BOUNDS ON THE DISTINGUISHING (CHROMATIC) NUMBER OF POSETS

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Abstract. In 2021, Collins and Trenk introduced the distinguishing number and distinguishing chromatic number of posets as analogs for the distinguishing (chromatic) number of graphs. A coloring c of a poset P is distinguishing if there are no nontrivial automorphisms of P preserving c. The distinguishing number $D(P)$ is the minimum number of colors in a distinguishing coloring. The distinguishing chromatic number $\chi_D(P)$ is the minimum number of colors in a proper distinguishing coloring. We present the bound $D(L) \leq |Q_L| - h(L) + 2$, where L is a lattice with join-irreducible set Q_L and height $h(L)$. For distributive lattices, Collins and Trenk showed that $\chi_D(L) \leq |Q_L| + \chi_D(Q_L) - 1$ when $\chi_D(Q_L) \geq 3$. We improve the bound to $|Q_L| + k$, where $k \leq 3$ is determined by Q_L. Our bound is sharp for boolean lattices. We also establish bounds on the distinguishing number of graded lattices via the motion lemma, and we compute the distinguishing (chromatic) number of the Young-Fibonacci lattice.

1. Introduction

Albertson and Collins [1] defined distinguishing colorings and the distinguishing number of graphs. A vertex coloring of a graph G is *distinguishing* if the only automorphism of the graph that preserves colors is the identity. The *distinguishing number* $D(G)$ of a graph G is the minimum number of colors in any distinguishing coloring of G. Collins and Trenk [9] introduced the *distinguishing chromatic number* $\chi_D(G)$ of a graph G, which is the minimum number of colors in any distinguishing and proper coloring of G, where a coloring is proper if no two adjacent vertices are the same color.

The distinguishing (chromatic) number of graphs has been studied extensively. For example, the distinguishing number of the hypercube [4], trees and forests [5], and the cartesian product of graphs [12] have been calculated. Furthermore, Collins, Hovey, and Trenk [8] bounded the distinguishing chromatic number of graph G by $\chi(G)$ and $\mathrm{Aut}(G)$, and Cavers and Seyffarth [6] characterized graphs with large distinguishing chromatic numbers.

A natural extension then is to define the distinguishing (chromatic) number of a poset, which was done by Collins and Trenk [10]. A coloring of a poset is *distinguishing* if the only automorphism of the poset that preserves the coloring is the identity. The *distinguishing number* $D(P)$ of a poset P is the minimum number of colors in any distinguishing coloring of P. Furthermore, a coloring is *proper* if no two comparable elements are the same color. The *distinguishing chromatic number* $\chi_D(P)$ of a poset P is the minimum number of colors in a proper distinguishing coloring of P.

The *comparability graph* G_P of poset P has elements of P as vertices and edges between two elements if and only if those elements are comparable. Comparing poset P to G_P, we have $D(P) \leq D(G_P)$ and $\chi_D(P) \leq \chi_D(G_P)$. However, Collins and Trenk [10, Proposition 22] demonstrated that $D(G_P) - D(P)$ and $\chi_D(G_P) - \chi_D(P)$ can be arbitrarily large. Hence,

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the ordering of posets alters the number of colors needed to break symmetries, and the distinguishing (chromatic) number of posets merits independent study.

Posets are a subset of directed graphs, and many results for the distinguishing (chromatic) number of graphs also hold for posets. For example, Choi [7] established analogues for posets for the bounds presented by Albertson and Collins [1] and Collins and Trenk [9].

We extend the work of Collins and Trenk [10] and Choi [7]. In Section 2, we formally define the distinguishing (chromatic) number and present elementary bounds. In Section 3, we establish a bound for $D(P)$ based on $\mathrm{Aut}(P)$, which is sharper than previous bounds by up to a factor of $h(P)$. In Section 4, we improve the bounds for the distinguishing chromatic number for distributive lattices. Our bound is sharp for boolean lattices. We also present bounds on the distinguishing (chromatic) number for arbitrary lattices. In Section 5, we establish a bound on the distinguishing number for graded lattices via the motion lemma. In Section 6, we compute the distinguishing (chromatic) number of the Young-Fibonacci lattice.

2. Preliminaries

2.1. **Posets.** A poset $P = (S, \preceq)$ is defined by a *ground set* of elements S and a reflexive, antisymmetric, transitive *order* $\preceq$ between the elements. We assume S is finite. An automorphism φ of P is a bijection on S preserving $\preceq$. For elements $\{s, t\} \subseteq S$, we write $s \prec t$ if $s \preceq t$ and $s \neq t$. If $s \preceq t$ or $t \preceq s$, then s and t are *comparable*. Otherwise, s and t are *incomparable*. If $s \prec t$ and there exists no element $r \in S$ such that $s \prec r \prec t$, then t *covers* s and we write $s \lessdot t$. The *comparability graph* of poset P, denoted by G_P, is the graph with vertex set S and edges drawn between any two comparable elements. The *Hasse diagram* of a poset P is a graph with vertex set S and edges drawn between $\{a, b\} \subseteq S$ if $a \gtrdot b$ or $a \lessdot b$. Furthermore, a is drawn with a higher y coordinate than b if $a \succeq b$.

For an element s of poset P, define $\deg_\uparrow(s)$ as the number of elements that cover s and define $\deg_\downarrow(s)$ as the number of elements that s covers.

A *chain* of length $h-1$ of a poset is a set of h pairwise comparable elements. An *antichain* of width w is a set of w pairwise incomparable elements. Let the *height* $h(P)$ be the length of the longest chain in P and let the *width* $w(P)$ be the width of the longest antichain.

An element $s \in S$ is *maximal* if there exists no element $t \in S$ such that $s \prec t$. An element $s \in S$ is *minimal* if there exists no element $t \in S$ such that $s \succ t$. Let $\max(P)$ and $\min(P)$ denote the set of maximal and minimal elements for P, respectively.

