THE CODOUBLE BOSONISATION OF THE FOMIN-KIRILLOV ALGEBRA

LUCAS HINDS

Abstract. In this paper, we discuss the codouble of the group algebra of S_3, as well as the Fomin-Kirillov algebra $\mathcal{FK}_3$. In particular, we give $\mathcal{FK}_3$ a left comodule structure over the codouble of S_3, so that we can construct its codouble bosonisation, which generalizes the usual quantum function algebra construction.

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Introduction

The study of Hopf algebras originated in algebraic topology and has since evolved into a useful tool in many different areas of mathematics, including quantum groups, category theory, and non-commutative geometry. In the 1980s, Drinfeld introduced the concept of a quantum group in his work on the quantum Yang–Baxter equation, leading to the development of the Drinfeld double, a quasitriangular Hopf algebra constructed from a finite-dimensional Hopf algebra and its dual [4]. For a finite group G, the Drinfeld double $D(G)$ contains both the group algebra kG and its dual k^G, along with nontrivial braiding arising from the conjugation action of G.

Closely related to these constructions are braided Hopf algebras, which generalize ordinary Hopf algebras to braided monoidal categories. Such structures arise naturally in the theory of Yetter–Drinfeld modules over a Hopf algebra, which serve as the setting for bosonisation and codouble bosonisation constructions introduced by Majid in [9] and later generalized by others [2]. These constructions are central to the theory of quantum groups and have found applications ranging from knot invariants to noncommutative geometry.

Another important object is the Fomin–Kirillov algebra $\mathcal{FK}_n$, introduced in the context of Schubert calculus and the cohomology of flag varieties. For $n = 3$, the algebra $\mathcal{FK}_3$ exhibits a rich structure that allows it to be interpreted as a braided Hopf algebra in suitable monoidal categories [1]. Understanding such algebras in relation to Hopf-theoretic constructions like $D(S_3)^*$ provides insight into their representation theory and categorical symmetry.

In this paper, we will look at the interaction between the Fomin–Kirillov algebra $\mathcal{FK}_3$ and the dual Drinfeld double $D(S_3)^*$, with the goal of equipping $\mathcal{FK}_3$ with a left comodule structure over $D(S_3)^*$. This allows us to treat $\mathcal{FK}_3$ as a braided Hopf algebra in the monoidal category of left $D(S_3)^*$-comodules. Our broader objective is to construct the codouble bosonisation of $\mathcal{FK}_3$ over $D(S_3)^*$, as introduced in [2].

We will begin by reviewing some essential preliminaries, including the definitions of Hopf algebras, Yetter–Drinfeld modules, and braided Hopf algebras. We will then examine the structure of the Drinfeld double $D(S_3)$ and its dual, along with the associated (co)quasitriangular structures. We will then analyze the algebraic structure of $\mathcal{FK}_3$ and demonstrate its realization as a braided Hopf algebra in the category of $D(S_3)^*$-comodules. Finally, we will derive a presentation by generators and relations for the codouble bosonisation.

Related work includes the generalized quantum group over $\mathcal{FK}_3$ studied by Vay in [10]. In that setting, the double bosonisation of $\mathcal{FK}_3$ yields a Hopf algebra whose dual can be described using codouble bosonisation. Analyzing this dual, specifically the simple modules of the dual, allows us to explore the simple comodules of the original quantum group and further illuminate the representation-theoretic structure underlying $\mathcal{FK}_3$.

1. Preliminaries

We will first recall some definitions relating to Hopf algebras, Yetter-Drinfeld modules, and braided Hopf algebras.

Definition 1.1. A tuple $(B, \mu, \eta, \Delta, \epsilon)$ is called a **bialgebra** over $\Bbbk$ if:

- B is a vector space over $\Bbbk$,
- (B, μ, η) is an algebra,
- (B, Δ, ϵ) is a coalgebra,
- The following diagrams commute:

$$\begin{array}{ccccc} B \otimes B & \xrightarrow{\mu} & B & \xrightarrow{\Delta} & B \otimes B \\ \downarrow{\scriptstyle \Delta\otimes\Delta} & & & & \uparrow{\scriptstyle \mu\otimes\mu} \\ B \otimes B \otimes B \otimes B & & \xrightarrow{\mathrm{id}\otimes\tau\otimes\mathrm{id}} & & B \otimes B \otimes B \otimes B \end{array} \tag{1.1}$$

Where $\tau\colon B\otimes B \to B\otimes B$ is the swap map $\tau(x\otimes y) = y\otimes x$.

$$\begin{array}{ccc} B\otimes B & \xrightarrow{\mu} & B \\ & {\scriptstyle \epsilon\otimes\epsilon}\searrow \quad \swarrow{\scriptstyle \epsilon} & \\ & K\otimes K\cong K & \end{array} \tag{1.2}$$

$$\begin{array}{ccc} & K\otimes K\cong K & \\ & {\scriptstyle \eta\otimes\eta}\swarrow \quad \searrow{\scriptstyle \eta} & \\ B\otimes B & \xleftarrow{\Delta} & B \end{array} \tag{1.3}$$

$$\begin{array}{ccc} \Bbbk & \xrightarrow{\mathrm{id}} & \Bbbk \\ & {\scriptstyle \eta}\searrow \quad \nearrow{\scriptstyle \epsilon} & \\ & B & \end{array} \tag{1.4}$$

Definition 1.2. A **Hopf Algebra** is a bialgebra H along with a map $S : H \to H$ called the antipode such that the following diagram commutes:

$$\begin{array}{ccccccc} & & H\otimes H & \xrightarrow{S\otimes \mathrm{id}} & H\otimes H & & \\ & {\scriptstyle\Delta}\nearrow & & & & \searrow{\scriptstyle\mu} & \\ H & & \xrightarrow{\epsilon} & \Bbbk & \xrightarrow{\eta} & & H \\ & {\scriptstyle\Delta}\searrow & & & & \nearrow{\scriptstyle\mu} & \\ & & H\otimes H & \xrightarrow{\mathrm{id}\otimes S} & H\otimes H & & \end{array} \tag{1.5}$$

Definition 1.3. A tuple (M, ρ, δ) is called a **Yetter-Drinfeld module** (YD-module) over a Hopf algebra H if:

- M is a module over H with the action ρ,
- M is a comodule over H with the coaction δ,
- The following compatibility condition holds:

$$\delta(\rho(h, v)) = h_{(1)}v_{(-1)}S(h_{(3)}) \otimes \rho(h_{(2)}, m_{(0)}).$$

We denote the category of YD-modules over H along with action and coaction preserving maps between them by ${}^{H}_{H}\mathcal{YD}$.

A natural way to generalize the concept of a Hopf algebra is to, instead of only considering objects in the category of vector spaces over $\Bbbk$, consider objects in any braided monoidal category, as shown below.

Definition 1.4. A tuple $(A, \mu, \eta, \Delta, \epsilon, S)$ in a braided monoidal category $(\mathbf{C}, \otimes, \mathbf{1}, \Psi)$ is called a **braided Hopf algebra** ([7] pg. 19) if

- (A, μ, η) is an algebra in $\mathbf{C}$,
- (A, Δ, ϵ) is a coalgebra in $\mathbf{C}$,
- Diagrams 1.1-1.5 commute, where the swap map τ is replaced with the braiding Ψ, the field $\Bbbk$ is replaced with $\mathbf{1}$, and the tensor product is replaced with the monoidal product.

Before we get started, we will establish some notation. First, we will use the following notation for categories of modules:

- The category of left modules over A is denoted ${}_A\mathcal{M}$.
- The category of left comodules over C is denoted ${}^C\mathcal{M}$.
- The category of right modules over A is denoted $\mathcal{M}_A$.
- The category of right comodules over C is denoted $\mathcal{M}^C$.

We will use Sweedler's notation for coproducts and coactions as follows

- For a coalgebra C with $c \in C$, we say:
$$\Delta(c) = c_{(1)} \otimes c_{(2)}.$$
- For a left comodule $M \in {}^C\mathcal{M}$, and $c \in C$ we say:
$$\delta(c) = c_{(1)} \otimes c_{(\infty)}.$$
- For a right comodule $M \in \mathcal{M}^C$, and $c \in C$ we say:
$$\delta(c) = c_{(0)} \otimes c_{(1)}.$$

Lastly, we will define the co-opposite comultiplication:

Definition 1.5. For a coalgebra C with comultiplication $\Delta(x) = x_{(1)} \otimes x_{(2)}$, define the co-opposite comultiplication as follows:

$$\Delta_{\text{cop}}(x) = x_{(2)} \otimes x_{(1)}.$$

We also define C^{cop} as the coalgebra with this co-opposite comultiplication, and, for a Hopf algebra H, we define H^{cop} as the Hopf algebra with the same algebra structure as H, but with co-opposite comultiplication.

With that, we will begin.

2. On the double of the group algebra of S_3

We will start by introducing the definition of the Drinfeld double.

Definition 2.1 (cf. [4], p. 816)**.** For any Hopf algebra H, the **Drinfeld Double** $D(H) = (H^*)^{\mathrm{cop}} \bowtie H$ is a Hopf algebra with underlying space $H^* \otimes H$, along with algebra and coalgebra structures, and an antipode as shown below:

$$\begin{aligned}
(f \bowtie a)(\tilde{f} \bowtie \tilde{b}) &= f(a_{(1)} \rightharpoonup \tilde{f}_{(2)}) \bowtie (a_{(2)} \leftharpoonup \tilde{f}_{(1)})b \\
\Delta_{D(H)}(f \bowtie a) &= f_{(2)} \bowtie a_{(1)} \otimes f_{(1)} \bowtie a_{(2)} \\
\epsilon_{D(H)}(f \bowtie a) &= f(1)\epsilon(a) \\
S_{D(H)}(f \bowtie a) &= (S_H(a_{(2)}) \triangleright S_{(H)^{cop}}(f_{(1)})) \bowtie (f_{(2)} \triangleright S_H(a_{(1)})) \\
1_{D(H)} &= \epsilon \bowtie 1
\end{aligned}$$

where $a \rightharpoonup f$, $a \leftharpoonup f$, $a \triangleright f$, and $f \triangleright a$ are defined as follows:

$$\begin{aligned}
a \rightharpoonup f &= \langle f_{(3)} S^{-1}(f_{(1)}), a\rangle f_{(2)}, & (a \triangleright f)(b) &= f(ab), \\
a \leftharpoonup f &= \langle f, S^{-1}(a_{(3)})a_{(1)}\rangle a_{(2)}, & f \triangleright a &= f(a_{(1)})a_{(2)}.
\end{aligned}$$

We would like to compute $D(S_3)$. To do this, we first establish the basis $\{\delta_h \bowtie g | h, g \in S_3\}$. Now, we will establish some properties that will prove to be helpful.

Lemma 2.2. *We have the following*

- $\Delta_{(\Bbbk S_3^*)^{cop}}(\delta_h) = \sum_{g \in S_3} \delta_g \otimes \delta_{hg^{-1}}$,
- $g \rightharpoonup \delta_h = \delta_{g^{-1}hg}$,
- $g \leftharpoonup \delta_h = \delta_h(e)g$.

Proof. To start, we see that:

$$\Delta_{(\Bbbk S_3^*)^{\mathrm{cop}}}(\delta_h)(g_1 \otimes g_2) = \delta_h(g_2 g_1) = \sum_{g \in S_3} \delta_g(g_1)\delta_{hg^{-1}}(g_2) = \sum_{g \in S_3} (\delta_g \otimes \delta_{hg^{-1}})(g_1 \otimes g_2).$$

To evaluate $g \rightharpoonup \delta_h$, we need to find $\Delta^2_{(\Bbbk S_3^*)^{\mathrm{cop}}}(\delta_h)$:

$$\begin{aligned}
\Delta^2_{(\Bbbk S_3^*)^{cop}}(\delta_h)(g_1 \otimes g_2 \otimes g_3) &= \sum_{g \in S_3} \Delta_{(\Bbbk S_3^*)^{cop}}(\delta_g)(g_1 \otimes g_2)\delta_{hg^{-1}}(g_3) \\
= \sum_{g, \tilde{g} \in S_3} \delta_{\tilde{g}}(g_1)\delta_{g\tilde{g}^{-1}}(g_2)\delta_{hg^{-1}}(g_3) &= \sum_{g, \tilde{g} \in S_3} (\delta_{\tilde{g}} \otimes \delta_{g\tilde{g}^{-1}} \otimes \delta_{hg^{-1}})(g_1 \otimes g_2 \otimes g_3).
\end{aligned}$$

Now we evaluate $g \rightharpoonup \delta_h$ using the definition:

$$\begin{aligned}
g \rightharpoonup \delta_h &= \sum_{g_1, g_2 \in S_3} \langle \delta_{hg_1^{-1}} S^{-1}(\delta_{g_2}), g\rangle \delta_{g_1 g_2{}^{-1}} \\
&= \sum_{g_1, g_2 \in S_3} (\delta_{hg_1^{-1}}(g) S^{-1}(\delta_{g_2})(g))\delta_{g_1 g_2{}^{-1}}
\end{aligned}$$

$$= \sum_{g_1, g_2 \in S_3} (\delta_{hg_1^{-1}}(g)\delta_{g_2}(g^{-1}))\delta_{g_1 g_2{}^{-1}}.$$

The coefficient $\delta_{hg_1^{-1}}(g)\delta_{g_2}(g^{-1})$ is only 1 when $g_1 = g^{-1}h$ and $g_2 = g^{-1}$. Thus,

$$g \rightharpoonup \delta_h = \delta_{g^{-1}hg}.$$

We can also evaluate $g \leftharpoonup \delta_h$ using the definition:

$$g \leftharpoonup \delta_h = \delta_h(S^{-1}(g)g)g = \delta_h(e)g.$$

□

Theorem 2.3. *The algebra $D(S_3)$ has the following presentation:*

$$\sigma^2 = \tau^3 = 1, \quad \sigma\tau = \tau^{-1}\sigma,$$
$$\delta_g\delta_h = \begin{cases} \delta_g & \text{if } g = h \\ 0 & \text{if } g \neq h \end{cases}, \quad \sum_{g \in S_3} \delta_g = 1,$$
$$\sigma\delta_h = \delta_{\sigma h \sigma^{-1}}\sigma, \quad \tau\delta_h = \delta_{\tau h \tau^{-1}},$$

where σ and τ are identified in $D(S_3)$ as $1 \bowtie \sigma$ and $1 \bowtie \tau$ respectively, and δ_g is identified as $\delta_g \bowtie 1$.

