\alpha)}(-1)(2(-1)^{e(\alpha)}-(-1)^{e(\alpha)}\,\mathrm{tr}(\phi(\alpha)))\\&=(-i)^{e(\alpha)}(\mathrm{tr}(\phi(\alpha))-2),\end{aligned}$$

as claimed. □

An interesting question to consider is which integer values $V_{\hat{\alpha}}(-1)$ can possibly take, and what type of link $\hat{\alpha}$ corresponds to. Here, we shall consider two extremal cases: when $\hat{\alpha}$ is split and when $\hat{\alpha}$ is a knot.

3.1 Split Links

From from Proposition 1.3 we have $V_L(-1) = \Delta_L(-1) = 0$ for all split links. We show that this condition is not sufficient.

Corollary 3.3. *Let $\alpha \in B_3$. Then $V_{\hat{\alpha}}(-1) = \Delta_{\hat{\alpha}}(-1) = 0$ if and only if $\operatorname{tr}\phi(\alpha) = 2$.*

Proof. Immediate by Proposition 3.2. □

The following result is due to Serre [8].

Proposition 3.4 ([8, Theorem 2.1.4])**.** *The group $\mathrm{SL}_2(\mathbb{Z})$ has presentation*

$$\mathrm{SL}_2(\mathbb{Z}) \cong \langle s, t \mid s^4 = 1, (st)^3 = s^2 \rangle$$

where the correspondence of matrices is given from (s, t) to

$$(S, T) = \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right).$$

Now we claim the following.

Lemma 3.5. *The following statements are true.*

(a) The matrices M_1 and M_2 generate the group $\mathrm{SL}_2(\mathbb{Z})$.

(b) The group $\mathrm{SL}_2(\mathbb{Z})$ admits the presentation

$$\mathrm{SL}_2(\mathbb{Z}) \cong \langle m_1, m_2 \mid (m_1 m_2 m_1)^4 = m_1 m_2 m_1 m_2^{-1} m_1^{-1} m_2^{-1} = 1 \rangle.$$

Proof. Retain the notation as in Proposition 3.4. Check that $(S, T) = (M_1 M_2 M_1, M_1^{-1})$ and $(M_1, M_2) = (T^{-1}, TST)$, so that any word in the alphabet $\{s, s^{-1}, t, t^{-1}\}$ can be translated to one in the alphabet $\{m_1, m_1^{-1}, m_2, m_2^{-1}\}$ and vice versa. This proves part (a) of the desired result.

Now, we can simply make the substitution $(s, t) = (m_1 m_2 m_1, m_1^{-1})$, which has the inverse substitution $(m_1, m_2) = (t^{-1}, tst)$, as follows from the above matrix transformations. Hence

$$\begin{aligned} \mathrm{SL}_2(\mathbb{Z}) &\cong \langle s, t \mid s^4 = 1, (st)^3 = s^2 \rangle \\ &\cong \langle m_1, m_2 \mid (m_1 m_2 m_1)^4 = 1, (m_1 m_2)^3 = (m_1 m_2 m_1)^2 \rangle \\ &\cong \langle m_1, m_2 \mid (m_1 m_2 m_1)^4 = m_1 m_2 m_1 m_2^{-1} m_1^{-1} m_2^{-1} = 1 \rangle, \end{aligned}$$

which proves part (b). □

Following Garside [9], define $\Delta_3 = \sigma_1 \sigma_2 \sigma_1 \in B_3$.

Proposition 3.6. *The map ϕ induces an isomorphism between the groups $B_3 / \langle \Delta_3^4 \rangle$ and $\mathrm{SL}_2(\mathbb{Z})$.*

Proof. From [9] we have that $Z(B_3) = \langle \Delta_3^2 \rangle$. In particular, quotienting by $\langle \Delta_3^4 \rangle$ simply introduces a corresponding relation, so by Lemma 3.5 we have

$$\langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1} = 1 \rangle / \langle \Delta_3^4 \rangle \cong \langle m_1, m_2 \mid (m_1m_2m_1)^4 = m_1m_2m_1m_2^{-1}m_1^{-1}m_2^{-1} = 1 \rangle$$

with an isomorphism sending σ_i to m_i.

Since $\phi(\Delta_3^4)$ acts as the identity, ϕ is constant over all elements within any coset of $\langle \Delta_3^4 \rangle$ as embedded in B_3, so the above isomorphism is well-defined. □

We can thus reduce our original task of finding 3-braids whose braid closures have Jones polynomial with root -1 with the task of finding conjugacy classes in $\mathrm{SL}_2(\mathbb{Z})$.