A poset P is *graded* if all maximal chains have the same length. Each graded poset has a *rank function* $\rho\colon S \to \{0, 1, \ldots, h\}$ which maps minimal elements to 0 and satisfies $\rho(t) = \rho(s) + 1$ if $t \gtrdot s$. Even if a poset P is not graded, the order of P allows us to assign indices to elements of P. Choi [7, Theorem 2.4] provides the following indexing.

Lemma 2.1 (Choi [7])**.** *For each poset $P = (S, \preceq)$, there exists an index $i(e)\colon S \to \{1, 2, \ldots, h(P) + 1\}$ which satisfies the following conditions.*

(i) The index is preserved under automorphism.
(ii) Distinct elements of the same index are incomparable.
(iii) If $\{e_1, e_2\} \subseteq S$ are two comparable elements satisfying $i(e_1) < i(e_2)$, then $e_1 \succeq e_2$.

2.2. **Lattices.** Let $P = (S, \preceq)$ be a poset. The *meet* of two elements $\{a, b\} \subseteq S$, denoted by $a \wedge b$, is the unique maximal element among those less than or equal to both a and b, if such an element exists. The *join* of two elements $\{a, b\} \subseteq S$, denoted by $a \vee b$, is the unique minimal element among those greater than or equal to both a and b.

Poset $P = (S, \preceq)$ is a *meet semi-lattice* if the meet exists for all pairs of elements $\{a, b\} \subseteq S$. Similarly, poset P is a *join semi-lattice* if the join exists for all pairs of elements $\{a, b\} \subseteq S$. Poset P is a *lattice* if it is both a meet and join semi-lattice. A lattice $L = (S, \preceq)$ is *distributive* if $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$ and $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$ for all triplets $\{a, b, c\} \subseteq S$.

An element of a lattice L with only one downward edge in the Hasse diagram of L is *join-irreducible*. Let Q_L denote the set of join-irreducible elements of lattice L. The following lemma shows that every element of L can be represented as the join of some subset of Q_L.

Lemma 2.2 (Davey and Priestley [11, Proposition 2.45]). *If L is a finite lattice and $a \in L$, then*

$$a = \bigvee \{x \in Q_L \mid x \preceq a\}.$$

For $a \in L$ define $J(a) = \{x \in Q_L \mid x \preceq a\}$ as the *join-irreducible rank* of a. For a poset $P = (S, \preceq)$ and subset a T of S, let $\text{down}(T) = \{s \in S \mid s \preceq t \text{ for some } t \in T\}$. Define $J(P)$ as the lattice with ground set $\{\text{down}(T) \mid T \subseteq S\}$ and ordering relation $\subseteq$. Birkhoff [3, Theorem 3] showed that distributive lattices can be represented by downsets.

Theorem 2.3 (Birkhoff [3, Theorem 3]). *If L is a distributive lattice, then $L \cong J(Q_L)$.*

We define two well-studied lattices, B_n and L_n. The *boolean lattice* B_n has ground set $\mathcal{P}(\{1, 2, \ldots, n\})$ and is ordered by inclusion. The boolean lattice B_3 is shown in Figure 1.

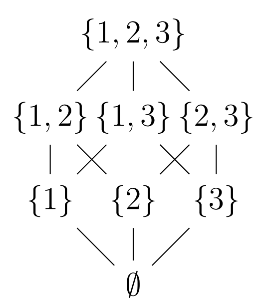

FIGURE 1. The boolean lattice B_3.

The *divisibility lattice* L_n has ground set as the divisors of n and is ordered by divisibility. The divisibility lattice L_{12} is shown in Figure 2.

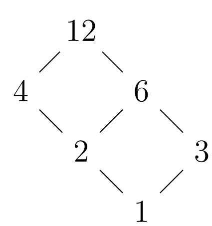

FIGURE 2. The divisibility lattice L_{12}.

Boolean lattices are isomorphic to divisibility lattices of products of distinct primes.

2.3. **The Distinguishing Number of Posets.** Collins and Trenk [10] label a coloring c of a poset $P = (S, \preceq)$ as *distinguishing* if no nontrivial automorphism of P preserves c. The *distinguishing number* of P, denoted by $D(P)$, is the smallest number of colors in any distinguishing coloring of P. An element of P is *pinned* by a coloring if all automorphisms preserving colors fix the element.

Collins and Trenk [10] noted that $D(P) \le D(G_P)$, but that $D(G_P)/D(P)$ can be arbitrarily large. For example, $D(P) = 1$ and $D(G_P) = k$ for a chain of length $k-1$.

Albertson and Collins [1, Corollary 1.1] established a bound for $D(G)$ in terms of $\operatorname{Aut}(G)$ for graphs G. Choi [7, Theorem 2.2] presented an analogous result for posets.

Theorem 2.4 (Choi [7, Theorem 2.2])**.** *If P is a poset with automorphism group $\operatorname{Aut}(P)$, then $D(P) \le 1 + \lfloor \log_2 |\operatorname{Aut}(P)| \rfloor$.*

By coloring according to index, we can bound the distinguishing number by the width of a poset.

Theorem 2.5. *If $P = (S, \preceq)$ is a poset, then $D(P) \le w(P)$.*

Proof. Assign indices to elements of S via Lemma 2.1. For each index, assign distinct colors to elements with that index so that no automorphism preserving colors can map one element of a given index to another element of that index. Automorphisms must preserve indices, so the coloring is distinguishing. Since indices form antichains, the coloring uses at most $w(P)$ colors. Hence, $D(P) \le w(P)$. □

As we add more comparison relations to a poset, the distinguishing number tends to decrease. For example, an antichain has large distinguishing number while a chain has distinguishing number one. Similar intuition does not hold for graphs as $D(K_n) = n$. Define $\rho(P) = \frac{|E(G_P)|}{\binom{|S|}{2}}$ as the *comparison density* of a poset P. The comparison density is the analogue of edge density for graphs. The following bound quantifies the heuristic that adding more comparisons decreases the distinguishing number of a poset.