Proof. To get the presentation, we first need a formula for the product. So, we evaluate.

$$(\delta_{h_1} \bowtie g_1)(\delta_{h_2} \bowtie g_2) = \sum_{\tilde{g} \in S_3} \delta_{h_1}(g_1 \rightharpoonup \delta_{h_2\tilde{g}^{-1}}) \bowtie (g_1 \leftharpoonup \delta_{\tilde{g}})g_2$$
$$= \sum_{\tilde{g} \in S_3} \delta_{h_1}\delta_{g_1^{-1}h_2\tilde{g}^{-1}g_1} \bowtie \delta_{\tilde{g}}(e)g_1g_2.$$

This is zero unless $\tilde{g} = e$, so we have:

$$(\delta_{h_1} \bowtie g_1)(\delta_{h_2} \bowtie g_2) = \delta_{h_1}\delta_{g_1^{-1}h_2g_1} \bowtie g_1g_2.$$

Now, to get our presentation, we first include the presentations of $\Bbbk S_3$ and $\Bbbk^{S_3}$:

$$\sigma^2 = \tau^3 = 1, \quad \sigma\tau = \tau^{-1}\sigma,$$
$$\delta_g\delta_h = \begin{cases} \delta_g & \text{if } g = h \\ 0 & \text{if } g \neq h \end{cases}, \quad \sum_{g \in S_3} \delta_g = 1.$$

But this is not enough, we need to add relations to describe the interaction between $\Bbbk S_3$ and $\Bbbk^{S_3}$. We can do this by describing what happens when we multiply $g\delta_h = (1 \bowtie g)(\delta_h \bowtie 1)$:

$$g\delta_h = (1 \bowtie g)(\delta_h \bowtie 1) = \delta_{g^{-1}hg} \bowtie g = \delta_{ghg^{-1}} g.$$

It remains to check that these relations are sufficient to describe the whole algebra. Every element of the algebra is a linear combination of products of the generators δ_g, σ, and τ. Using the cross relations, we may rewrite any such product by moving all δ_g to the left of the words in σ and τ. Since the δ_g form a basis of $\Bbbk^{S_3}$, and σ, τ generate $\Bbbk S_3$, any element can therefore be written as a linear combination of terms of the form $\delta_g \sigma^a \tau^b$, where $g \in S_3$, $a \in \{0, 1\}$, and $b \in \{0, 1, 2\}$.

There are 36 such elements, which equals $\dim(D(S_3))$, so the relations generate all of $D(S_3)$. □

To complete our analysis of the structure of $D(S_3)$, we will also find the structure of the coproduct, the unit, and the counit.

Theorem 2.4. *The coproduct, unit, counit, and antipode of $D(S_3)$ have the following structure:*

$$\begin{aligned}
\Delta_{D(S_3)}(\delta_h \bowtie g) &= \sum_{\tilde{g} \in S_3}, (\delta_{h\tilde{g}^{-1}} \bowtie g) \otimes (\delta_{\tilde{g}} \bowtie g), \\
1_{D(S_3)} &= \sum_{h \in S_3} \delta_h \bowtie e, \\
\epsilon_{D(S_3)}(\delta_h \bowtie g) &= \delta_h(e), \\
S_{D(S_3)}(\delta_h \bowtie g) &= \delta_{ghg^{-1}} \bowtie g^{-1}.
\end{aligned}$$

Proof. We simply apply the definition of each of the operations in the Drinfeld double, starting with the coproduct,

$$\Delta_{D(S_3)}(\delta_h \bowtie g) = \sum_{\tilde{g} \in S_3} (\delta_{h\tilde{g}^{-1}} \bowtie g) \otimes (\delta_{\tilde{g}} \bowtie g),$$

then the unit,

$$1_{D(S_3)} = \epsilon \bowtie e = \sum_{g \in G} \delta_g \bowtie e,$$

then the counit:

$$\epsilon_{D(S_3)}(\delta_h \bowtie g) = \delta_h(e)\epsilon(g) = \delta_h(e),$$

and finally the antipode:

$$\begin{aligned}
S_{D(S_3)}(\delta_h \bowtie g) &= \sum_{k \in S_3} (S_{\Bbbk S_3}(g) \triangleright S_{(\Bbbk S_3^*)^{cop}}(\delta_k)) \bowtie (\delta_{hk^{-1}} \triangleright S_{\Bbbk S_3}(g)) \\
&= \sum_{k \in S_3} (g^{-1} \triangleright \delta_{k^{-1}}) \bowtie (\delta_{hk^{-1}} \triangleright g^{-1}) \\
&= \sum_{k \in S_3} \delta_{kg^{-1}} \bowtie \delta_{hk^{-1}}(g^{-1}) g^{-1} = \delta_{ghg^{-1}} \bowtie g^{-1}.
\end{aligned}$$

□

3. On the codouble of the group algebra of S_3

Before we make any computations, we define a basis for the codouble $D(S_3)^*$:

Definition 3.1. Let $\chi_{h,g}$ be defined by:

$$\chi_{h,g}(\delta_{h_1} \bowtie g_1) = \delta_h(h_1)\delta_g(g_1).$$

Then, define the following elements:

$$A = \sum_{g \in S_3} \chi_{\sigma,g}, \quad B = \sum_{g \in S_3} \chi_{\tau,g}, \quad E_g = \chi_{e,g},$$

where $\sigma = (12)$ and $\tau = (123)$

The set $\{\chi_{h,g} | h, g \in S_3\}$ is the standard basis of the dual space. We can now compute the product, coproduct, and antipode on this space, starting with the product:

Lemma 3.2. *For $\chi_{h_1,g_1}, \chi_{h_2,g_2} \in D(S_3)^*$, we have:*

$$\chi_{h_1,g_1}\chi_{h_2,g_2} = \delta_{g_1}(g_2)\chi_{h_1h_2,g_1}.$$

Proof. We simply apply the definition of the dual algebra:

$$\begin{aligned}(\chi_{h_1,g_1}\chi_{h_2,g_2})(\delta_h \bowtie g) &= \sum_{\tilde{g} \in S_3} \chi_{h_1,g_1}(\delta_{h\tilde{g}^{-1}} \bowtie g)\chi_{h_2,g_2}(\delta_{\tilde{g}} \bowtie g) \\ &= \sum_{\tilde{g} \in S_3} \delta_{h_1}(h\tilde{g}^{-1})\delta_{g_1}(g)\delta_{h_2}(\tilde{g})\delta_{g_2}(g).\end{aligned}$$

This is only nonzero when $\tilde{g} = h_2$, so we have:

$$\begin{gathered}(\chi_{h_1,g_1}\chi_{h_2,g_2})(\delta_h \bowtie g) = \delta_{h_1}(hh_2^{-1})\delta_{g_1}(g)\delta_{g_2}(g) \\ = \delta_{g_1}(g_2)\delta_{h_1h_2}(h)\delta_{g_1}(g) = \delta_{g_1}(g_2)\chi_{h_1h_2,g_1}(\delta_h \bowtie g).\end{gathered}$$

□

Another thing we will need to know is the unit in $D(S_3)^*$. We will compute the unit as well as the counit in the following lemma:

Lemma 3.3. *The unit and counit of $D(S_3)^*$ are as follows:*

$$1_{D(S_3)^*} = \sum_{g \in S_3} \chi_{e,g} = \sum_{g \in S_3} E_g,$$
$$\epsilon_{D(S_3)^*}(\chi_{h,g}) = \delta_g(e).$$

Proof. We simply use the definition to compute the unit and counit, starting with the unit.

$$\begin{aligned} 1_{D(S_3)^*}(\delta_h \bowtie g) &= \epsilon_{D(S_3)}(\delta_h \bowtie g) = \delta_h(e) = \delta_e(h) \sum_{\tilde{g} \in S_3} \delta_{\tilde{g}}(g) \\ &= \sum_{\tilde{g} \in S_3} \chi_{e,\tilde{g}}(\delta_h \bowtie g). \end{aligned}$$

For the counit, we get

$$\begin{aligned} \epsilon_{D(S_3)^*}(\chi_{h,g}) &= \chi_{h,g}(1_{D(S_3)}) = \chi_{h,g}\left(\sum_{\tilde{g} \in S_3} \delta_{\tilde{g}} \bowtie e\right) \\ &= \sum_{\tilde{g} \in S_3} \chi_{h,g}(\delta_{\tilde{g}} \bowtie e) = \sum_{\tilde{g} \in S_3} \delta_h(\tilde{g})\delta_g(e). \end{aligned}$$

Since $\delta_h(\tilde{g})\delta_g(e) = 0$ unless $\tilde{g} = h$, we have:

$$\epsilon_{D(S_3)^*}(\chi_{h,g}) = \delta_g(e).$$

□

Now we can see the following presentation for $D(S_3)^*$

Theorem 3.4. *The dual algebra $D(S_3)^*$ has a presentation with generators A, B, and E_g for all $g \in S_3$, and the relations as follows:*

$$\begin{aligned} &A^2 = 1, \quad B^3 = 1, \quad AB = B^{-1}A, \\ &E_g E_h = \delta_{g,h} E_g, \quad \sum_{g \in S_3} E_g = 1, \\ &AE_g = E_g A, \quad BE_g = E_g B. \end{aligned}$$

Proof. First we verify the relations involving only the A and B terms:

$$A^2 = \sum_{g,h \in S_3} \chi_{\sigma,g}\chi_{\sigma,h} = \sum_{g,h \in S_3} \delta_g(h)\chi_{\sigma^2,g} = \sum_{g \in S_3} \chi_{e,g} = 1,$$
$$B^3 = \sum_{g,h,k \in S_3} \chi_{\tau,g}\chi_{\tau,h}\chi_{\tau,k} = \sum_{g,h,k \in S_3} \delta_g(h)\delta_h(k)\chi_{\tau^3,g} = \sum_{g \in S_3} \chi_{e,g} = 1,$$
$$AB = \sum_{g,h, \in S_3} \chi_{\sigma,g}\chi_{\tau,h} = \sum_{g,h \in S_3} \delta_g(h)\chi_{\sigma\tau,g} = \sum_{g \in S_3} \chi_{\tau^{-1}\sigma,g},$$
$$B^{-1}A = \sum_{g,h, \in S_3} \chi_{\tau^{-1},g}\chi_{\sigma,h} = \sum_{g,h \in S_3} \delta_g(h)\chi_{\tau^{-1}\sigma,g} = \sum_{g \in S_3} \chi_{\tau^{-1}\sigma,g}.$$

Now we can verify the relation involving only the E_h terms:

$$E_g E_h = \chi_{e,g}\chi_{e,h} = \delta_g(h)\chi_{e,g} = \delta_g(h)E_g,$$
$$\sum_{g \in G} E_g = \sum_{g \in G} \chi_{e,g} = 1.$$

Next we will verify the cross-relations:

$$AE_g = \sum_{h \in S_3} \chi_{\sigma,h}\chi_{e,g} = \sum_{h \in S_3} \delta_h(g)\chi_{\sigma,h} = \chi_{\sigma,g},$$
$$E_g A = \sum_{h \in S_3} \chi_{e,g}\chi_{\sigma,h} = \sum_{h \in S_3} \delta_g(h)\chi_{\sigma,h} = \chi_{\sigma.g},$$
$$BE_g = \sum_{h \in S_3} \chi_{\tau,h}\chi_{e,g} = \sum_{h \in S_3} \delta_h(g)\chi_{\tau,h} = \chi_{\tau,g},$$
$$E_g B = \sum_{h \in S_3} \chi_{e,g}\chi_{\tau,h} = \sum_{h \in S_3} \delta_g(h)\chi_{\tau,h} = \chi_{\tau,g}.$$