Proposition 3.7. *Let $\alpha \in B_3$ satisfy $\phi(\alpha) = 2$. Then there exists some $\alpha_0 \in B_3$ and $k \in \mathbb{Z}$ such that $\phi(\alpha_0) = M_2^k$ and $\hat{\alpha} = \hat{\alpha}_0$.*

Proof. It suffices to show that, given some matrix $A \in \mathrm{SL}_2(\mathbb{Z})$ such that $\operatorname{tr} A = 2$, there exists some $B \in \mathrm{SL}_2(\mathbb{Z})$ such that $BAB^{-1} = M_2^k$ for some $k \in \mathbb{Z}$. Indeed, let $A = \phi(\alpha)$. By Lemma 3.5, the M_i generate $\mathrm{SL}_2(\mathbb{Z})$, so there exists some $\beta \in B_3$ such that $B = \phi(\beta)$. Define $\alpha_0 \in B_3$ to be $\alpha_0 = \beta\alpha\beta^{-1}$. Then $\phi(\alpha_0) = BAB^{-1} = M_2^k$ and $\hat{\alpha}_0 = \hat{\alpha}$ as we can simply rotate the braid closure of α_0 such that the components corresponding to β and β^{-1} cancel out.

We now give an explicit construction of our above claim. Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Since $\operatorname{tr} A = 2$ and $\det A = 1$, the eigenvalue of A is 1 with multiplicity 2. Thus, there exists some $(x, y, z, w) \in \mathbb{Z}^4$ such that $\gcd(x, y) = 1$, $xw - yz = 1$, and $A(x\mathbf{e}_1 + y\mathbf{e}_2) = x\mathbf{e}_1 + y\mathbf{e}_2$. Then define the matrix

$$B = \begin{pmatrix} x & z \\ y & w \end{pmatrix}^{-1} = \begin{pmatrix} w & -z \\ -y & x \end{pmatrix},$$

which is an element of $\mathrm{SL}_2(\mathbb{Z})$ by construction. Letting $\{\mathbf{e}_1, \mathbf{e}_2\}$ be the canonical basis for $\mathbb{Z}^2$, we can now check that

$$BAB^{-1}\mathbf{e}_1 = BA(x\mathbf{e}_1 + y\mathbf{e}_2) = B(x\mathbf{e}_1 + y\mathbf{e}_2) = \mathbf{e}_1$$

and

$$(BAB^{-1}\mathbf{e}_2) \cdot \mathbf{e}_2 = w(dx - by) + z(cx - ay) = \det B(\operatorname{tr} A - 1) = 1.$$

Hence, for some $k \in \mathbb{Z}$,

$$BAB^{-1} = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} = M_2^k,$$

as desired. □

Now we introduce the main results of this subsection.

Theorem 3.8. *Let $\alpha \in B_3$ satisfy $V_{\hat{\alpha}}(-1) = \Delta_{\hat{\alpha}}(-1) = 0$. Then there exist integers m and k for which $\hat{\alpha}$ is equivalent to the braid closure of $\sigma_2^k \Delta_3^{4m}$.*

Proof. By Corollary 3.3 and Proposition 3.7, there exists some $\alpha_0 \in B_3$ and $k \in \mathbb{Z}$ satisfying $\phi(\alpha_0) = M_2^k$ and $\hat{\alpha} = \hat{\alpha}_0$. Thus $\sigma_2^{-k}\alpha_0$ is sent by ϕ to the identity, from which by Proposition 3.6 it follows that $\sigma_2^{-k}\alpha_0 \in \langle \Delta_3^4 \rangle$. Then α_0 is of the form $\sigma_2^k \Delta_3^{4m}$ for some m. □

Theorem 3.9. *For $m, k \in \mathbb{Z}$, let $\gamma_{m,k}$ denote the braid $\sigma_2^k \Delta_3^{4m}$, so that $\hat{\gamma}_{m,k}$ denotes its oriented link closure.*

(a) The link $\hat{\gamma}_{m,k}$ is split if and only if $m = 0$.

(b) For $m_1, m_2, k_1, k_2 \in \mathbb{Z}$, the links $\hat{\gamma}_{m_1,k_1}$ and $\hat{\gamma}_{m_2,k_2}$ are equivalent iff $(m_1, k_1) = (m_2, k_2)$.