Theorem 2.6. *If $P = (S, \preceq)$ is a poset, then*

$$D(P) \le |S| \sqrt{1 - \rho \cdot \frac{|S|-1}{|S|}}.$$

Proof. Assign indices to elements of S via Lemma 2.1. Let $w_1, w_2, \ldots, w_{h(P)+1}$ be the number of elements of each index. Since $|E(G_P)| \le \sum_{1 \le i < j \le h(P)+1} w_i w_j$, we have

$$D(P)^2 \le (\max_i w_i)^2 \le \sum_{i=1}^{h(P)+1} w_i^2 = \left(\sum_{i=1}^{h(P)+1} w_i \right)^2 - 2 \left(\sum_{1 \le i < j \le h(P)+1} w_i w_j \right).$$

Hence, $D(P)$ is at most

$$\sqrt{|S|^2 - 2|E(G_P)|} = |S| \sqrt{1 - \frac{|E(G_P)|}{\frac{|S|^2}{2}}} = |S| \sqrt{1 - \rho \cdot \frac{|S|-1}{|S|}},$$

as desired. □

Remark 2.7. Theorem 2.6 does not have any nontrivial equality cases, and does not approximate $D(P)$ well. For example, our bound yields $D(B_3) \le 5.1$ when $D(B_3) = 2$.

2.4. **The Distinguishing Chromatic Number of Posets.** The *distinguishing chromatic number* $\chi_D(P)$ of poset P is the least number of colors needed for a proper distinguishing coloring of P. Choi [7, Proposition 2.3] observed that the ratio $\chi_D(G_P)/\chi_D(P)$ can be arbitrarily large. However, Choi [7, Theorem 2.4] showed that $\chi_D(G_P)$ is bounded as a function of $\chi_D(P)$.

Theorem 2.8 (Choi [7, Theorem 2.4])**.** *If $P = (S, \preceq)$ is a poset, $\chi_D(G_P) \leq \chi_D(P)^2$.*

Theorem 2.8 also establishes a lower bound on $\chi_D(P)$. On the other hand, Choi [7, Proposition 2.5] shows that there are no meaningful upper bounds on $\chi_D(P)$ in terms of $|S|$.

Proposition 2.9 (Choi [7, Proposition 2.5])**.** *Given any ground set S and constant c satisfying $2 \leq c \leq |S|$, there exists a poset $P = (S, \preceq)$ such that $\chi_D(P) = c$.*

3. From the Automorphism Group of the Poset

Albertson and Collins [1] established bounds on the distinguishing number of graphs based on the automorphism group of graphs. In this section, we refine Theorem 2.4 by proving similar bounds for posets. Theorem 2.4 does not consider the structure of posets beyond the automorphism group. For example, the poset of n disjoint chains of the same length and the poset of n points have the same automorphism group S_n. Yet the former poset has a smaller distinguishing number than the latter poset.

The structure of posets allows us to strengthen Theorem 2.4. Partition $P = (S, \preceq)$ into indices as in Lemma 2.1, which partitions S into indices P_i for i satisfying $1 \leq i \leq h(P)+1$. For $1 \leq i \leq h(P)$, let $\varphi_i \colon \operatorname{Aut}(P) \to \operatorname{Aut}(P_i)$ map $g \in \operatorname{Aut}(P)$ to $g|_{P_i}$. Note φ_i is well-defined as $gP_i = P_i$. Let $k(P)$ denote the number of values for which φ_i is faithful. The following proof improves Theorem 2.4 by extending the coloring argument of Albertson and Collins [1, Theorem 1] and Collins, Hovey, and Trenk [8, Theorem 4.3].

Theorem 3.1. *If P is a poset with $k(P) \geq 1$, then*

$$D(P) \leq 1 + \left\lceil \frac{\log_2 |\operatorname{Aut}(P)|}{k(P)} \right\rceil.$$

Proof. Let $G = \operatorname{Aut}(P)$. Without loss of generality, suppose φ_i for i satisfying $1 \leq i \leq k$ are faithful. We write $k(P)$ as k. Let $t = \lceil \frac{N}{k} \rceil$, where N is the sum of the exponents in the prime factorization of $|G|$. We inductively define colorings $c_{i,j}$ for $0 \leq i \leq k-1$ and $0 \leq j \leq t$. Let $G_{i,j}$ be the subgroup of G that preserves $c_{i,j}$. If $G_{i,j}$ is trivial for any i and j, the result follows. Henceforth, assume that $G_{i,j}$ is nontrivial for all i and j. Let $N_{i,j}$ be the sum of the exponents in the prime factorization of $|G_{i,j}|$.

We claim by induction that for $0 \leq i \leq k-1$ and $0 \leq j \leq t$, there exists a coloring $c_{i,j}$ such that $N_{i,j} \leq N - it - j$. For the base case $i = 0$ and $j = 0$, define $c_{0,0}$ as coloring all vertices with color 0, which yields $G_{0,0} = G$.

We first define $c_{i,j+1}$ from $c_{i,j}$ for $0 \leq i \leq k-1$ and $0 \leq j \leq t-1$. Since we assume $G_{i,j}$ is not trivial, there exists $g \in G_{i,j}$ that is not the identity. Since φ_i is faithful, the automorphism $\varphi_i(g)$ is not the identity. Therefore, there exists $x \in P_i$ such that $\varphi_i(g)x \neq x$. Note that $G_{i,j}$ preserves $c_{i,j}$ and that $c_{i,j}$ uses each color at most once for each P_i. Thus, all points colored by $c_{i,j}$ are fixed, so x is not already colored. We define $c_{i,j+1}$ to be the coloring $c_{i,j}$ with x colored with color $j+1$. Then, $G_{i,j+1} = \{g \in G_{i,j} \mid \varphi_i(g) \in \operatorname{Stab}(x)\}$. Notice $G_{i,j+1}$ is a subgroup of $G_{i,j}$. Since $g \notin G_{i,j+1}$, we have $G_{i,j+1} < G_{i,j}$, so $N_{i,j+1} \leq N_{i,j} - 1 \leq N - it - (j+1)$.