Next, we need to show that these generators actually generate $D(S_3)^*$. To see this, we will just show that every element of the basis $\{\chi_{h,g}\}_{g,h \in S_3}$ is generated.

$$E_g = \chi_{e,g},$$
$$BE_g = \chi_{\tau,g},$$
$$B^2 E_g = B\chi_{\tau,g} = \sum_{h \in S_3} \chi_{\tau,h}\chi_{\tau,g} = \chi_{\tau^2,g},$$
$$AE_g = \chi_{\sigma,g},$$

$$BAE_g = B\chi_{\sigma,g} = \sum_{h\in S_3} \chi_{\tau,h}\chi_{\sigma,g} = \chi_{\tau\sigma,g},$$
$$B^2AE_g = B\chi_{\tau\sigma,g} = \sum_{h\in S_3} \chi_{\tau,h}\chi_{\tau\sigma,g} = \chi_{\tau^2\sigma,g}.$$

Finally, we need to show that our relations are enough. To see this, let the algebra generated by A, B and E_g for all $g \in S_3$ with the relations given be $\mathbf{A}$. We can see that every element in $\mathbf{A}$ can be written as a linear combination of words in $\{A, B\} \cup \{E_g\}_{g\in S_3}$. Given an arbitrary word in $\{A, B\} \cup \{E_g\}_{g\in S_3}$, we can use the cross relations to move all of the E_g's to the right side of the word. So, every word in $\{A, B\} \cup \{E_g\}_{g\in S_3}$ can be written as the product of a word in $\{A, B\}$ and a word in $\{E_g\}_{g\in S_3}$. Since the subalgebra generated by $\{A, B\}$ is clearly the group algebra $\Bbbk S_3$, and the subalgebra generated by $\{E_g\}_{g\in S_3}$ is clearly the algebra $\Bbbk^{S_3}$, we can write this word as a constant times an element of the form $B^bA^aE_g$ for some $b \in \{0, 1, 2\}$, $a \in \{0, 1\}$ and $g \in S_3$. Since there are 36 elements of this form, we see that $\dim \mathbf{A} \le 36$. Thus, $D(S_3)^* \subset \mathbf{A}$, we must have that $\mathbf{A} = D(S_3)^*$.

□

We can also compute the coproduct and antipode for $D(S_3)^*$.

Lemma 3.5. *The formulas for the coproduct and antipode on $D(S_3)^*$ can be seen as follows:*

$$\Delta_{(D(S_3))^*}(\chi_{h,g}) = \sum_{\tilde{g}\in S_3} \chi_{h,g\tilde{g}^{-1}} \otimes \chi_{g\tilde{g}^{-1}h\tilde{g}g^{-1},\tilde{g}},$$
$$S_{(D(S_3))^*}(\chi_{h,g}) = \chi_{ghg^{-1},g^{-1}}.$$

Proof. We simply evaluate.

$$\begin{aligned}
&\Delta_{(D(S_3))^*}(\chi_{h,g})(\delta_{h_1} \bowtie g_1 \otimes \delta_{h_2} \bowtie g_2)\\
&= \delta_h(h_1)\delta_g(g_1g_2)\delta_{h_1}(g_1^{-1}h_2g_1)\\
&= \sum_{\tilde{g}_1,\tilde{g}_2,\tilde{h}_1,\tilde{h}_2\in S_3} \delta_{\tilde{g}_1}(g_1)\delta_{\tilde{g}_2}(g_2)\delta_{\tilde{h}_1}(h_1)\delta_{\tilde{h}_2}(h_2)\delta_h(h_1)\delta_g(g_1g_2)\delta_{h_1}(g_1^{-1}h_2g_1).
\end{aligned}$$

This is only nonzero when $\tilde{h}_1 = \tilde{g}_1^{-1}\tilde{h}_2\tilde{g}_1$, so we can simplify:

$$= \sum_{\tilde{g}_1,\tilde{g}_2,\tilde{h}_2\in S_3} \delta_{\tilde{g}_1}(g_1)\delta_{\tilde{g}_2}(g_2)\delta_{\tilde{g}_1^{-1}\tilde{h}_2\tilde{g}_1}(h_1)\delta_{\tilde{h}_2}(h_2)\delta_h(h_1)\delta_g(g_1g_2).$$

This is only nonzero when $\tilde{g}_1\tilde{g}_2 = g$, so we can set $\tilde{g}_1 = g\tilde{g}_2^{-1}$:

$$= \sum_{\tilde{g}_2, \tilde{h}_2 \in S_3} \delta_{g\tilde{g}_2^{-1}}(g_1)\delta_{\tilde{g}_2}(g_2)\delta_{\tilde{g}_2 g^{-1}\tilde{h}_2 g\tilde{g}_2^{-1}}(h_1)\delta_{\tilde{h}_2}(h_2)\delta_h(h_1).$$

Finally, this is only nonzero when $h = \tilde{g}_2 g^{-1}\tilde{h}_2 g\tilde{g}_2^{-1}$, so we can set $\tilde{h}_2 = g\tilde{g}_2^{-1}h\tilde{g}_2 g^{-1}$:

$$\begin{aligned}
&= \sum_{\tilde{g}_2 \in S_3} \delta_{g\tilde{g}_2^{-1}}(g_1)\delta_{\tilde{g}_2}(g_2)\delta_h(h_1)\delta_{g\tilde{g}_2^{-1}h\tilde{g}_2 g^{-1}}(h_2)\\
&= \sum_{\tilde{g} \in S_3} \delta_{g\tilde{g}^{-1}}(g_1)\delta_{\tilde{g}}(g_2)\delta_h(h_1)\delta_{g\tilde{g}^{-1}h\tilde{g}g^{-1}}(h_2)\\
&= \sum_{\tilde{g} \in S_3} \chi_{h,g\tilde{g}^{-1}}(\delta_{h_1} \bowtie g_1)\chi_{g\tilde{g}^{-1}h\tilde{g}g^{-1},\tilde{g}}(\delta_{h_2} \bowtie g_2).
\end{aligned}$$

Evaluating the antipode, we get

$$\begin{aligned}
&S_{(D(S_3))^*}(\chi_{h_1,g_1})(\delta_{h_2} \bowtie g_2) = \chi_{h_1,g_1}(S_{D(S_3)}(\delta_{h_2} \bowtie g_2))\\
&= \chi_{h_1,g_1}(\delta_{g_2 h g_2^{-1}} \bowtie g_2^{-1}) = \delta_{h_1}(g_2 h_2 g_2^{-1})\delta_{g_1}(g_2^{-1})\\
&= \delta_{h_1}(g_2 h_2 g_2^{-1})\delta_{g_1^{-1}}(g_2) = \delta_{h_1}(g_1^{-1} h_2 g_1)\delta_{g_1^{-1}}(g_2)\\
&= \delta_{g_1 h_1 g_1^{-1}}(h_2)\delta_{g_1^{-1}}(g_2) = \chi_{g_1 h_1 g_1^{-1}, g_1^{-1}}(\delta_{h_2} \bowtie g_2).
\end{aligned}$$

□

Theorem 3.6. *The coproduct structure and antipode for $D(S_3)^*$ can be seen as follows:*

$$\Delta(A)$$

$$= A(E_e \otimes A + E_\sigma \otimes A + E_\tau \otimes B^{-1}A + E_{\tau\sigma} \otimes B^{-1}A + E_{\tau^{-1}} \otimes BA + E_{\tau^{-1}\sigma} \otimes BA),$$

$$\Delta(B)$$

$$= B(E_e \otimes B + E_\tau \otimes B + E_{\tau^{-1}} \otimes B + E_\sigma \otimes B^{-1} + E_{\tau\sigma} \otimes B^{-1} + E_{\tau^{-1}\sigma} \otimes B^{-1}),$$

$$\Delta(E_g) = \sum_{h \in S_3} E_{gh^{-1}} \otimes E_h,$$

$$\begin{aligned}
S(A) &= AE_e + AE_\sigma + B^{-1}AE_{\tau^{-1}} + B^{-1}AE_{\tau\sigma} + BAE_\tau + BAE_{\tau^{-1}\sigma},\\
S(B) &= BE_e + B^{-1}E_\sigma + BE_\tau + B^{-1}E_{\tau\sigma} + BE_{\tau^{-1}} + B^{-1}E_{\tau^{-1}\sigma},\\
S(E_g) &= E_{g^{-1}}.
\end{aligned}$$

Proof. We will use Lemma 3.5:

$$\Delta(A) = \Delta\left(\sum_{g\in S_3}\chi_{\sigma,g}\right) = \sum_{g\in S_3}\Delta(\chi_{\sigma,g}) = \sum_{g,\tilde{g}\in S_3}\chi_{\sigma,g\tilde{g}^{-1}}\otimes\chi_{g\tilde{g}^{-1}\sigma\tilde{g}g^{-1},\tilde{g}}.$$

We will make the substitution $h = g\tilde{g}^{-1}$ and $g = \tilde{g}$, so we have:

$$\begin{aligned}
&= \sum_{g,h\in S_3}\chi_{\sigma,h}\otimes\chi_{h\sigma h^{-1},g} = \sum_{g,h\in S_3} AE_h\otimes\chi_{h\sigma h^{-1},g}\\
&= \sum_{g\in S_3} AE_e\otimes\chi_{\sigma,g} + AE_\sigma\otimes\chi_{\sigma,g} + AE_\tau\otimes\chi_{\tau^{-1}\sigma,g}\\
&+ AE_{\tau\sigma}\otimes\chi_{\tau^{-1}\sigma,g} + AE_{\tau^{-1}}\otimes\chi_{\tau\sigma,g} + AE_{\tau^{-1}\sigma}\otimes\chi_{\tau\sigma,g}\\
&= AE_e\otimes A + AE_\sigma\otimes A + AE_\tau\otimes B^{-1}A\\
&+ AE_{\tau\sigma}\otimes B^{-1}A + AE_{\tau^{-1}}\otimes BA + AE_{\tau^{-1}\sigma}\otimes BA.
\end{aligned}$$

We will do the same to calculate $\Delta(B)$:

$$\Delta(B) = \Delta\left(\sum_{g\in S_3}\chi_{\tau,g}\right) = \sum_{g\in S_3}\Delta(\chi_{\tau,g}) = \sum_{g,\tilde{g}\in S_3}\chi_{\tau,g\tilde{g}^{-1}}\otimes\chi_{g\tilde{g}^{-1}\tau\tilde{g}g^{-1},\tilde{g}}.$$

We will again make the substitution $h = g\tilde{g}^{-1}$ and $g = \tilde{g}$, so we have:

$$\begin{aligned}
&= \sum_{g,h\in S_3}\chi_{\tau,h}\otimes\chi_{h\tau h^{-1},g} = \sum_{g,h\in S_3} BE_h\otimes\chi_{h\tau h^{-1},g}\\
&= \sum_{g\in S_3} BE_e\otimes\chi_{\tau,g} + BE_\tau\otimes\chi_{\tau,g} + BE_{\tau^{-1}}\otimes\chi_{\tau,g}\\
&+ BE_\sigma\otimes\chi_{\tau^{-1},g} + BE_{\tau\sigma}\otimes\chi_{\tau^{-1},g} + BE_{\tau^{-1}\sigma}\otimes\chi_{\tau^{-1},g}\\
&= BE_e\otimes B + BE_\tau\otimes B + BE_{\tau^{-1}}\otimes B\\
&+ BE_\sigma\otimes B^{-1} + BE_{\tau\sigma}\otimes B^{-1} + BE_{\tau^{-1}\sigma}\otimes B^{-1}.
\end{aligned}$$

Next we calculate $\Delta(E_g)$:

$$\Delta(E_g) = \Delta(\chi_{e,g}) = \sum_{\tilde{g}\in S_3}\chi_{e,g\tilde{g}^{-1}}\otimes\chi_{e,\tilde{g}} = \sum_{h\in S_3} E_{gh^{-1}}\otimes E_h.$$

Finally, we calculate the antipode on the generators:

$$S(A) = S\left(\sum_{g\in S_3}\chi_{\sigma,g}\right) = \sum_{g\in S_3}S(\chi_{\sigma,g}) = \sum_{g\in S_3}\chi_{g\sigma g^{-1},g^{-1}}$$

$$= \chi_{\sigma,e} + \chi_{\sigma,\sigma} + \chi_{\tau^{-1}\sigma,\tau^{-1}} + \chi_{\tau^{-1}\sigma,\tau\sigma} + \chi_{\tau\sigma,\tau} + \chi_{\tau\sigma,\tau^{-1}\sigma}$$
$$= AE_e + AE_\sigma + B^{-1}AE_{\tau^{-1}} + B^{-1}AE_{\tau\sigma} + BAE_\tau + BAE_{\tau^{-1}\sigma},$$
$$S(B) = S\left(\sum_{g\in S_3}\chi_{\tau,g}\right) = \sum_{g\in S_3} S(\chi_{\tau,g}) = \sum_{g\in S_3}\chi_{g\tau g^{-1},g^{-1}}$$
$$= \chi_{\tau,e} + \chi_{\tau^{-1},\sigma} + \chi_{\tau,\tau^{-1}} + \chi_{\tau^{-1},\tau\sigma} + \chi_{\tau,\tau} + \chi_{\tau^{-1},\tau^{-1}\sigma}$$
$$= BE_e + B^{-1}E_\sigma + BE_\tau + B^{-1}E_{\tau\sigma} + BE_{\tau^{-1}} + B^{-1}E_{\tau^{-1}\sigma},$$
$$S(E_g) = S(\chi_{e,g}) = \chi_{e,g^{-1}} = E_{g^{-1}}.$$

□

4. On the Quasitriangular and Coquasitriangular Structures

Recall the definition of a quasitriangular Hopf algebra:

Definition 4.1 (cf. [4], p. 811)**.** A Hopf algebra H is **quasitriangular** if there exists an invertible element $R \in H \otimes H$ such that:

- $R\Delta(a) = \Delta^{\mathrm{cop}}(a)R$, for all $a \in H$, and
- $(\Delta \otimes \mathrm{id})(R) = R_{13}R_{23}$, $(\mathrm{id} \otimes \Delta)(R) = R_{13}R_{12}$

R is called the universal R-matrix for H

We will now look at the quasitriangular structure on $D(S_3)$:

Theorem 4.2. *The matrix:*

$$R = \sum_{g\in G} \epsilon \bowtie g \otimes \delta_g \bowtie e$$

is a universal R-matrix for $D(S_3)$.