Proof. First, suppose that $2 \mid k$. Consider the homomorphism $\psi\colon B_3 \to \mathfrak{S}_3$ with $\psi(\sigma_i) = (i\ i+1)$. Then $\hat{\gamma}_{m,k}$ has three circles as $\psi(\gamma_{m,k}) = (2\ 3)^k((1\ 2)(2\ 3)(1\ 2))^{4m} = 1$. By inspection, one can check that removing the strand unaffected by σ_2 from $\gamma_{m,k}$ results in the braid $\sigma_1^{4m+k} \in B_2$, while if any other strand is removed, the resulting braid is $\sigma_1^{4m} \in B_2$. For part (a), if $\hat{\gamma}_{m,k}$ is split, then at least two out of the three elements in the multiset $\{4m, 4m, 4m+k\}$ are equal to zero, forcing $m = 0$. For part (b), if $\hat{\gamma}_{m_1,k_1}$ and $\hat{\gamma}_{m_2,k_2}$ are equivalent for even k_i, then the multisets $\{4m_1, 4m_1, 4m_1+k_1\}$ and $\{4m_2, 4m_2, 4m_2+k_2\}$ are equal, implying that $(m_1, k_1) = (m_2, k_2)$.

Otherwise, if $2 \nmid k$, then $\hat{\gamma}_{m,k}$ is a link of two circles as $\psi(\gamma_{m,k}) = (2\,3)^k = (2\,3)$. Then the two strands affected by σ_2 form a $T(2, 4m+k)$ torus link. Let W be the bounding torus of this link. Then the third strand is an unknot that wraps $2m$ times around a cross-sectional disk of W. Hence, for part (a), if $\hat{\gamma}_{m,k}$ is split, then the third strand cannot wrap around the bounding torus at all, again forcing $m = 0$. For part (b), if $\hat{\gamma}_{m_1,k_1}$ and $\hat{\gamma}_{m_2,k_2}$ are equivalent for odd n_i, then $(2m_1, 4m_1+k_1) = (2m_2, 4m_2+k_2)$, and the desired conclusion follows.

Note for part (b) that counting circles gives $k_1 \equiv k_2 \pmod 2$, as subsumed above. □

Thus we have an infinite family of distinct links with no splitting whose Jones and Alexander polynomials vanish at $t = -1$. It turns out that certain pairs of links fail to be detected by the HOMFLYPT polynomial.

Theorem 3.10. *If $6 \mid k$, then $X_{\hat{\gamma}_{m,k}}(q, \Lambda) = X_{\hat{\gamma}_{m+\frac{k}{6},-k}}(q, \Lambda)$.*

Proof. We first compute $\chi_{(2,1)}(\gamma_{m,k})$. The Burau representation lets σ_i act as $-L_i$, where

$$(L_1, L_2) = \left(\begin{pmatrix} -q & 0 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -q \\ 0 & -q \end{pmatrix} \right).$$

One can check that $(L_1L_2L_1)^4$ acts as scalar multiplication by q^6. Also note that $L_2^k \mathbf{e}_1 = \mathbf{e}_1$ as $\mathbf{e}_1$ is an eigenvector and $(L_2^k \mathbf{e}_2) \cdot \mathbf{e}_2 = -q(L_2^{k-1}(\mathbf{e}_1 + \mathbf{e}_2)) \cdot \mathbf{e}_2 = -q(\mathbf{e}_1 + L_2^{k-1}\mathbf{e}_2) \cdot \mathbf{e}_2 =$

$-q(L_2^{k-1}\mathbf{e}_2)\cdot\mathbf{e}_2$. Thus $(L_2^k\mathbf{e}_2)\cdot\mathbf{e}_2 = (-q)^k$ by induction, so it follows that $\operatorname{tr}(L_2^k) = 1+(-q)^k$. We thus have

$$\chi_{(2,1)}(\gamma_{m,k}) = (-1)^{e(\gamma_{m,k})}\operatorname{tr}(L_2^k(L_1L_2L_1)^{4m}) = (-1)^{e(\gamma_{m,k})}q^{6m}(1+(-q)^k).$$

Since $e(\gamma_{m,k}) = e(\gamma_{m+\frac{k}{6},-k}) = 12m+k$, one checks that $\chi_{(2,1)}(\gamma_{m,k}) = \chi_{(2,1)}(\gamma_{m+\frac{k}{6},-k})$.