We now define $c_{i+1,0}$ from $c_{i,t}$ for $0 \leq i \leq k-2$. Since $G_{i,t}$ is a nontrivial subgroup of G and φ_{i+1} is faithful, there must exist $g \in G_{i,t}$ such that $\varphi_{i+1}(g)$ is not the identity. Then, an analogous argument to the previous paragraph yields $N_{i+1,0} \leq N_{i,t} - 1 \leq N - (i+1)t$.

Our inductive step is complete, and we have $N_{k-1,t} \leq N-(k-1)t-t \leq 0$, so $G_{k-1,t}$ is trivial. Color all uncolored elements with a new color. We use colors $1, 2, \ldots, t$ for each index, and $c_{0,0}$ uses color 0. Hence, we have $t+1$ total colors, and $D(P) \leq 1 + \left\lceil \frac{N}{k} \right\rceil \leq 1 + \left\lceil \frac{\log_2 |\operatorname{Aut}(P)|}{k} \right\rceil$. $\square$

We return to the question that motivated our result. Let P_1 be the poset consisting of n chains of length $n-1$ and let P_2 be the poset consisting of an antichain of width n. The bound given by Theorem 2.4 yields the upper bound of $1 + \log_2(n!) \in \mathcal{O}(n \log n)$ for both $D(P_1) = 2$ and $D(P_2) = n$. Although Theorem 3.1 does not improve Theorem 2.4 for $\operatorname{Aut}(P_2)$, our bound does give a better estimate for $\operatorname{Aut}(P_1)$. For each index of P_1, the automorphism subgroup induced by elements of $\operatorname{Aut}(P_1)$ is $S_n \cong \operatorname{Aut}(P_1)$. Hence, every index is faithful, so Theorem 3.1 gives $D(P_1) \leq 1 + \left\lceil \frac{\log_2(n!)}{n} \right\rceil \in \mathcal{O}(\log n)$.

Theorem 3.1 also provides a better estimate for $D(B_n)$. Since $\operatorname{Aut}(B_n) \cong S_n$, the bound from Theorem 2.4 is in $\mathcal{O}(n \log n)$. Index each element of B_n with its cardinality and define φ_i as in Theorem 3.1. The automorphisms of B_n can be represented as permutations of $\{1, 2, \ldots, n\}$. Each permutation induces an automorphism on $\mathcal{P}(B_n)$. Consider $g \in \ker \varphi_i$. If $1 \leq i \leq \frac{n}{2}$, consider $\{1, 2, \ldots, i\}$ and $\{1, i+1, \ldots, 2i-1\}$. Both must map to themselves under $\varphi_i(g)$, so $g1 = 1$. Similarly, g fixes all the elements of $\{1, 2, \ldots, n\}$, so g is the identity. If $\frac{n}{2} < i \leq n-1$, consider $\{2, 3, \ldots, i\}$ and $\{n, n-1, \ldots, n-i+1\}$, and the same argument suffices. Hence, φ_i is faithful for all $1 \leq i \leq n-1$, so $k(P) = n-1$. Therefore, $D(B_n) \in \mathcal{O}\left(\frac{\log_2(n!)}{n-1}\right) = \mathcal{O}(\log n)$.

The bound is weaker than the true value of $D(B_n) = 2$, but our bound illustrates how considering the structure of a poset yields stronger bounds based on the automorphism group than Theorem 2.4.

4. Lattices

Collins and Trenk [10] characterize the distinguishing number of distributive lattices and provide a bound on the distinguishing chromatic number of distributive lattices. In this section, we establish a bound for the distinguishing (chromatic) number of general lattices and strengthen the bound for the distinguishing chromatic number of distributive lattices. Collins and Trenk [10, Theorem 14] showed that distributive lattices can be distinguished with two colors.

Theorem 4.1 (Collins and Trenk [10, Theorem 14]). *If L is a distributive lattice, $D(L) \leq 2$.*

However, general lattices can have arbitrarily large distinguishing numbers. For example, consider the lattice M_n (Figure 3), which consists of an antichain of size n bounded by $\hat{0}$ and $\hat{1}$. If $n \geq 3$, then $D(M_n) = n$.

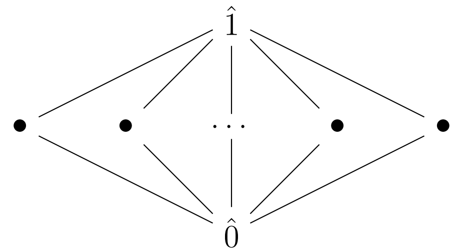

FIGURE 3. The lattice M_n.

We establish a bound for $D(L)$ that holds for arbitrary lattices L.

Theorem 4.2. *If L is a lattice, then $D(L) \leq |Q_L| - h(L) + 2$.*

Proof. Let $a_1 \prec a_2 \prec \cdots \prec a_{h(L)+1}$ be a maximal chain of L. Pin all elements of the chain by coloring the chain with color 1. Notice $J(a_i)$ maps to itself under any automorphism. We pin all elements of $J(a_i)$ for $i \geq 1$ via induction. The base case is clear as $J(a_1)$ is empty. For the inductive step, we know $J(a_i)$ is pinned, and using colors 2 through $|J(a_{i+1}) \setminus J(a_i)| + 1$ can pin the rest of $J(a_{i+1})$. Hence, the induction is complete and $J(a_{h(L)+1}) = Q_L$ is pinned. Pinning Q_L suffices to break all nontrivial automorphisms by Lemma 2.2. Color the remaining elements with color 2. We know

$$\sum_{i=1}^{h(L)} |J(a_{i+1}) \setminus J(a_i)| = |Q_L|,$$

and $|J(a_{i+1}) \setminus J(a_i)| \geq 1$ for all i satisfying $1 \leq i \leq h(L)$, so

$$\max_{1 \leq i \leq h(L)} |J(a_{i+1}) \setminus J(a_i)| \leq |Q_L| - (h(L) - 1).$$

Hence, the coloring uses at most $|Q_L| - h(L) + 2$ total colors. □

Remark 4.3. Theorem 4.2 is sharp for M_n.