Proof. To prove this, we must prove both properties of the R-matrix.
(i) Let $a = \delta_h \bowtie x$. Then:

$$\Delta(a) = \sum_{r\in G} \delta_{hr^{-1}} \bowtie x \otimes \delta_r \bowtie x,$$

and

$$R\Delta(a) = \sum_{g,r\in G} (\epsilon \bowtie g)(\delta_{hr^{-1}} \bowtie x) \otimes (\delta_g \bowtie e)(\delta_r \bowtie x).$$

We compute:

$$(\epsilon \bowtie g)(\delta_{hr^{-1}} \bowtie x) = \delta_{g^{-1}hr^{-1}g} \bowtie gx, \qquad (\delta_g \bowtie e)(\delta_r \bowtie x) = \delta_{g,r}\delta_r \bowtie x.$$

So:

$$R\Delta(a) = \sum_{g,r\in S_3} \delta_{g^{-1}hr^{-1}g} \bowtie gx \otimes \delta_{g,r}\delta_r \bowtie x.$$
$$= \sum_{g\in S_3} \delta_{g^{-1}h} \bowtie gx \otimes \delta_g \bowtie x.$$

Now we compute $\Delta^{\mathrm{cop}}(a)R$. We have:

$$\Delta^{\mathrm{cop}}(a) = \sum_{r\in G} \delta_r \bowtie x \otimes \delta_{hr^{-1}} \bowtie x,$$

and

$$\Delta^{\mathrm{cop}}(a)R = \sum_{r,g\in G} (\delta_r \bowtie x)(\epsilon \bowtie g) \otimes (\delta_{hr^{-1}} \bowtie x)(\delta_g \bowtie e).$$

We see that:

$$(\delta_r \bowtie x)(\epsilon \bowtie g) = \delta_r \bowtie xg, \qquad (\delta_{hr^{-1}} \bowtie x)(\delta_g \bowtie e) = \delta_{hr^{-1},x^{-1}gx}\delta_{hr^{-1}} \bowtie x.$$

So:

$$\Delta^{\mathrm{cop}}(a)R = \sum_{g,r\in S_3} \delta_r \bowtie xg \otimes \delta_{hr^{-1},x^{-1}gx}\delta_{hr^{-1}} \bowtie x.$$
$$= \sum_{g\in S_3} \delta_{x^{-1}g^{-1}xh} \bowtie xg \otimes \delta_{x^{-1}gx} \bowtie x$$

Now we change variables. Set $s = x^{-1}gx$ and we get:

$$\sum_{s\in S_3} \delta_{s^{-1}h} \bowtie gs \otimes \delta_s \bowtie x.$$

This is the same as the earlier expression for $R\Delta(a)$. Hence, condition (i) is satisfied.

(ii) We simply evaluate both sides to get:

$$(\Delta \otimes \mathrm{id})(R) = \sum_{g\in G} (\epsilon \bowtie g) \otimes (\epsilon \bowtie g) \otimes (\delta_g \bowtie e)$$
$$R_{13}R_{23} = \sum_{g,h\in G} (\epsilon \bowtie g) \otimes (\epsilon \bowtie h) \otimes (\delta_g \bowtie e)(\delta_h \bowtie e)$$
$$= \sum_{g,h\in G} (\epsilon \bowtie g) \otimes (\epsilon \bowtie h) \otimes (\delta_{g,h}\delta_g \bowtie e)$$
$$= \sum_{g\in G} (\epsilon \bowtie g) \otimes (\epsilon \bowtie g) \otimes (\delta_g \bowtie e)$$

Doing this to the other equation we get:

$$\begin{aligned}(\mathrm{id}\otimes\Delta)(R) &= \sum_{g,h\in G}(\epsilon\bowtie g)\otimes(\delta_{gh^{-1}}\bowtie e)\otimes(\delta_h\bowtie e)\\ R_{13}R_{12} &= \sum_{g,h\in G}(\epsilon\bowtie gh)\otimes(\delta_g\bowtie e)\otimes(\delta_h\bowtie e)\\ &= \sum_{g,h\in G}(\epsilon\bowtie g)\otimes(\delta_{gh^{-1}}\bowtie e)\otimes(\delta_h\bowtie e)\end{aligned}$$

□

By thinking of the R-matrix as a map $R\colon \Bbbk \to H\otimes H$, we can dualize the notion of quasitriangularity, as shown below:

Definition 4.3 (cf. [8] Def. 2.2.1)**.** A Hopf algebra H is **coquasitriangular (or dual quasitriangular)** if there exists a convolution invertible map $\mathcal{R}\colon H\otimes H\to \Bbbk$ such that:

(i) $R(ab\otimes c)=\sum R(a\otimes c_{(1)})R(b\otimes c_{(2)})$,
(ii) $R(a\otimes bc)=\sum R(a_{(1)}\otimes c)R(a_{(2)}\otimes b)$, and
(iii) $\sum b_{(1)}a_{(1)}R(a_{(2)}\otimes b_{(2)})=\sum R(a_{(1)}\otimes b_{(1)})a_{(2)}b_{(2)}$.

The map R is called the co-R-matrix of H.

We can easily obtain the coquasitriangular structure on $D(S_3)^*$ by using the following dualization:

Theorem 4.4 (cf. [6] Lemma 7.2.1)**.** *For any quasitriangular Hopf algebra H with R-matrix $R=\sum a_i\otimes b_i$, the dual H^* is coquasitriangular with $\mathcal{R}(f\otimes g)=\sum f(a_i)g(b_i)$.*

Corollary 4.5. *The Hopf algebra $D(S_3)^*$ is coquasitriangular. The co-R-matrix is given by the map*

$$\mathcal{R}: D(S_3)^*\otimes D(S_3)^*\to \Bbbk$$

defined on basis elements $\chi_{h,g},\chi_{k,\ell}\in D(S_3)^*$ by

$$\mathcal{R}(\chi_{h,g}\otimes\chi_{k,\ell})=\delta_{g,k}\delta_{\ell,e}.$$

Proof. This follows easily from Theorem 4.4:

$$\begin{aligned}\mathcal{R}(\chi_{h,g}\otimes\chi_{k,\ell}) &= \sum_{\tilde g\in S_3}\chi_{h,g}(\epsilon\bowtie\tilde g)\chi_{k,\ell}(\delta_{\tilde g}\bowtie e)\\ &= \sum_{\tilde g\in S_3}\delta_g(\tilde g)\delta_k(\tilde g)\delta_\ell(e)=\delta_{g,k}\delta_{\ell,e}.\end{aligned}$$

□

Corollary 4.6. *The co-R-matrix on $D(S_3)^*$ satisfies the following:*

$$\mathcal{R}(B^bA^a \otimes B^dA^c) = 1,$$
$$\mathcal{R}(B^bA^aE_g \otimes B^dA^c) = \delta_{g,\tau^d\sigma^c},$$
$$\mathcal{R}(B^bA^a \otimes B^dA^cE_h) = \delta_{h,e},$$
$$\mathcal{R}(B^bA^aE_g \otimes B^dA^cE_h) = \delta_{g,\tau^d\sigma^c}\delta_{h,e}.$$

Proof. We simply apply our formula:

$$\mathcal{R}(B^bA^a \otimes B^dA^c) = \sum_{g,h\in S_3} \mathcal{R}(\chi_{\tau^b\sigma^a,g} \otimes \chi_{\tau^d\sigma^c,h}) = \sum_{g,h\in S_3} \delta_{g,\tau^d\sigma^c}\delta_{h,e} = 1,$$
$$\mathcal{R}(B^bA^aE_g \otimes B^dA^c) = \sum_{h\in S_3} \mathcal{R}(\chi_{\tau^b\sigma^a,g} \otimes \chi_{\tau^d\sigma^c,h}) = \sum_{h\in S_3} \delta_{g,\tau^d\sigma^c}\delta_{h,e} = \delta_{g,\tau^d\sigma^c},$$
$$\mathcal{R}(B^bA^a \otimes B^dA^cE_h) = \sum_{g\in S_3} \mathcal{R}(\chi_{\tau^b\sigma^a,g} \otimes \chi_{\tau^d\sigma^c,h}) = \sum_{g\in S_3} \delta_{g,\tau^d\sigma^c}\delta_{h,e} = \delta_{h,e},$$
$$\mathcal{R}(B^bA^aE_g \otimes B^dA^cE_h) = \mathcal{R}(\chi_{\tau^b\sigma^a,g} \otimes \chi_{\tau^d\sigma^c,h}) = \delta_{g,\tau^d\sigma^c}\delta_{h,e} = \delta_{g,\tau^d\sigma^c}\delta_{h,e}.$$

□

Note that the quasitriangular structure of $D(S_3)$ gives us a braiding in the category of left $D(S_3)$-modules, and the coquasitriangular structure of $D(S_3)^*$ gives us a braiding in the category of right $D(S_3)^*$-comodules.

5. $\mathcal{FK}_3$ as a braided Hopf algebra in ${}^{D(S_3)^*}\mathcal{M}$

To start, we will look at the definition of the algebra $\mathcal{FK}_3$:

Definition 5.1 (cf. [1])**.** The **Fomin-Kirillov algebra** $\mathcal{FK}_3$ is the algebra generated by the elements x_{ij} with $1 \leq i < j \leq 3$ subject to the following relations:

$$0 = x_{ij}^2 \text{ for all } i \neq j,$$
$$0 = x_{12}x_{23} + x_{23}x_{13} + x_{13}x_{12},$$
$$0 = x_{23}x_{12} + x_{13}x_{23} + x_{12}x_{13}.$$

We can consider $\mathcal{FK}_3$ as a Yetter-Drinfeld module over $\Bbbk S_3$.

Definition 5.2. The algebra $\mathcal{FK}_3$ is a YD-module over $\Bbbk S_3$ with the following action and coaction:

$$g \rightharpoonup x_{ij} = \operatorname{sgn}(g)x_{g(ij)g^{-1}},$$
$$\delta(x_{ij}) = (ij) \otimes x_{ij}.$$

Where (ij) is the corresponding transposition in S_3.

There is an equivalence of the categories ${}^{\Bbbk S_3}_{\Bbbk S_3}\mathcal{YD}$ and ${}_{\Bbbk S_3}\mathcal{M}$, as shown below:

Theorem 5.3 (cf. [5] Theorem XIII.5.1)**.** *Let H be a finite-dimensional Hopf algebra over a field $\Bbbk$. Then, the categories ${}^{H}_{H}\mathcal{YD}$ and ${}_{D(H)}\mathcal{M}$ are equivalent. In particular, there is an equivalence $F\colon {}^{H}_{H}\mathcal{YD} \to {}_{D(H)}\mathcal{M}$ which maps the YD module M with action $\rightharpoonup\colon H \otimes M \to M$ and coaction $\delta : M \to H \otimes M$ to the module M with action defined on $1 \bowtie x$ and $f \bowtie 1$ by:*

$$\begin{aligned}(1 \bowtie x)(m) &= x \rightharpoonup m,\\ (f \bowtie 1)(m) &= \langle f, m_{(-1)}\rangle m_{(0)}.\end{aligned}$$

Thus, we can transform $\mathcal{FK}_3$ into a left module over $D(S_3)$.