In addition, it is clear that $\chi_{(1,1,1)}(\gamma_{m,k}) = \chi_{(1,1,1)}(\gamma_{m+\frac{k}{6},-k})$ and $\chi_{(3)}(\gamma_{m,k}) = \chi_{(3)}(\gamma_{m+\frac{k}{6},-k})$. The desired result then follows by Definition 2.18. □

We finish this subsection by constructing the Jones and Alexander polynomials of $\hat{\gamma}_{m,k}$.

Proposition 3.11. *The Jones and Alexander polynomials of $\hat{\gamma}_{m,k}$ are*

$$V_{\hat{\gamma}_{m,k}}(t) = (-\sqrt{t})^{12m+k}\left(t^{6m}((-t)^k+1)+\frac{1}{t}+t\right)$$

and

$$\Delta_{\hat{\gamma}_{m,k}}(t) = \frac{(-t)^{12m+k} - t^{6m}((-t)^k+1)+1}{(-\sqrt{t})^{12m+k-2}(t^2+t+1)}.$$

Proof. From Theorem 3.10, we have that $\chi_{(2,1)}(\gamma_{m,k}) = (-1)^k q^{6m}(1+(-q)^k)$ and $e(\gamma_{m,k}) = 12m+k$. It follows from Examples 2.19 and 2.20 that

$$\begin{aligned} V_{\hat{\gamma}_{m,k}}(t) &= (\sqrt{t})^{e(\gamma_{m,k})-2}((t^2+1)\chi_{(1,1,1)}(\gamma_{m,k}) + t\chi_{(2,1)}(\gamma_{m,k})) \\ &= (-\sqrt{t})^{12m+k}\left(t^{6m}((-t)^k+1)+t+\frac{1}{t}\right) \end{aligned}$$

and

$$\begin{aligned} \Delta_{\hat{\gamma}_{m,k}}(t) &= \frac{\chi_{(1,1,1)}(\gamma_{m,k}) - \chi_{(2,1)}(\gamma_{m,k}) + \chi_{(3)}(\gamma_{m,k})}{(\sqrt{t})^{e(\gamma_{m,k})-2}(t^2+t+1)} \\ &= \frac{(-t)^{12m+k} - t^{6m}((-t)^k+1)+1}{(-\sqrt{t})^{12m+k-2}(t^2+t+1)}. \end{aligned}$$

In both cases, we have substituted t for q. □

Remark. For even k, Proposition 3.11 simplifies to

$$V_{\hat{\gamma}_{m,k}}(t) = t^{6m+\frac{k}{2}-1} + t^{6m+\frac{k}{2}+1} + t^{12m+\frac{k}{2}} + t^{12m+\frac{3k}{2}}$$

and

$$\Delta_{\hat{\gamma}_{m,k}}(t) = \frac{t^{6m+\frac{k}{2}+1} - t^{\frac{k}{2}+1} - t^{-\frac{k}{2}+1} + t^{-6m-\frac{k}{2}+1}}{t^2+t+1}.$$

3.2 Knots

Recall from Proposition 3.9 the natural homomorphism $\psi\colon B_3 \to \mathfrak{S}_3$ from the braid group to the symmetric group, given by $\psi(\sigma_i) = (i\ i+1)$. The number of circles in the link $\hat{\alpha}$ is determined by the cycle type of $\psi(\alpha)$, with $\hat{\alpha}$ being a knot iff $\psi(\alpha)$ is a 3-cycle.

Proposition 3.12. *Braids* $\alpha_1, \alpha_2 \in B_3$ *satisfy* $\psi(\alpha_1) = \psi(\alpha_2)$ *iff* $\phi(\alpha_1) \equiv \phi(\alpha_2) \pmod 2$.

Proof. It suffices to show that $\alpha \in B_3$ satisfies $\psi(\alpha) = e$ iff $\phi(\alpha) \equiv I \pmod 2$.

First, we have by definition that

$$B_3 \cong \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1} = 1\rangle$$

while

$$\mathfrak{S}_3 \cong \langle s_1, s_2 \mid s_1^2 = s_2^2 = s_1s_2s_1s_2^{-1}s_1^{-1}s_2^{-1} = e\rangle \cong \langle s_1, s_2 \mid s_1s_2s_1s_2^{-1}s_1^{-1}s_2^{-1} = e\rangle / \langle s_1^2, s_2^2\rangle.$$

Hence, $\psi(\alpha) = e$ iff we can reduce α to the trivial braid by deleting instances of σ_i^2.