Since $|Q_L| = h(L)$ for distributive lattices by Birkhoff's Theorem [3, Theorem 3], Theorem 4.1 follows from Theorem 4.2. We also establish an upper bound on $\chi_D(L)$ for lattices L. Our proof is similar to the argument given by Collins and Trenk [10, Lemma 25].

Theorem 4.4. *If L is a join semi-lattice with least element $\hat{0}$, then $\chi_D(L) \leq h(L) + \chi_D(Q_L)$.*

Proof. Index L by Lemma 2.1. For $1 \leq i \leq h(L) + 1$, color the elements with index i with color i, which is a proper coloring. Assign a distinguishing chromatic coloring to Q_L by recoloring Q_L with colors $h(L) + 1$ to $h(L) + \chi_D(Q_L)$.

Since L has a least element $\hat{0}$, there is only one element with index $h(L) + 1$. We proceed by induction to show that elements of index j for $1 \leq j \leq h(L)$ are pinned. For the base case, let e be an element of L with index $h(L)$. If e covers at least two elements, it must cover an element $d \neq \hat{0}$. Then, $i(\hat{0}) > i(d) > h(L)$, so $i(\hat{0}) \geq h(L) + 2$, a contradiction. Hence, the elements of index $h(L)$ are a subset of Q_L and are hence pinned. For the inductive step, let a be an element of index j. If $a \in Q_L$, then a is pinned, so assume there exist distinct elements b and c such that $a \gtrdot b$ and $a \gtrdot c$ and $a = b \vee c$. Then, $\min\{i(b), i(c)\} > i(a) = j$, so b and c are pinned. Since a is the unique join of b and c, it is also pinned. Hence, the induction is complete, and there exists a proper distinguishing coloring using $h(L) + \chi_D(Q_L)$ colors. □

Since a lattice is also a join semi-lattice, we can establish an upper bound for the distinguishing chromatic number of lattices.

Corollary 4.5. *If L is a lattice, then $\chi_D(L) \leq h(L) + \chi_D(Q_L)$.*

We discuss the optimality of Corollary 4.5. Let $f(x, y)$ be a polynomial such that $\chi_D(L) \leq f(h(L), \chi_D(Q_L))$. The lattice M_n in Figure 3 has $h(L) = 2$, $\chi_D(Q_L) = n$, and $\chi_D(L) = n+2$. Therefore, $\deg_y(f) \geq 1$. Consider the distributive lattice $L \cong J(Q_L)$, where Q_L consists of the disjoint union of $k!$ chains of length $k-1$ as shown in Figure 4.

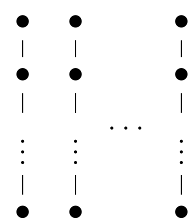

FIGURE 4. Join-irreducible elements Q_L consisting of $k!$ chains of length $k-1$.

Note that $\chi_D(Q_L) = k$ as there are $k!$ colorings of a chain of length $k-1$ with k colors. Since $L \cong J(Q_L)$ is the divisibility lattice of $n = \prod_{i=1}^{k!} p_i^k$, we have $h(L) = k \cdot k!$ and $\chi_D(L) = k \cdot k! + 2$ for $k \geq 2$, as calculated by Choi [7, Theorem 6.5]. Therefore, $\deg_x(f) \geq 1$. Hence, for polynomial bounds on $h(L)$ and $\chi_D(Q_L)$, Corollary 4.5 is optimal up to constants.

Distributive lattices have more structure than arbitrary lattices, so we can establish sharper bounds for the distinguishing (chromatic) number of distributive lattices.

Lemma 4.6 (Collins and Trenk [10, Lemma 25])**.** *If L is a distributive lattice, then $\chi_D(L) \leq |Q_L| + \chi_D(Q_L)$.*

Theorem 4.7 (Collins and Trenk [10, Theorem 26])**.** *If L is a distributive lattice such that $\chi_D(Q_L) \geq 3$, then $\chi_D(L) \leq |Q_L| + \chi_D(Q_L) - 1$.*

We know that $\chi_D(L) - |Q_L| = \chi_D(L) - h(L) \geq 0$. Both of the results presented by Collins and Trenk [10, Lemma 25, Theorem 26] bound $\chi_D(L) - |Q_L|$ from above in terms of $\chi_D(Q_L)$. We bound $\chi_D(L) - |Q_L|$ from above by a constant. The coloring for $m \geq 6$ in the following proof is the same coloring Choi [7, Theorem 6.2] defined to calculate $\chi_D(B_n)$.

Theorem 4.8. *If L is a distributive lattice, then $\chi_D(L) \leq |Q_L| + k$, where*

$$k = \begin{cases} 1 & \text{if } |\max(Q_L)| = 1, \\ 2 & \text{if } |\max(Q_L)| \neq 1, 4, \\ 3 & \text{if } |\max(Q_L)| = 4. \end{cases}$$

Proof. By Lemma 2.2, it suffices to pin the elements of Q_L. Furthermore, elements of Q_L cover at most one element, so pinning the elements of $\max(Q_L)$ pins all of Q_L. Write $\max(Q_L) = \{1, 2, \ldots, m\}$. By Birkhoff's Theorem, we work with $J(Q_L)$. Color elements of $J(Q_L)$ by rank using colors 1 through $|Q_L| + 1$. Let $Q_L' = Q_L \setminus \max(Q_L)$ and $J' = \{S \in J(Q_L) \mid Q_L' \subseteq S\}$. Label each element S of J' with $S' = S \cap \max(Q_L)$. Each subset of $\{1, 2, \ldots, m\}$ is assigned to exactly one set as $S = S' \sqcup Q_L' \subseteq J'$.