Corollary 5.4. *The module $F(\mathcal{FK}_3)$ where $\mathcal{FK}_3$ is given the YD-module structure described earlier, and F is the functor described in Theorem 5.3, gives a left module structure for $\mathcal{FK}_3$ over $D(S_3)$ with the following action defined on $1 \bowtie g$ and $f \bowtie 1$:*

$$\begin{aligned}(1 \bowtie g)(x_{ij}) &= \operatorname{sgn}(g)x_{g(ij)g^{-1}},\\ (\delta_h \bowtie 1)(x_{ij}) &= \delta_h\left((ij)\right)x_{ij}.\end{aligned}$$

Proof. We simply use the definition of F from the previous theorem.

$$\begin{aligned}(1 \bowtie g)(x_{ij}) &= g \rightharpoonup x_{ij} = \operatorname{sgn}(g)x_{g(ij)g^{-1}},\\ (\delta_h \bowtie 1)(x_{ij}) &= \delta_h\left((ij)\right)x_{ij}.\end{aligned}$$

□

Now, recall that there is an equivalence between left modules on an algebra A and right comodules on the dual A^* when A is finite dimensional.

Proposition 5.5. *For any finite dimensional algebra A, the categories ${}_A\mathcal{M}$ and $\mathcal{M}^{A^*}$ are equivalent. Particularly, there is an equivalence of the two categories $F\colon {}_A\mathcal{M} \to \mathcal{M}^{A^*}$ which maps the module M to the comodule M with coaction $\rho(m) = \sum_i (a_i \cdot m) \otimes a^i$, where $\{a_i\}$ is a basis for A, and a^i is the corresponding dual basis with $a^i(a_j) = \delta_i(j)$.*

Proof. The proof is clear due to duality. □

We will now calculate the coaction on $F(\mathcal{FK}_3)$ so we can represent $\mathcal{FK}_3$ as a right comodule on $D(S_3)^*$.

Corollary 5.6. *The comodule $F(\mathcal{FK}_3)$ where $\mathcal{FK}_3$ is given the left module structure described earlier, and F is the functor described in proposition 5.5, gives a right comodule structure for $\mathcal{FK}_3$ over $D(S_3)^*$ with the following coaction defined on the generators of $\mathcal{FK}_3$:*

$$\delta(x_{12}) =$$
$$x_{12} \otimes (AE_e - AE_\sigma) + x_{13} \otimes (BAE_{\tau^{-1}} - BAE_{\tau^{-1}\sigma}) + x_{23} \otimes (B^{-1}AE_\tau - B^{-1}AE_{\tau\sigma}),$$
$$\delta(x_{13}) =$$
$$x_{12} \otimes (AE_\tau - AE_{\tau^{-1}\sigma}) + x_{13} \otimes (BAE_e - BAE_{\tau\sigma}) + x_{23} \otimes (B^{-1}AE_{\tau^{-1}} - B^{-1}AE_\sigma),$$
$$\delta(x_{23}) =$$
$$x_{12} \otimes (AE_{\tau^{-1}} - AE_{\tau\sigma}) + x_{13} \otimes (BAE_\tau - BAE_\sigma) + x_{23} \otimes (B^{-1}AE_e - B^{-1}AE_{\tau^{-1}\sigma}).$$

Proof. To do this, we simply use the definition of the functor F. We will use our standard basis for $D(S_3)$, $\{\delta_h \bowtie g | g, h \in S_3\}$ to apply the definition. Doing so gives us

$$\begin{aligned}
\delta(x_{12}) &= \sum_{g,h\in S_3} ((\delta_h \bowtie g) \cdot x_{12}) \otimes \chi_{h,g} = \sum_{g,h\in S_3} ((\delta_h \bowtie 1)(1 \bowtie g) \cdot x_{12}) \otimes \chi_{h,g} \\
&= \sum_{g,h\in S_3} \left((\delta_h \bowtie 1) \cdot \operatorname{sgn}(g) x_{g(12)g^{-1}}\right) \otimes \chi_{h,g} \\
&= \sum_{g,h\in S_3} \left(\operatorname{sgn}(g)\delta_h\left(g(12)g^{-1}\right) \cdot x_{g(12)g^{-1}}\right) \otimes \chi_{h,g} \\
&= x_{12} \otimes \chi_{(12),e} - x_{12} \otimes \chi_{(12),(12)} - x_{23} \otimes \chi_{(23),(13)} \\
&- x_{13} \otimes \chi_{(13),(23)} + x_{23} \otimes \chi_{(23),(123)} + x_{13} \otimes \chi_{(13),(132)} \\
&= x_{12} \otimes \chi_{\sigma,e} - x_{12} \otimes \chi_{\sigma,\sigma} - x_{23} \otimes \chi_{\tau^{-1}\sigma,\tau\sigma} \\
&- x_{13} \otimes \chi_{\tau\sigma,\tau^{-1}\sigma} + x_{23} \otimes \chi_{\tau^{-1}\sigma,\tau} + x_{13} \otimes \chi_{\tau\sigma,\tau^{-1}} =
\end{aligned}$$
$$x_{12} \otimes (AE_e - AE_\sigma) + x_{13} \otimes (BAE_{\tau^{-1}} - BAE_{\tau^{-1}\sigma}) + x_{23} \otimes (B^{-1}AE_\tau - B^{-1}AE_{\tau\sigma}),$$

$$\begin{aligned}
\delta(x_{13}) &= \sum_{g,h\in S_3} ((\delta_h \bowtie g) \cdot x_{13}) \otimes \chi_{h,g} = \sum_{g,h\in S_3} ((\delta_h \bowtie 1)(1 \bowtie g) \cdot x_{13}) \otimes \chi_{h,g} \\
&= \sum_{g,h\in S_3} \left((\delta_h \bowtie 1) \cdot \operatorname{sgn}(g) x_{g(13)g^{-1}}\right) \otimes \chi_{h,g} \\
&= \sum_{g,h\in S_3} \left(\operatorname{sgn}(g)\delta_h\left(g(13)g^{-1}\right) \cdot x_{g(13)g^{-1}}\right) \otimes \chi_{h,g} \\
&= x_{13} \otimes \chi_{(13),e} - x_{23} \otimes \chi_{(23),(12)} - x_{13} \otimes \chi_{(13),(13)} \\
&- x_{12} \otimes \chi_{(12),(23)} + x_{12} \otimes \chi_{(12),(123)} + x_{23} \otimes \chi_{(23),(132)}
\end{aligned}$$

$$
\begin{aligned}
&= x_{13} \otimes \chi_{\tau\sigma,e} - x_{23} \otimes \chi_{\tau^{-1}\sigma,\sigma} - x_{13} \otimes \chi_{\tau\sigma,\tau\sigma} \\
&- x_{12} \otimes \chi_{\sigma,\tau^{-1}\sigma} + x_{12} \otimes \chi_{\sigma,\tau} + x_{23} \otimes \chi_{\tau^{-1}\sigma,\tau^{-1}} =
\end{aligned}
$$

$$
x_{12} \otimes (AE_\tau - AE_{\tau^{-1}\sigma}) + x_{13} \otimes (BAE_e - BAE_{\tau\sigma}) + x_{23} \otimes (B^{-1}AE_{\tau^{-1}} - B^{-1}AE_\sigma),
$$

$$
\begin{aligned}
\delta(x_{23}) &= \sum_{g,h\in S_3} ((\delta_h \bowtie g) \cdot x_{23}) \otimes \chi_{h,g} = \sum_{g,h\in S_3} ((\delta_h \bowtie 1)(1 \bowtie g) \cdot x_{23}) \otimes \chi_{h,g} \\
&= \sum_{g,h\in S_3} \left((\delta_h \bowtie 1) \cdot \operatorname{sgn}(g) x_{g(23)g^{-1}}\right) \otimes \chi_{h,g} \\
&= \sum_{g,h\in S_3} \left(\operatorname{sgn}(g)\delta_h \left(g(23)g^{-1}\right) \cdot x_{g(23)g^{-1}}\right) \otimes \chi_{h,g} \\
&= x_{23} \otimes \chi_{(23),e} - x_{13} \otimes \chi_{(13),(12)} - x_{12} \otimes \chi_{(12),(13)} \\
&- x_{23} \otimes \chi_{(23),(23)} + x_{13} \otimes \chi_{(13),(123)} + x_{12} \otimes \chi_{(12),(132)} \\
&= x_{23} \otimes \chi_{\tau^{-1}\sigma,e} - x_{13} \otimes \chi_{\tau\sigma,\sigma} - x_{12} \otimes \chi_{\sigma,\tau\sigma} \\
&- x_{23} \otimes \chi_{\tau^{-1}\sigma,\tau^{-1}\sigma} + x_{13} \otimes \chi_{\tau\sigma,\tau} + x_{12} \otimes \chi_{\sigma,\tau^{-1}} =
\end{aligned}
$$

$$
x_{12} \otimes (AE_{\tau^{-1}} - AE_{\tau\sigma}) + x_{13} \otimes (BAE_\tau - BAE_\sigma) + x_{23} \otimes (B^{-1}AE_e - B^{-1}AE_{\tau^{-1}\sigma}).
$$

□

Next, we recall that there is an equivalence between the categories $\mathcal{M}^{D(S_3)^*}$ and ${}^{D(S_3)^*}\mathcal{M}$.

Proposition 5.7 (cf. [3] Prop. 4.2.14)**.** *For any finite dimensional Hopf algebra H, the categories $\mathcal{M}^H$ and ${}^H\mathcal{M}$ are equivalent. In particular, there is an equivalence $F\colon \mathcal{M}^H \to {}^H\mathcal{M}$ which maps the right comodule M with coaction δ to the left comodule M with coaction $\bar{\delta}(x) = S(x_{(1)}) \otimes x_{(0)}$.*

With this, we can finally give $\mathcal{FK}_3$ a left comodule structure over $D(S_3)^*$, but first we will evaluate the antipode on some elements of $D(S_3)^*$ that will help us compute the left comodule structure:

Lemma 5.8. *The antipode acts as follows on B^{-1}, BA, and $B^{-1}A$:*

$$
\begin{aligned}
S(B^{-1}) &= B^{-1}E_e + BE_\sigma + B^{-1}E_{\tau^{-1}} + BE_{\tau\sigma} + B^{-1}E_\tau + BE_{\tau^{-1}\sigma}, \\
S(BA) &= ABE_e + AB^{-1}E_\sigma + AE_\tau + AE_{\tau\sigma} + AB^{-1}E_{\tau^{-1}} + ABE_{\tau^{-1}\sigma}, \\
S(B^{-1}A) &= AB^{-1}E_e + ABE_\sigma + ABE_\tau + AB^{-1}E_{\tau\sigma} + AE_{\tau^{-1}} + AE_{\tau^{-1}\sigma}.
\end{aligned}
$$

Proof. We simply evaluate:

$$\begin{aligned} S(B^{-1}) &= S\left(\sum_{g\in S_3}\chi_{\tau^{-1},g}\right) = \sum_{g\in S_3} S(\chi_{\tau^{-1},g}) = \sum_{g\in S_3}\chi_{g\tau^{-1}g^{-1},g^{-1}} \\ &= \chi_{\tau^{-1},e} + \chi_{\tau,\sigma} + \chi_{\tau^{-1},\tau^{-1}} + \chi_{\tau,\tau\sigma} + \chi_{\tau^{-1},\tau} + \chi_{\tau,\tau^{-1}\sigma} \\ &= B^{-1}E_e + BE_\sigma + B^{-1}E_{\tau^{-1}} + BE_{\tau\sigma} + B^{-1}E_\tau + BE_{\tau^{-1}\sigma}, \end{aligned}$$

$$\begin{aligned} S(BA) &= S(A)S(B) \\ &= (AE_e + AE_\sigma + AB^{-1}E_\tau + ABE_{\tau\sigma} + ABE_{\tau^{-1}} + AB^{-1}E_{\tau^{-1}\sigma}) \\ &\quad (BE_e + B^{-1}E_\sigma + BE_\tau + B^{-1}E_{\tau\sigma} + BE_{\tau^{-1}} + B^{-1}E_{\tau^{-1}\sigma}) \\ &= ABE_e + AB^{-1}E_\sigma + AE_\tau + AE_{\tau\sigma} + AB^{-1}E_{\tau^{-1}} + ABE_{\tau^{-1}\sigma}, \end{aligned}$$

$$\begin{aligned} S(B^{-1}A) &= S(A)S(B^{-1}) \\ &= (AE_e + AE_\sigma + AB^{-1}E_\tau + ABE_{\tau\sigma} + ABE_{\tau^{-1}} + AB^{-1}E_{\tau^{-1}\sigma}) \\ &\quad (B^{-1}E_e + BE_\sigma + B^{-1}E_{\tau^{-1}} + BE_{\tau\sigma} + B^{-1}E_\tau + BE_{\tau^{-1}\sigma}) \\ &= AB^{-1}E_e + ABE_\sigma + ABE_\tau + AB^{-1}E_{\tau\sigma} + AE_{\tau^{-1}} + AE_{\tau^{-1}\sigma}. \end{aligned}$$