Next, $M_1^2 \equiv M_2^2 \equiv I \pmod 2$. Then one checks that

$$\mathrm{SL}_2(\mathbb{Z}/2\mathbb{Z}) \cong \langle m_1, m_2 \mid m_1^2 = m_2^2 = m_1m_2m_1m_2^{-1}m_1^{-1}m_2^{-1} = 1\rangle,$$

where again m_i is represented by the matrix M_i. The relation $(m_1m_2m_1)^4 = 1$ may be dropped as it is implied by setting $m_1^2 = m_2^2 = 1$. We find again that $\phi(\alpha) \equiv I \pmod 2$ iff we can reduce α to the trivial braid by deleting instances of σ_i^2, as claimed. □

Remark. Observe that $\mathfrak{S}_3$ and $\mathrm{SL}_2(\mathbb{Z}/2\mathbb{Z})$ are isomorphic, as both groups are nonabelian of order 6. The isomorphism map $\theta\colon \mathfrak{S}_3 \to \mathrm{SL}_2(\mathbb{Z}/2\mathbb{Z})$ is canonically constructed via the permutation action of $\mathrm{SL}_2(\mathbb{Z}/2\mathbb{Z})$ on $\{\binom{0}{1}, \binom{1}{0}, \binom{1}{1}\} \subset \mathbb{F}_2^2$. Let $\pi_2\colon \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/2\mathbb{Z})$ be the usual projection. Then Proposition 3.12 induces θ to be precisely the lift from ψ to $\pi_2 \circ \phi$, as in the following commutative diagram.

$$\begin{array}{ccc} B_3 & \xrightarrow{\ \phi\ } & \mathrm{SL}_2(\mathbb{Z}) \\ {\scriptstyle\psi}\big\downarrow & & \big\downarrow{\scriptstyle\pi_2} \\ \mathfrak{S}_3 & \xrightarrow[\ \theta\ ]{} & \mathrm{SL}_2(\mathbb{Z}/2\mathbb{Z}) \end{array}$$

Note that Proposition 3.12 implies that the number of circles in the link $\hat{\alpha}$ depends only on the parity of the entries of $\phi(\alpha)$.

Corollary 3.13. *If* $\hat{\alpha}$ *is a knot for* $\alpha \in B_3$*, then* $\phi(\alpha) \in \{\left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)\} \pmod 2$. *In particular,* $\mathrm{tr}(\phi(\alpha))$ *is odd.*

Proof. Recall that $\hat{\alpha}$ is a knot iff $\psi(\alpha) \in \{(1\ 2\ 3), (1\ 3\ 2)\} = \{\psi(\sigma_1\sigma_2), \psi(\sigma_2\sigma_1)\}$. Then by Proposition 3.12 it follows that $\phi(\alpha)$ is congruent to either M_1M_2 or M_2M_1 modulo 2, which is exactly the desired claim. □

Recall from Proposition 3.2 that

$$V_{\hat{\alpha}}(-1) = (-i)^{e(\alpha)}(\mathrm{tr}(\phi(\alpha)) - 2)$$

and $\Delta_{\hat{\alpha}}(-1) = (-1)^{e(\alpha)} V_{\hat{\alpha}}(-1)$. Thus, the residue class of $e(\alpha)$ modulo 4 may affect the true value of $V_{\hat{\alpha}}(-1)$ and $\Delta_{\hat{\alpha}}(-1)$.

Proposition 3.14. *For $\alpha \in B_3$, the trace $\mathrm{tr}(\phi(\alpha))$ is odd only when $e(\alpha)$ is even. In this case, $\mathrm{tr}(\phi(\alpha)) \equiv e(\alpha) - 1 \pmod 4$.*

Proof. Express α as a word in σ_1, σ_2, σ_1^{-1}, and σ_2^{-1}. Correspondingly, write $\phi(\alpha)$ in terms of M_1, M_2, M_1^{-1} and M_2^{-1}. The signed length of this word is exactly $e(\alpha)$. Now we can replace each instance of M_i^{-1} with a word in only M_1 and M_2 of length 11, as $M_1^{-1} = (M_2 M_1^2)^3 M_2 M_1$ and $M_2^{-1} = (M_1 M_2^2)^3 M_1 M_2$. This gives us a new word whose length is congruent to $e(\alpha)$ modulo 4. It suffices to look at the trace of words of length k that are generated by $\{M_1, M_2\}$.