For $m = 1$, there is one possible lattice L with $\chi_D(L) = 2 \leq |Q_L| + 1$.

For $m = 2$, recolor one of the elements of $\max(Q_L)$ with color $|Q_L|+2$, which fixes $\max(Q_L)$ and therefore Q_L with $|Q_L| + 2$ colors.

For $m = 3$, we have $\chi_D(Q_L) \leq |\max(Q_L)| = 3$. If $\chi_D(Q_L) < 3$, we have $\chi_D(L) \leq \chi_D(Q_L) + |Q_L| \leq 2+|Q_L|$ by Lemma 4.6. If $\chi_D(Q_L) = 3$, we have $\chi_D(L) \leq |Q_L|+\chi_D(Q_L)-1 \leq |Q_L|+2$ by Theorem 4.7.

Similarly, for $m = 4$, we have $\chi_D(Q_L) \leq |\max(Q_L)| = 4$. If $\chi_D(Q_L) < 3$, we have $\chi_D(L) \leq \chi_D(Q_L)+|Q_L| \leq 2+|Q_L|$ by Lemma 4.6. Otherwise, $\chi_D(L) \leq |Q_L|+\chi_D(Q_L)-1 \leq |Q_L|+3$ by Theorem 4.7.

For $m = 5$, recolor elements of J' labeled with $\{1, 2\}$, $\{1, 4\}$, $\{2, 4\}$, $\{3, 4\}$, and $\{1, 3, 5\}$ with color $|Q_L| + 2$. The integer 1 is the only one in precisely three colored sets, of sizes 2, 2, and 3. The integer 2 is the only one in precisely two colored sets, of sizes 2 and 2. The integer 3 is the only one in precisely two colored sets, of sizes 2 and 3. The integer 4 is the only one in precisely three colored sets, of sizes 2, 2, and 2. The integer 5 is the only one in precisely one colored set, of size 3. The coloring for $m = 5$ is shown in Figure 5.

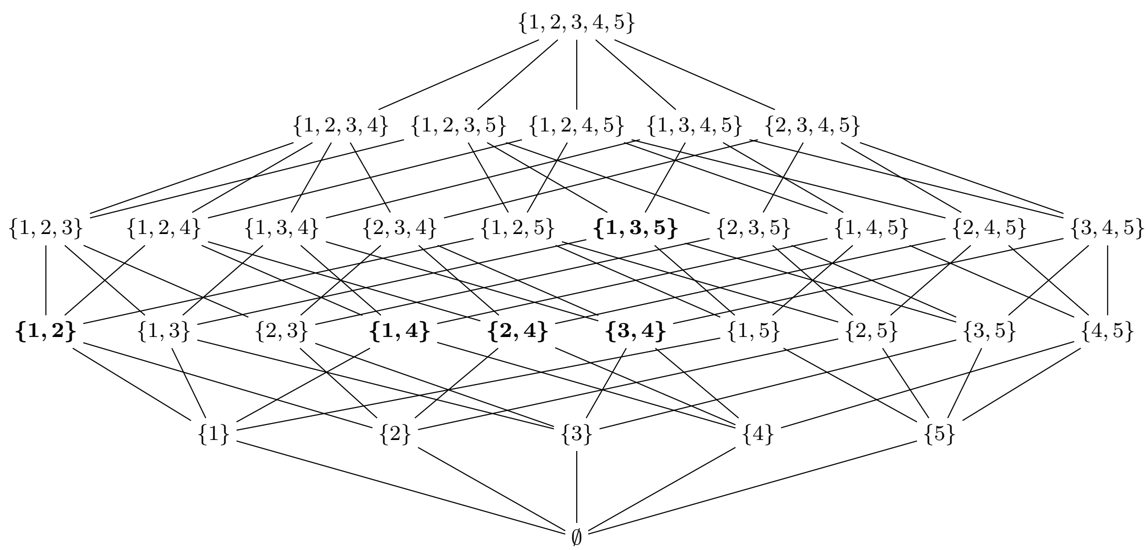

FIGURE 5. Coloring of labels of J' for $m = 5$.

For $m = 6$, the same coloring suffices as fixing 1, 2, 3, 4, and 5 also fixes 6.

For $m \geq 7$, recolor elements of J' with color $|Q_L| + 2$ as follows.

- Color the set labeled with all odd integers.
- For $i = 1, 2, \ldots, \lfloor \frac{m-1}{2} \rfloor$, color the set labeled with the ith odd integer and the first i even integers.

No colored set includes another one. If $m \geq 7$, then 1 is the unique integer in precisely two colored label sets of sizes 2 and $\lfloor \frac{m+1}{2} \rfloor$. Hence, 1 is pinned. Then, the odd integers are the ones in a label set with size $\lfloor \frac{m+1}{2} \rfloor$ containing 1, so the odd integers are distinguished from the even ones. The odd integers can be distinguished from each other by the size of the set they are in, and the even integers can be distinguished from each other by how many sets they are in. Hence, all elements of $\max(Q_L)$ are pinned with $|Q_L| + 2$ colors. □

We discuss the sharpness of Theorem 4.8. Choi [7, Theorem 6.2, Theorem 6.5] calculated $\chi_D(B_n)$ and $\chi_D(L_n)$.