□

Theorem 5.9. *The comodule $F(\mathcal{FK}_3)$ where $\mathcal{FK}_3$ is given the right comodule structure described earlier, and F is the functor described in proposition 5.7, gives a left comodule structure for $\mathcal{FK}_3$ over $D(S_3)^*$ with the following coaction $\bar{\delta}$ defined on the generators of $\mathcal{FK}_3$:*

$$\begin{aligned} \bar{\delta}(x_{12}) &= (AE_e - AE_\sigma) \otimes x_{12} \\ &\quad + (AE_\tau - B^{-1}AE_{\tau^{-1}\sigma}) \otimes x_{13} \\ &\quad + (AE_{\tau^{-1}} - BAE_{\tau\sigma}) \otimes x_{23}, \end{aligned}$$

$$\begin{aligned} \bar{\delta}(x_{13}) &= (B^{-1}AE_{\tau^{-1}} - BAE_{\tau^{-1}\sigma}) \otimes x_{12} \\ &\quad + (B^{-1}AE_e - AE_{\tau\sigma}) \otimes x_{13} \\ &\quad + (B^{-1}AE_\tau - B^{-1}AE_\sigma) \otimes x_{23}, \end{aligned}$$

$$\bar{\delta}(x_{23}) = (BAE_\tau - B^{-1}AE_{\tau\sigma}) \otimes x_{12}$$

$$+ (BAE_{\tau^{-1}} - BAE_{\sigma}) \otimes x_{13}$$
$$+ (BAE_e - AE_{\tau^{-1}\sigma}) \otimes x_{23}.$$

Proof. We simply use the definition of the functor F, and our formulas from Lemma 5.8.

$$\begin{aligned}
\bar{\delta}(x_{12}) &= S(AE_e - AE_{\sigma}) \otimes x_{12} + S(BAE_{\tau^{-1}} - BAE_{\tau^{-1}\sigma}) \otimes x_{13} \\
&+ S(B^{-1}AE_{\tau} - B^{-1}AE_{\tau\sigma}) \otimes x_{23} \\
&= S(E_e - E_{\sigma})S(A) \otimes x_{12} + S(E_{\tau^{-1}} - E_{\tau^{-1}\sigma})S(BA) \otimes x_{13} \\
&+ S(E_{\tau} - E_{\tau\sigma})S(B^{-1}A) \otimes x_{23}
\end{aligned}$$

$$\begin{aligned}
&= (E_e - E_{\sigma})(AE_e + AE_{\sigma} + B^{-1}AE_{\tau^{-1}} + B^{-1}AE_{\tau\sigma} + BAE_{\tau} + BAE_{\tau^{-1}\sigma}) \otimes x_{12} \\
&+(E_{\tau} - E_{\tau^{-1}\sigma})(ABE_e + AB^{-1}E_{\sigma} + AE_{\tau} + AE_{\tau\sigma} + AB^{-1}E_{\tau^{-1}} + ABE_{\tau^{-1}\sigma}) \otimes x_{13} \\
&+(E_{\tau^{-1}} - E_{\tau\sigma})(AB^{-1}E_e + ABE_{\sigma} + ABE_{\tau} + AB^{-1}E_{\tau\sigma} + AE_{\tau^{-1}} + AE_{\tau^{-1}\sigma}) \otimes x_{23} \\
&= (AE_e - AE_{\sigma}) \otimes x_{12} + (AE_{\tau} - ABE_{\tau^{-1}\sigma}) \otimes x_{13} + (AE_{\tau^{-1}} - AB^{-1}E_{\tau\sigma}) \otimes x_{23} \\
&= (AE_e - AE_{\sigma}) \otimes x_{12} + (AE_{\tau} - B^{-1}AE_{\tau^{-1}\sigma}) \otimes x_{13} + (AE_{\tau^{-1}} - BAE_{\tau\sigma}) \otimes x_{23},
\end{aligned}$$

$$\begin{aligned}
\bar{\delta}(x_{13}) &= S(AE_{\tau} - AE_{\tau^{-1}\sigma}) \otimes x_{12} + S(BAE_e - BAE_{\tau\sigma}) \otimes x_{13} \\
&+ S(B^{-1}AE_{\tau^{-1}} - B^{-1}AE_{\sigma}) \otimes x_{23} \\
&= S(E_{\tau} - E_{\tau^{-1}\sigma})S(A) \otimes x_{12} + S(E_e - E_{\tau\sigma})S(BA) \otimes x_{13} \\
&+ S(E_{\tau^{-1}} - E_{\sigma})S(B^{-1}A) \otimes x_{23} \\
&= (E_{\tau^{-1}} - E_{\tau^{-1}\sigma})(AE_e + AE_{\sigma} + B^{-1}AE_{\tau^{-1}} + B^{-1}AE_{\tau\sigma} + BAE_{\tau} + BAE_{\tau^{-1}\sigma}) \otimes x_{12} \\
&+ (E_e - E_{\tau\sigma})(ABE_e + AB^{-1}E_{\sigma} + AE_{\tau} + AE_{\tau\sigma} + AB^{-1}E_{\tau^{-1}} + ABE_{\tau^{-1}\sigma}) \otimes x_{13} \\
&+ (E_{\tau} - E_{\sigma})(AB^{-1}E_e + ABE_{\sigma} + ABE_{\tau} + AB^{-1}E_{\tau\sigma} + AE_{\tau^{-1}} + AE_{\tau^{-1}\sigma}) \otimes x_{23} \\
&= (B^{-1}AE_{\tau^{-1}} - BAE_{\tau^{-1}\sigma}) \otimes x_{12} + (ABE_e - AE_{\tau\sigma}) \otimes x_{13} \\
&+ (ABE_{\tau} - ABE_{\sigma}) \otimes x_{23} \\
&= (B^{-1}AE_{\tau^{-1}} - BAE_{\tau^{-1}\sigma}) \otimes x_{12} + (B^{-1}AE_e - AE_{\tau\sigma}) \otimes x_{13} \\
&+ (B^{-1}AE_{\tau} - B^{-1}AE_{\sigma}) \otimes x_{23},
\end{aligned}$$

$$\begin{aligned}
\bar{\delta}(x_{23}) &= S(AE_{\tau^{-1}} - AE_{\tau\sigma}) \otimes x_{12} + S(BAE_{\tau} - BAE_{\sigma}) \otimes x_{13} \\
&+ S(B^{-1}AE_e - B^{-1}AE_{\tau^{-1}\sigma}) \otimes x_{23} \\
&= S(E_{\tau^{-1}} - E_{\tau\sigma})S(A) \otimes x_{12} + S(E_{\tau} - E_{\sigma})S(BA) \otimes x_{13} \\
&+ S(E_e - E_{\tau^{-1}\sigma})S(B^{-1}A) \otimes x_{23}
\end{aligned}$$

$$
\begin{aligned}
&= (E_\tau - E_{\tau\sigma})(AE_e + AE_\sigma + B^{-1}AE_{\tau^{-1}} + B^{-1}AE_{\tau\sigma} + BAE_\tau + BAE_{\tau^{-1}\sigma}) \otimes x_{12} \\
&+ (E_{\tau^{-1}} - E_\sigma)(ABE_e + AB^{-1}E_\sigma + AE_\tau + AE_{\tau\sigma} + AB^{-1}E_{\tau^{-1}} + ABE_{\tau^{-1}\sigma}) \otimes x_{13} \\
&+ (E_e - E_{\tau^{-1}\sigma})(AB^{-1}E_e + ABE_\sigma + ABE_\tau + AB^{-1}E_{\tau\sigma} + AE_{\tau^{-1}} + AE_{\tau^{-1}\sigma}) \otimes x_{23} \\
&= (BAE_\tau - B^{-1}AE_{\tau\sigma}) \otimes x_{12} + (AB^{-1}E_{\tau^{-1}} - AB^{-1}E_\sigma) \otimes x_{13} \\
&+ (AB^{-1}E_e - AE_{\tau^{-1}\sigma}) \otimes x_{23} \\
&= (BAE_\tau - B^{-1}AE_{\tau\sigma}) \otimes x_{12} + (BAE_{\tau^{-1}} - BAE_\sigma) \otimes x_{13} \\
&+ (BAE_e - AE_{\tau^{-1}\sigma}) \otimes x_{23}.
\end{aligned}
$$

□

Before we move on, we need to find the braiding and coproduct structures for $\mathcal{FK}_3$. Before we do this though, we will define $x_1 = x_{12}, x_2 = x_{13}$, and $x_3 = x_{23}$.

Theorem 5.10. *The braiding on $\mathcal{FK}_3$ is given by:*

$$
\begin{aligned}
c(x_i \otimes x_i) &= -x_i \otimes x_i, \text{ and} \\
c(x_i \otimes x_j) &= -x_k \otimes x_i \text{ for } i, j, k \text{ distinct.}
\end{aligned}
$$

Proof. We will use the standard formula $c(v \otimes w) = (\deg v) \cdot w \otimes v$. Doing this gives:

$$
\begin{aligned}
c(x_1 \otimes x_1) &= (12) \cdot x_1 \otimes x_1 = -x_1 \otimes x_1, \\
c(x_2 \otimes x_2) &= (13) \cdot x_2 \otimes x_2 = -x_2 \otimes x_2, \\
c(x_3 \otimes x_3) &= (23) \cdot x_3 \otimes x_3 = -x_3 \otimes x_3, \\
c(x_1 \otimes x_2) &= (12) \cdot x_2 \otimes x_1 = -x_3 \otimes x_1, \\
c(x_1 \otimes x_3) &= (12) \cdot x_3 \otimes x_1 = -x_2 \otimes x_1, \\
c(x_2 \otimes x_1) &= (13) \cdot x_1 \otimes x_2 = -x_3 \otimes x_2, \\
c(x_2 \otimes x_3) &= (13) \cdot x_3 \otimes x_2 = -x_1 \otimes x_2, \\
c(x_3 \otimes x_1) &= (23) \cdot x_1 \otimes x_3 = -x_2 \otimes x_3, \\
c(x_3 \otimes x_2) &= (23) \cdot x_2 \otimes x_3 = -x_1 \otimes x_3.
\end{aligned}
$$

□

Next we need the coproducts on $\mathcal{FK}_3$.

Theorem 5.11. *The coproducts on $\mathcal{FK}_3$ are as follows:*

$$\Delta(x_1) = x_1 \otimes 1 + 1 \otimes x_1,$$

$$\begin{aligned}
\Delta(x_2) &= x_2 \otimes 1 + 1 \otimes x_2, \\
\Delta(x_3) &= x_3 \otimes 1 + 1 \otimes x_3, \\
\Delta(x_1x_2) &= x_1x_2 \otimes 1 + x_1 \otimes x_2 - x_3 \otimes x_1 + 1 \otimes x_1x_2, \\
\Delta(x_1x_3) &= x_1x_3 \otimes 1 + x_1 \otimes x_3 - x_2 \otimes x_1 + 1 \otimes x_1x_3, \\
\Delta(x_2x_1) &= x_2x_1 \otimes 1 + x_2 \otimes x_1 - x_3 \otimes x_2 + 1 \otimes x_2x_1, \\
\Delta(x_2x_3) &= x_2x_3 \otimes 1 + x_2 \otimes x_3 - x_1 \otimes x_2 + 1 \otimes x_2x_3, \\
\Delta(x_1x_2x_1) &= x_1x_2x_1 \otimes 1 - x_1x_3 \otimes x_2 + x_3x_1 \otimes x_1 + x_2 \otimes x_1x_2 \\
&\quad + x_1x_2 \otimes x_1 + x_1 \otimes x_2x_1 + 1 \otimes x_1x_2x_1, \\
\Delta(x_1x_2x_3) &= x_1x_2x_3 \otimes 1 + x_3x_2 \otimes x_1 + x_1 \otimes x_1x_2 + x_1x_2 \otimes x_3 \\
&\quad + x_1 \otimes x_2x_3 - x_3 \otimes x_1x_3 + 1 \otimes x_1x_2x_3, \\
\Delta(x_2x_3x_1) &= x_2x_3x_1 \otimes 1 + x_1x_3 \otimes x_2 - x_2 \otimes x_2x_3 + x_2x_3 \otimes x_1 \\
&\quad + x_2 \otimes x_3x_1 - x_1 \otimes x_2x_1 + 1 \otimes x_2x_3x_1, \\
\Delta(x_1x_2x_1x_3) &= x_1x_2x_1x_3 \otimes 1 + x_1x_2x_3 \otimes x_2 - x_2x_1x_3 \otimes x_1 + x_2x_1 \otimes x_1x_2 \\
&\quad + x_1x_2 \otimes x_2x_1 - x_3 \otimes x_1x_2x_1 + x_1x_2x_1 \otimes x_3 - x_1x_3 \otimes x_2x_3 \\
&\quad + x_3x_1 \otimes x_1x_3 + x_2 \otimes x_1x_2x_3 + x_1x_2 \otimes x_1x_3 + x_1 \otimes x_2x_1x_3 \\
&\quad + 1 \otimes x_1x_2x_1x_3.
\end{aligned}$$

Proof. The first three identities are true since the x_i are primitive elements of $\mathcal{FK}_3$, and the rest can be derived through multiplication. □

We may now begin computing the codouble bosonisation of $\mathcal{FK}_3$.