For $i \leq 3$, let $X_i \subset \mathrm{SL}_2(\mathbb{Z}/4\mathbb{Z})$ be the set containing all such words with length congruent to i modulo 4. We check that X_0 is a subgroup and isomorphic to the alternating group A_4, while the X_i are its cosets. By enumerating the X_i for each j we find that $\mathrm{tr}(\phi(\alpha))$ is 0 or 2 modulo 4 for $\phi(\alpha) \in X_1 \cup X_3$. For $\phi(\alpha) \in X_0$ we find that $\mathrm{tr}(\phi(\alpha))$ is 2 or 3 modulo 4, and for $\phi(\alpha) \in X_2$ we find that $\mathrm{tr}(\phi(\alpha))$ is 1 or 2 modulo 4. Either way, we are done. □

Remark. Observe the semidirect decomposition $\mathrm{SL}_2(\mathbb{Z}/4\mathbb{Z}) \cong X_0 \rtimes \mathbb{Z}/4\mathbb{Z}$, and $X_0 \cong A_4$ is the commutator of $\mathrm{SL}_2(\mathbb{Z}/4\mathbb{Z})$. This implies the epimorphism $\eta\colon \mathrm{SL}_2(\mathbb{Z}/4\mathbb{Z}) \to \mathbb{Z}/4\mathbb{Z}$. In the case of Proposition 3.14, $\eta(m) = i$ if $m \in X_i$. Letting $\pi_4\colon \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/4\mathbb{Z})$ be the usual projection, the composition of all relevant maps from B_3 to $\mathbb{Z}/4\mathbb{Z}$ is exactly the braid exponent modulo 4, yielding the following commutative diagram.

$$\begin{array}{ccc} B_3 & \xrightarrow{e \bmod 4} & \mathbb{Z}/4\mathbb{Z} \\ {\scriptstyle\phi}\big\downarrow & & \big\uparrow{\scriptstyle\eta} \\ \mathrm{SL}_2(\mathbb{Z}) & \xrightarrow[\pi_4]{} & \mathrm{SL}_2(\mathbb{Z}/4\mathbb{Z}) \end{array}$$

The main result of this subsection now follows.

Theorem 3.15. *If L is a knot with index at most 3, then $V_L(-1) \equiv \Delta_L(-1) \equiv 1 \pmod 4$.*

Proof. If $L = \hat{\alpha}$ then $\mathrm{tr}(\phi(\alpha))$ is odd by Corollary 3.13. Let $\mathrm{tr}(\phi(\alpha)) \equiv 2b - 1 \pmod 4$ where $b \in \{0, 1\}$. By Proposition 3.14 we have $e(\alpha) \equiv 2b \pmod 4$, so

$$V_L(-1) = (-i)^{e(\alpha)}(\mathrm{tr}(\phi(\alpha)) - 2) \equiv (-1)^b(2b - 3) \equiv 1 \pmod 4,$$

and $\Delta_L(-1) \equiv (-1)^{2b} V_L(-1) \equiv 1 \pmod 4$, so we are done. □

3.3 4-braids and beyond

Consider the Jones polynomial of 4-braid closures. In particular, Example 2.19 yields that

$$V_{\hat{\alpha}}(t) = -\frac{(\sqrt{t})^{e(\alpha)}((1+t+t^2+t^3+t^4)\chi_{(1,1,1,1)}(\alpha) + (t+t^2+t^3)\chi_{(2,1,1)}(\alpha) + t^2\chi_{(2,2)}(\alpha))}{t^{3/2}+t^{5/2}}$$

for $\alpha \in B_4$. Substituting $t = -1$, it must then be the case that

$$\chi_{(1,1,1,1)}(\alpha) - \chi_{(2,1,1)}(\alpha) + \chi_{(2,2)}(\alpha) = 0$$

for all $\alpha \in B_4$ upon specialization to $t = -1$, which can also seen to be true as, applying categorification, we have the long exact sequence

$$0 \longrightarrow S_{-1}^{(1,1,1,1)} \longrightarrow S_{-1}^{(2,1,1)} \longrightarrow S_{-1}^{(2,2)} \longrightarrow 0$$

due to one of the main results developed in [4]. We may possibly circumvent this issue is to instead look at the braid $\alpha' = \alpha\sigma_4 \in B_5$, as $\hat{\alpha}$ and $\hat{\alpha}'$ are equivalent links. Thus, we can avoid discussing 4-braids and instead discuss 5-braids, where the singularity of the Jones polynomial at $t = -1$ is easily removable.

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