Theorem 4.9 (Choi [7, Theorem 6.2])**.** *We have that*

$$\chi_D(B_n) = \begin{cases} n+1 & \text{if } n = 1, \\ n+2 & \text{if } n \neq 1, 4, \\ n+3 & \text{if } n = 4. \end{cases}$$

Theorem 4.10 (Choi [7, Theorem 6.5])**.** *For $n = p_1^{q_1} p_2^{q_2} \dots p_k^{q_k}$, we have that*

$$\chi_D(L_n) = \begin{cases} h(L_n)+1 & \text{if } q_1, q_2, \dots, q_k \text{ mutually distinct,} \\ h(L_n)+3 & \text{if } n = p_1p_2p_3p_4, \text{ and} \\ h(L_n)+2 & \text{otherwise} \end{cases}$$

Hence, Theorem 4.8 is sharp for B_n and Theorem 4.8 is sharp for L_n when $k > 4$ and q_i are not all distinct. We have bounded $\chi_D(L) - |Q_L|$ by a constant. On the other hand, the example in Figure 4 shows that we cannot bound $\chi_D(L) - \chi_D(Q_L)$ or $\chi_D(L)/\chi_D(Q_L)$ by a constant.

5. The Motion Lemma

Collins and Trenk [10, Question 6.2] asked whether their bound for the distinguishing number of distributive lattices could be shown using the motion lemma [15, Theorem 2]. We prove a bound for the distinguishing number of graded lattices via the motion lemma.

The *motion* $m(\varphi)$ of an automorphism φ of a graph G is the number of vertices of G which are not fixed by φ. Then, the motion lemma yields the following bound on $D(G)$.

Lemma 5.1 (Russell and Sundaram [15, Theorem 2])**.** *If* $(\ln d) \min_{\varphi \in \mathrm{Aut}(G)\setminus\{\mathrm{id}\}} m(\varphi) > \ln|\mathrm{Aut}(G)|$*, then* $D(G) \leq d$*.*

The motion lemma also holds for posets P if we define the *motion* $m(\varphi)$ of an automorphism $\varphi \in \mathrm{Aut}(P)$ as the number of elements of P which are not fixed by φ.

Lemma 5.2. *If* $(\ln d) \min_{\varphi \in \mathrm{Aut}(P)\setminus\{\mathrm{id}\}} m(\varphi) > \ln|\mathrm{Aut}(P)|$*, then* $D(P) \leq d$*.*

We establish a bound on graded lattices via the motion lemma for posets.

Theorem 5.3. *Let L be a graded lattice of height h. Suppose that for every element e with $1 \leq \rho(e) \leq h-2$, we have $\deg_\uparrow(e) \geq 2$, and for every element e with $2 \leq \rho(e) \leq h-1$, we have $\deg_\downarrow(e) \geq 2$. Then, if $|\mathrm{Aut}(L)| < d^{2(h-1)}$, we have $D(L) \leq d$.*

Proof. Consider a nontrivial automorphism φ of L. Suppose there exist distinct elements a and b of rank r such that $\varphi(a) = b$. If $r \leq h-2$, there exist distinct elements a_1 and a_2 that cover a. Both $\varphi(a_1)$ and $\varphi(a_2)$ cover b. If both a_1 and a_2 are fixed under φ, then a_1 and a_2 cover both a and b. Then, $a \vee b$ would not exist, a contradiction. Hence, there are elements of rank $r+1$ that are not fixed. Therefore, if φ does not fix elements of rank $r \leq h-2$, it also does not fix elements of rank $r+1$. Similarly, if φ does not fix elements of rank $r \geq 2$, it also does not fix elements of rank $r-1$.

Notice there must be an element of rank $1 \leq s \leq h-1$ which is not fixed by φ. Then, by induction, every rank $1 \leq t \leq h-1$ must have an element which is not fixed by φ. Thus, we have $m(\varphi) \geq 2(h-1)$ as each element which is not fixed has a distinct image which is also not fixed. Our result follows from the motion lemma for posets. $\square$

Remark 5.4. If $n \geq 4$, then B_n satisfies the preconditions of Theorem 5.3. Our bound yields $D(B_n) \leq \frac{\ln(n!)}{2(n-1)} \in \mathcal{O}(\ln n)$. Hence, Theorem 5.3 has the same complexity as Theorem 3.1 for B_n.

6. Special Non-Distributive Lattices

Collins and Trenk [10] established bounds for the distinguishing number and distinguishing chromatic number of distributive lattices. In Section 4, we computed bounds for the distinguishing (chromatic) number of general lattices. In this section, we compute the distinguishing (chromatic) number of special non-distributive lattices.

The Tamari Lattice T_n of order n consists of ways to parenthesize $n+1$ factors, with ordering given by the right associativity rule. Bennett and Birkhoff [2, Corollary 2] establishes that T_n has no nontrivial automorphisms, so $D(T_n) = 1$.

The Young-Fibonacci lattice $\mathbb{YF}$ consists of finite words with alphabet $\{1, 2\}$. An element a of $\mathbb{YF}$ is covered by an element b if b is obtained from a by inserting a 1 to the left of the leftmost 1 in a or changing the leftmost 1 of a into a 2. Let $\mathbb{YF}_{\leq r}$ be the poset consisting of the elements of $\mathbb{YF}$ with rank at most r. We compute the distinguishing number of $\mathbb{YF}_{\leq r}$.

Proposition 6.1. *If $r \geq 2$, then $D(\mathbb{YF}_{\leq r}) = 2$.*

Proof. We claim that the join-irreducibles of $\mathbb{YF}_{\leq r}$ are 2 and the strings of the form $1s$. The only string that a string of the form $1s$ can cover is s. If a string is of the form $21s$ (where s may be the empty string), then the string covers $11s$ and $2s$. If a string is of the form $22s$ (where s may be the empty string), then the string covers $12s$ and $21s$. Hence, strings starting with 2 that have length at least two cannot be join-irreducibles.

We claim that no element except 1 is covered by at least two join-irreducibles. Consider a join-irreducible $1s$ which covers only s. If s is of the form $1a$, then s is covered by only $11a = 1s$ and $2s$, the latter of which is not a join-irreducible. If s is of the form $2a$, then s is covered by $12a = 1s$ and elements that start with 2. Hence, s is never covered by join-irreducibles other than $1s$, as desired.