6. The algebra structure of the codouble bosonisation of $\mathcal{FK}_3$

We now look at the definition of the codouble bosonisation:

Definition 6.1 (cf. [2] Theorem 3.1)**.** Let B be a finite-dimensional braided group in ${}^{A}\mathcal{M}$ with basis $\{e_a\}$. Denote its dual by $B^* \in \mathcal{M}^A$ with dual basis $\{f^a\}$. Then there is an ordinary Hopf algebra $B^{\underline{\mathrm{op}}} \blacktriangleright\!\!\!\triangleleft A \triangleright\!\!\!\blacktriangleleft B^*$, the *co-double bosonisation*, built on the vector space $B^{\underline{\mathrm{op}}} \otimes A \otimes B^*$ with

$$(x \bowtie k \bowtie y)(w \bowtie \ell \bowtie z) = x \cdot_{\underline{\mathrm{op}}} w^{(\bar{\infty})} \bowtie k_{(2)}\ell_{(1)} \bowtie y^{(\bar{0})} z\ \mathcal{R}(y^{(\bar{1})}, \ell_{(2)})\mathcal{R}(Sk_{(1)}, w^{(\bar{1})}),$$

$$\begin{aligned}
&\Delta(x \bowtie k \bowtie y) \\
&= \sum_a x_{\underline{(1)}} \bowtie x_{\underline{(2)}}{}^{(\bar{1})}{}_{(1)}k_{(1)} \bowtie f^a \otimes e_{a\underline{(1)}}{}^{(\bar{\infty})} \cdot_{\underline{\mathrm{op}}} x_{\underline{(2)}}{}^{(\bar{\infty})} \cdot_{\underline{\mathrm{op}}} \bar{S}e_{a\underline{(3)}}{}^{(\bar{\infty})} \bowtie k_{(4)}y_{\underline{(1)}}{}^{(\bar{1})}{}_{(2)} \bowtie y_{\underline{(2)}} \\
&\quad \mathcal{R}(e_{a\underline{(1)}}{}^{(\bar{1})}, x_{\underline{(2)}}{}^{(\bar{1})}{}_{(2)}k_{(2)})\mathcal{R}(S(k_{(3)}y_{\underline{(1)}}{}^{(\bar{1})}{}_{(1)}), e_{a\underline{(3)}}{}^{(\bar{1})})\ \langle y_{\underline{(1)}}{}^{(\bar{0})}, e_{a\underline{(2)}}\rangle
\end{aligned}$$

for all $x, w \in B^{\underline{\mathrm{op}}}$, $k, \ell \in A$, and $y, z \in B^*$.

To find the codouble bosonisation of $\mathcal{FK}_3$ in ${}^{D(S_3)^*}\mathcal{M}$, we first need to compute the structure of $\mathcal{FK}_3{}^*$ and its coaction, as well as the structure of $\mathcal{FK}_3{}^{\underline{\mathrm{op}}}$. We will start with $\mathcal{FK}_3{}^*$.

Definition 6.2. Let $y_1, y_2, y_3 \in \mathcal{FK}_3{}^*$ be the dual elements of x_1, x_2, x_3 respectively.

With this, we can talk about the structure of $\mathcal{FK}_3{}^*$. But first, we need to build to a lemma relating the duals of the basis elements of $\mathcal{FK}_3$ to the generators of $\mathcal{FK}_3{}^*$.

Lemma 6.3. *We have the following:*

$$\begin{aligned} y_1y_2 &= (x_1x_2)^* - (x_2x_3)^*, & y_1y_1 &= 0,\\ y_2y_1 &= (x_2x_1)^* - (x_1x_3)^*, & y_2y_2 &= 0,\\ y_1y_3 &= (x_1x_3)^*, & y_3y_3 &= 0,\\ y_2y_3 &= (x_2x_3)^*, & y_1y_2y_1 &= (x_1x_2x_1)^*,\\ y_3y_1 &= -(x_1x_2)^*, & y_1y_2y_3 &= (x_1x_2x_3)'*\\ y_3y_2 &= -(x_2x_1)^*, & y_2y_1y_3 &= (x_2x_1x_3)^*,\\ y_1y_2y_1y_3 &= (x_1x_2x_1x_3)^*. \end{aligned}$$

Proof. We will let $\{e_a\}$ be the basis of $\mathcal{FK}_3$ with $\{f_a\}$ being the corresponding dual basis of $\mathcal{FK}_3$ So, we have:

$$y_1y_2 = \sum_a \langle y_1y_2, e_a\rangle f^a.$$

Since $\mathcal{FK}_3$ is a Nichols algebra, evaluations will be zero unless the degrees of the terms agree. So, we have:

$$\begin{aligned} y_1y_2 = {} & \langle y_1y_2, x_1x_2\rangle (x_1x_2)^* + \langle y_1y_2, x_1x_3\rangle (x_1x_3)^* \\ & + \langle y_1y_2, x_2x_1\rangle (x_2x_1)^* + \langle y_1y_2, x_2x_3\rangle (x_2x_3)^*. \end{aligned}$$

Using the definition of the product in the dual algebra and simplifying, we get:

$$y_1y_2 = (x_1x_2)^* - (x_2x_3)^*.$$

The rest of the identities follow similarly

□

We can now derive the relations for the dual algebra

Theorem 6.4. *The dual algebra* ${\mathcal{FK}_3}^*$ *is generated by* y_1, y_2, y_3 *with relations as follows:*

$$\begin{aligned} y_1y_2 + y_2y_3 + y_3y_1 &= 0, \\ y_3y_2 + y_2y_1 + y_1y_3 &= 0, \\ y_1^2 = y_2^2 = y_3^2 &= 0. \end{aligned}$$

Proof. To start, we will show that these relations are satisfied:

$$\begin{aligned} y_1y_2 + y_2y_3 + y_3y_1 &= (x_1x_2)^* - (x_2x_3)^* + (x_2x_3)^* - (x_1x_2)^*, \\ y_3y_2 + y_2y_1 + y_1y_3 &= -(x_2x_1)^* + (x_2x_1)^* - (x_1x_3)^* + (x_1x_3)^*. \end{aligned}$$

Let the algebra generated by y_1, y_2 and y_3 under these relations be A. We wish to prove that ${\mathcal{FK}_3}^* \cong A$. Since ${\mathcal{FK}_3}^*$ satisfies the relations from A, there exists an injective algebra homomorphism from ${\mathcal{FK}_3}^*$ into A. Thus, if we can show that $\dim A \leq \dim {\mathcal{FK}_3}^*$, we are done, as this algebra homomorphism would have to be an isomorphism. We can see this by the fact that $\dim {\mathcal{FK}_3}^* = \dim \mathcal{FK}_3 = \dim A$ since $A \cong \mathcal{FK}_3$. So, we are done. ☐

Next, we will find the coactions on $\mathcal{FK}_3^*$. We will need the following theorem to find them.

Definition 6.5. Given a left comodule $B \in {}^A\mathcal{M}$ with coaction

$$\delta(x) = x_{(1)} \otimes x_{(\infty)},$$

we define $B^* \in \mathcal{M}^A$ as follows:

$$\delta_{B^*}(y) = \sum_i y_i \langle y, (x_i)_{(\infty)} \rangle \otimes S^{-1}((x_i)_{(1)}),$$

where x_i is a basis for B and y_i is the dual basis.

Now we can use this definition to find our comodule structure.

Theorem 6.6. *The comodule structure of* ${\mathcal{FK}_3}^*$ *in* $\mathcal{M}^{D(S_3)^*}$ *is given as follows:*

$$\begin{aligned} \delta(y_1) &= y_1 \otimes (AE_e - AE_\sigma) \\ &\quad + y_2 \otimes (B^{-1}AE_{\tau^{-1}} - B^{-1}AE_{\tau^{-1}\sigma}) \\ &\quad + y_3 \otimes (BAE_\tau - BAE_{\tau\sigma}), \\ \delta(y_2) &= y_1 \otimes (AE_\tau - AE_{\tau^{-1}\sigma}) \\ &\quad + y_2 \otimes (B^{-1}AE_e - B^{-1}AE_{\tau\sigma}) \\ &\quad + y_3 \otimes (BAE_{\tau^{-1}} - BAE_\sigma), \end{aligned}$$

$$\begin{aligned}\delta(y_3) &= y_1 \otimes (AE_{\tau^{-1}} - AE_{\tau\sigma}) \\ &+ y_2 \otimes (B^{-1}AE_{\tau} - B^{-1}AE_{\sigma}) \\ &+ y_3 \otimes (BAE_e - BAE_{\tau^{-1}\sigma}).\end{aligned}$$

Proof. We will simply use Theorem 6.5. By simply plugging in, and noting that the left coaction on a degree n element gives a degree n element in the $\mathcal{FK}_3$ tensor, we can conclude that:

$$\delta(y_k) = \sum_{i,j=1}^{3} y_i \langle y_k, x_j \rangle \otimes S^{-1}(h_{ij}) = \sum_{i=1}^{3} y_i \otimes S^{-1}(h_{ik}).$$

Evaluating gives us:

$$\begin{aligned}\delta(y_1) &= y_1 \otimes S^{-1}(AE_e - AE_{\sigma}) \\ &+ y_2 \otimes S^{-1}(AE_{\tau} - B^{-1}AE_{\tau^{-1}\sigma}) \\ &+ y_3 \otimes S^{-1}(AE_{\tau^{-1}} - BAE_{\tau\sigma}), \\ \delta(y_2) &= y_1 \otimes S^{-1}(B^{-1}AE_{\tau^{-1}} - BAE_{\tau^{-1}\sigma}) \\ &+ y_2 \otimes S^{-1}(B^{-1}AE_e - AE_{\tau\sigma}) \\ &+ y_3 \otimes S^{-1}(B^{-1}AE_{\tau} - B^{-1}AE_{\sigma}), \\ \delta(y_3) &= y_1 \otimes S^{-1}(BAE_{\tau} - B^{-1}AE_{\tau\sigma}) \\ &+ y_2 \otimes S^{-1}(BAE_{\tau^{-1}} - BAE_{\sigma}) \\ &+ y_3 \otimes S^{-1}(BAE_e - AE_{\tau^{-1}\sigma}).\end{aligned}$$

And simplifying gives us:

$$\begin{aligned}\delta(y_1) &= y_1 \otimes (AE_e - AE_{\sigma}) \\ &+ y_2 \otimes (B^{-1}AE_{\tau^{-1}} - B^{-1}AE_{\tau^{-1}\sigma}) \\ &+ y_3 \otimes (BAE_{\tau} - BAE_{\tau\sigma}), \\ \delta(y_2) &= y_1 \otimes (AE_{\tau} - AE_{\tau^{-1}\sigma}) \\ &+ y_2 \otimes (B^{-1}AE_e - B^{-1}AE_{\tau\sigma}) \\ &+ y_3 \otimes (BAE_{\tau^{-1}} - BAE_{\sigma}), \\ \delta(y_3) &= y_1 \otimes (AE_{\tau^{-1}} - AE_{\tau\sigma}) \\ &+ y_2 \otimes (B^{-1}AE_{\tau} - B^{-1}AE_{\sigma}) \\ &+ y_3 \otimes (BAE_e - BAE_{\tau^{-1}\sigma}).\end{aligned}$$

□

Now we need to find the structure of $\mathcal{FK}_3^{\text{op}}$. To do this, first let us recall the definition of the braided opposite braided group.

Definition 6.7 (cf. [2] 2.3)**.** If B is a braided group in C, with invertible antipode, then $B^{\underline{\text{op}}}$ has the same coalgebra structure but with braided opposite product and antipode given by:

$$\cdot_{\underline{\text{op}}} = \cdot \circ \Psi_{B,B}^{-1}, \quad \bar{S} = \underline{S}^{-1}.$$

Using this we can conclude the following:

Theorem 6.8. *The algebra $\mathcal{FK}_3^{\underline{op}}$ has a presentation with generators x_1, x_2 and x_3 with relations:*

$$x_1 \cdot_{\underline{op}} x_2 + x_2 \cdot_{\underline{op}} x_3 + x_3 \cdot_{\underline{op}} x_1 = 0,$$
$$x_3 \cdot_{\underline{op}} x_2 + x_2 \cdot_{\underline{op}} x_1 + x_1 \cdot_{\underline{op}} x_3 = 0,$$
$$x_1 \cdot_{\underline{op}} x_1 = x_2 \cdot_{\underline{op}} x_1 = x_3 \cdot_{\underline{op}} x_3 = 0.$$

Proof. To start, we will show that these relations are satisfied:

$$x_1 \cdot_{\underline{\text{op}}} x_2 + x_2 \cdot_{\underline{\text{op}}} x_3 + x_3 \cdot_{\underline{\text{op}}} x_1 = -x_3x_1 - x_1x_2 - x_2x_3 = 0,$$
$$x_3 \cdot_{\underline{\text{op}}} x_2 + x_2 \cdot_{\underline{\text{op}}} x_1 + x_1 \cdot_{\underline{\text{op}}} x_3 = -x_1x_3 - x_3x_2 - x_2x_1 = 0,$$
$$x_1 \cdot_{\underline{\text{op}}} x_1 = -x_1^2 = 0,$$
$$x_2 \cdot_{\underline{\text{op}}} x_2 = -x_2^2 = 0,$$
$$x_3 \cdot_{\underline{\text{op}}} x_3 = -x_3^2 = 0.$$

Now, let the algebra generated by x_1, x_2, and x_3 under these relations be A. We wish to prove that $\mathcal{FK}_3 \cong A$. Since $\mathcal{FK}_3{}^*$ satisfies the relations from A, there exists an injective algebra homomorphism from $\mathcal{FK}_3{}^*$ into A. Thus, if we can show that $\dim A \leq \dim \mathcal{FK}_3{}^*$, we are done, as this algebra homomorphism would have to be an isomorphism. We can see this by the fact that $\dim \mathcal{FK}_3{}^* = \dim \mathcal{FK}_3 = \dim A$ since $A \cong \mathcal{FK}_3$. So, we are done.