Now, we can show $D(\mathbb{YF}_{\leq r}) \leq 2$ for $r \geq 2$. Consider the coloring in Figure 6, which colors 2 with color 1 and the rest of the elements with color 2.

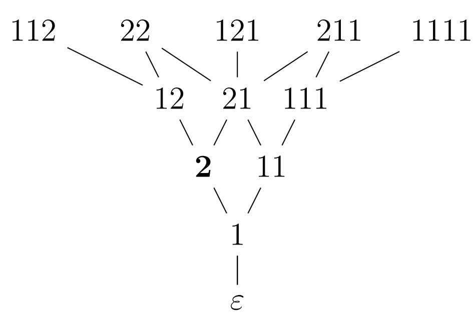

Figure 6. A distinguishing coloring of $\mathbb{YF}_{\leq 4}$.

We proceed by induction to show that our coloring is distinguishing for $r \geq 2$. The base case is clear. For the inductive step, all elements of rank at most $r-1$ are pinned by the inductive hypothesis. All elements of rank r which are not join-irreducibles are pinned by

Lemma 2.2. Furthermore, join-irreducibles of rank r are pinned as we showed that no two join-irreducibles can cover the same element. Hence, our induction is complete.

Let φ map strings of the form $s11$ to $s2$, strings of the form $s2$ to $s11$, and strings of the form $s21$ to themselves. We claim that φ is a nontrivial automorphism of $\mathbb{YF}_{\leq r}$. Suppose that a and b are elements of $\mathbb{YF}_{\leq r}$ such that $a \gtrdot b$. We have three cases.

If b is of the form $s2$, then s is a string of twos and $a = s21$ (we insert a 1 at the end, which requires all previous digits to be 2) or a is a string of the form $t2$, where $t \gtrdot s$. In the former subcase, $\varphi(a) = s21 \gtrdot s11 = \varphi(b)$. In the latter subcase, $\varphi(a) = t11 \gtrdot s11 = \varphi(b)$.

If b is of the form $s11$, then s is a string of twos and $a = s21$ or $a = t11$, where $t \gtrdot s$. In the former subcase, $\varphi(a) = s21 \gtrdot s2 = \varphi(b)$. In the latter subcase, $\varphi(a) = t2 \gtrdot s2 = \varphi(b)$.

If b is of the form $s21$, then s is a string of twos and $a = s22$ or s is a string of twos and $a = s211$ or $a = t21$, where $t \gtrdot s$. In the first subcase, we have $\varphi(a) = s211 \gtrdot s21 = \varphi(b)$. In the second subcase, we have $\varphi(a) = s22 \gtrdot s21 = \varphi(b)$. In the third subcase, we have $\varphi(a) = t21 \gtrdot s21 = \varphi(b)$.

Hence, we always have $\varphi(a) \gtrdot \varphi(b)$. Also, $D(\mathbb{YF}_{\leq r}) > 1$ as φ is nontrivial for $r \geq 2$. Therefore, $D(\mathbb{YF}_{\leq r}) = 2$. □

We compute the distinguishing chromatic number of $\mathbb{YF}_{\leq r}$ as well.

Proposition 6.2. *If $r \geq 2$, then $\chi_D(\mathbb{YF}_{\leq r}) = r + 2$.*

Proof. Color $\mathbb{YF}_{\leq r}$ by rank and color the string 2 with color $r + 1$. By the argument in Proposition 6.1, the coloring is distinguishing and proper. Therefore, $\chi_D(\mathbb{YF}_{\leq r}) \leq r + 2$.

Since $h(\mathbb{YF}_{\leq r}) = r$, we have $\chi_D(\mathbb{YF}_{\leq r}) \geq r+1$. Suppose for the sake of contradiction that $\chi_D(\mathbb{YF}_{\leq r}) = r + 1$. We claim that any two elements of the same rank must be the same color. Call two elements a and b of rank s *equivalent* if there exists a chain $c_0, c_1, \ldots, c_{s-1}$ such that $a \gtrdot c_{s-1}$ and $b \gtrdot c_{s-1}$. Notice it suffices to show the existence of an element c_{s-1} of rank $s - 1$ that is covered by both a and b.

We claim by induction on s that equivalent elements a and b of rank s are colored the same. Our base case is $s = r$. Note a and b cannot have the same colors as any of $c_0, c_1, \ldots, c_{s-1}$, so there is only one color left for both a and b. Hence, they are the same color. For the inductive step, ranks $s + 1, \ldots, r$ eliminate $r - s$ colors for a and b. On the other hand, $c_0, c_1, \ldots, c_{s-1}$ eliminate s colors for a and b so there is only one color for both a and b.

Consider a string a with at least two ones. Remove the leftmost 1 of a to form string s. Then, turn the leftmost 1 of s into a 2, forming string b. We know $a \gtrdot s$ and $b \gtrdot s$ so a and b are equivalent. Call the operation *merging*.

Next, consider a string a with at least one 1. Remove the leftmost 1 from a to form string s. Then, let $b = 1s$. Notice a and b are equivalent. Call the operation from a to b *shifting*.

Consider rank s. If s is even, there is always an even number of ones. Hence, repeated merging allows us to turn any string of rank s into a string of twos of length $\frac{s}{2}$. All strings of rank s are equivalent to this string, so all strings of rank s have the same color. If s is odd, repeated merging allows us to turn any string of rank s into a string of twos with a 1 inserted into it. Then, shifting allows us to move the 1 to the left of the string. We have formed a string starting with 1 and followed by a string of length $\frac{s-1}{2}$ consisting of twos. Hence, all strings of rank s have the same color. Thus, the only coloring with $r + 1$ colors is a coloring by rank. Then, the automorphism from Proposition 6.1 preserves colors, a contradiction. □

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