□

We can now find the product structure of the codouble bosonisation.

Theorem 6.9. *The codouble bosonisation $\mathcal{FK}_3^{\underline{\text{op}}} {>\!\!\!\triangleleft} D(S_3)^* {\triangleright\!\!\!<} \mathcal{FK}_3^*$ has the following presentation by generators and relations:*

Generators:

$$x_1, x_2, x_3, A, B, E_g, y_1, y_2, y_3.$$

Relations:

$$x_1^2 = x_2^2 = x_3^2 = 0, \quad x_1x_3 + x_3x_2 + x_2x_1 = 0, \quad x_3x_1 + x_2x_3 + x_1x_2 = 0,$$

$$y_1^2 = y_2^2 = y_3^2 = 0, \quad y_1y_3 + y_3y_2 + y_2y_1 = 0, \quad y_3y_1 + y_2y_3 + y_1y_2 = 0,$$

$$A^2 = B^3 = 1, \quad E_gE_h = \delta_{g,h}E_g, \quad AB = BA^{-1},$$
$$AE_g = E_gA, \quad BE_g = E_gB, \quad \sum_{g\in S_3} E_g = 1,$$

$$\begin{aligned}
y_1A &= -A(E_ey_1 + E_\sigma y_1 + E_{\tau^{-1}}y_3 + E_{\tau^{-1}\sigma}y_3 + E_\tau y_2 + E_{\tau\sigma}y_2),\\
y_2A &= -A(E_ey_3 + E_\sigma y_3 + E_{\tau^{-1}}y_2 + E_{\tau^{-1}\sigma}y_2 + E_\tau y_1 + E_{\tau\sigma}y_1),\\
y_3A &= -A(E_ey_2 + E_\sigma y_2 + E_{\tau^{-1}}y_1 + E_{\tau^{-1}\sigma}y_1 + E_\tau y_3 + E_{\tau\sigma}y_3),\\
y_1B &= B(E_ey_3 + E_\tau y_3 + E_{\tau^{-1}}y_3 + E_\sigma y_2 + E_{\tau\sigma}y_2 + E_{\tau^{-1}\sigma}y_2),\\
y_2B &= B(E_ey_1 + E_\tau y_1 + E_{\tau^{-1}}y_1 + E_\sigma y_3 + E_{\tau\sigma}y_3 + E_{\tau^{-1}\sigma}y_3),\\
y_3B &= B(E_ey_2 + E_\tau y_2 + E_{\tau^{-1}}y_2 + E_\sigma y_1 + E_{\tau\sigma}y_1 + E_{\tau^{-1}\sigma}y_1),
\end{aligned}$$

$$Ax_1 = x_1A, \quad Ax_2 = x_2BA, \quad Ax_3 = x_3B^{-1}A,$$
$$E_gx_1 = x_1E_{\sigma g}, \quad E_gx_2 = x_2E_{\tau^{-1}\sigma g}, \quad E_gx_3 = x_3E_{\tau\sigma g},$$
$$y_iE_g = E_gy_i, \quad Bx_i = x_iB^{-1}, \quad y_ix_i = x_iy_i.$$

Proof. To start, we will define our generators:

$$x_1 = x_1 \bowtie 1 \bowtie 1, \quad x_2 = x_2 \bowtie 1 \bowtie 1, \quad x_3 = x_3 \bowtie 1 \bowtie 1,$$
$$A = 1 \bowtie A \bowtie 1, \quad B = 1 \bowtie B \bowtie 1, \quad E_g = 1 \bowtie E_g \bowtie 1,$$
$$y_1 = 1 \bowtie 1 \bowtie y_1, \quad y_2 = 1 \bowtie 1 \bowtie y_2, \quad y_3 = 1 \bowtie 1 \bowtie y_3.$$

It is clear that these generators generate the entire algebra. Now, to find the relations, it is sufficient to find the relations for each individual tensor, and the cross-relations. The relations for each individual tensor are as follows:

$$x_1^2 = x_2^2 = x_3^2 = 0, \quad x_1x_3 + x_3x_2 + x_2x_1 = 0, \quad x_3x_1 + x_2x_3 + x_1x_2 = 0,$$
$$y_1^2 = y_2^2 = y_3^2 = 0, \quad y_1y_3 + y_3y_2 + y_2y_1 = 0, \quad y_3y_1 + y_2y_3 + y_1y_2 = 0,$$

$$A^2 = B^3 = 1, \quad E_gE_h = \delta_{g,h}E_g, \quad AB = BA^{-1},$$
$$AE_g = E_gA, \quad BE_g = E_gB, \quad \sum_{g\in S_3} E_g = 1.$$

These come directly from the relations for the subalgebras. Now, we will use the formula to derive cross relations. To start, we will look at y_1A.

$$\begin{aligned}
y_1 A &= (1 \bowtie 1 \bowtie y_1)(1 \bowtie A \bowtie 1) \\
&= 1 \bowtie A_{(1)} \bowtie y_1^{(0)} \mathcal{R}(y_1^{(1)}, A_{(2)}) \\
&= 1 \bowtie (\chi_{\sigma,e} + \chi_{\sigma,\sigma}) \bowtie y_1^{(0)} \mathcal{R}(y_1^{(1)}, A) \\
&+ 1 \bowtie (\chi_{\sigma,\tau} + \chi_{\sigma,\tau^{-1}\sigma}) \bowtie y_1^{(0)} \mathcal{R}(y_1^{(1)}, BA) \\
&+ 1 \bowtie (\chi_{\sigma,\tau^{-1}} + \chi_{\sigma,\tau\sigma}) \bowtie y_1^{(0)} \mathcal{R}(y_1^{(1)}, B^{-1}A) \\
&= -1 \bowtie (\chi_{\sigma,e} + \chi_{\sigma,\sigma}) \bowtie y_1 \\
&- 1 \bowtie (\chi_{\sigma,\tau} + \chi_{\sigma,\tau^{-1}\sigma}) \bowtie y_2 \\
&- 1 \bowtie (\chi_{\sigma,\tau^{-1}} + \chi_{\sigma,\tau\sigma}) \bowtie y_3 \\
&= -AE_e y_1 - AE_\sigma y_1 - AE_\tau y_2 - AE_{\tau^{-1}\sigma} y_2 - AE_{\tau^{-1}} y_3 - AE_{\tau\sigma} y_3.
\end{aligned}$$

The other $y_i A$ and $y_i B$ cross relations follow similarly:

$$\begin{aligned}
y_2 A &= -A(E_e y_3 + E_\sigma y_3 + E_{\tau^{-1}} y_2 + E_{\tau^{-1}\sigma} y_2 + E_\tau y_1 + E_{\tau\sigma} y_1), \\
y_3 A &= -A(E_e y_2 + E_\sigma y_2 + E_{\tau^{-1}} y_1 + E_{\tau^{-1}\sigma} y_1 + E_\tau y_3 + E_{\tau\sigma} y_3), \\
y_1 B &= B(E_e y_3 + E_\tau y_3 + E_{\tau^{-1}} y_3 + E_\sigma y_2 + E_{\tau\sigma} y_2 + E_{\tau^{-1}\sigma} y_2), \\
y_2 B &= B(E_e y_1 + E_\tau y_1 + E_{\tau^{-1}} y_1 + E_\sigma y_3 + E_{\tau\sigma} y_3 + E_{\tau^{-1}\sigma} y_3), \\
y_3 B &= B(E_e y_2 + E_\tau y_2 + E_{\tau^{-1}} y_2 + E_\sigma y_1 + E_{\tau\sigma} y_1 + E_{\tau^{-1}\sigma} y_1).
\end{aligned}$$

The $y_i E_g$ cross relation is a bit different.

$$\begin{aligned}
y_i E_g &= (1 \bowtie 1 \bowtie y_i)(1 \bowtie E_g \bowtie 1) \\
&= 1 \bowtie E_{g_{(1)}} \bowtie y_i^{(0)} \mathcal{R}(y_i^{(1)}, E_{g_{(2)}}) \\
&= \sum_{h \in S_3} 1 \bowtie E_{gh^{-1}} \bowtie y_i^{(0)} \mathcal{R}(y_i^{(1)}, E_h) \\
&= 1 \bowtie E_g \bowtie y_i^{(0)} \mathcal{R}(y_i^{(1)}, 1) \\
&= 1 \bowtie E_g \bowtie y_i = E_g y_i.
\end{aligned}$$

Now we need to find the cross relations between the x_i's and A, B, and E_g.

$$\begin{aligned}
A x_1 &= (1 \bowtie A \bowtie 1)(x_1 \bowtie 1 \bowtie 1) \\
&= x_1^{(\overline{\infty})} \bowtie A_{(2)} \bowtie \mathcal{R}(S(A_{(1)}), x_1^{(\bar{1})}) \\
&= x_1 \bowtie A_{(2)} \bowtie \mathcal{R}(S(A_{(1)}), \chi_{\sigma,e})
\end{aligned}$$

$$
\begin{aligned}
&= x_1 \bowtie A \bowtie \mathcal{R}(S(\chi_{\sigma,e} + \chi_{\sigma,\sigma}), \chi_{\sigma,e}) \\
&+ x_1 \bowtie BA \bowtie \mathcal{R}(S(\chi_{\sigma,\tau^{-1}} + \chi_{\sigma,\tau^{-1}\sigma}), \chi_{\sigma,e}) \\
&+ x_1 \bowtie B^{-1}A \bowtie \mathcal{R}(S(\chi_{\sigma,\tau} + \chi_{\sigma,\tau\sigma}), \chi_{\sigma,e}) \\
&= x_1 \bowtie A \bowtie 1 = x_1 A.
\end{aligned}
$$

Similarly we can obtain the other Ax_i and Bx_i cross relations:

$$
Ax_2 = x_2 BA, \quad Ax_3 = x_3 B^{-1}A, \quad Bx_i = x_i B^{-1}.
$$

Now we will find the $E_g x_1$ cross relation:

$$
\begin{aligned}
E_g x_1 &= (1 \bowtie E_g \bowtie 1)(x_1 \bowtie 1 \bowtie 1) \\
&= x_1^{(\overline{\infty})} \bowtie E_{g_{(2)}} \bowtie \mathcal{R}(S(E_{g_{(1)}}), x_1^{(\overline{1})}) \\
&= \sum_{h \in S_3} x_1^{(\infty)} \bowtie E_h \bowtie \mathcal{R}(S(E_{gh^{-1}}), x_1^{(\overline{1})}) \\
&= \sum_{h \in S_3} x_1 \bowtie E_h \bowtie \mathcal{R}(E_{hg^{-1}}, \chi_{\sigma,e}) \\
&= x_1 \bowtie E_{\sigma g} \bowtie 1 = x_1 E_{\sigma g}.
\end{aligned}
$$

The other $E_g x_i$ cross relations follow similarly:

$$
E_g x_2 = x_2 E_{\tau^{-1}\sigma g}, \quad E_g x_3 = x_3 E_{\tau\sigma g}.
$$

Finally, we have only the $y_i x_i$ cross relations left:

$$
\begin{aligned}
y_i x_i &= (1 \bowtie 1 \bowtie y_i)(x_i \bowtie 1 \bowtie 1) \\
&= x_i^{(\overline{\infty})} \bowtie 1 \bowtie y_i^{(\overline{0})} \mathcal{R}(y_i^{(\overline{1})}, 1) \mathcal{R}(1, x_i^{(\overline{1})}) \\
&= x_i^{(\overline{\infty})} \bowtie 1 \bowtie y_i^{(\overline{0})} \epsilon(y_i^{(\overline{1})}) \epsilon(x_i^{(\overline{1})}) \\
&= x_i \bowtie 1 \bowtie y_i = x_i y_i.
\end{aligned}
$$

With that, our presentation is complete.

□

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