Curvature on Riemannian Manifolds

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Manifolds

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Earth (2-dimensional manifold)

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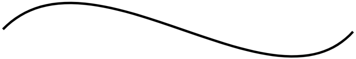

Curve (1-dimensional manifold)

Manifolds

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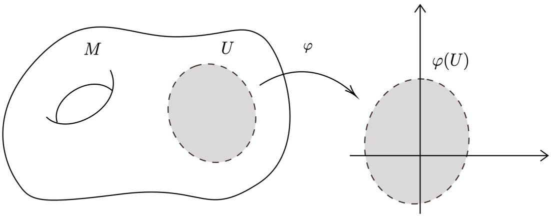

Example chart (U, φ)

Manifolds

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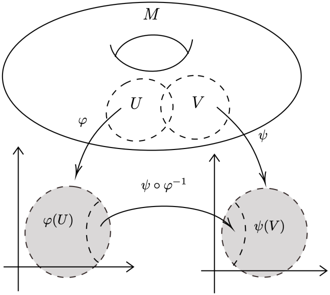

Compatible charts (U, φ) and (V, ψ)

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Let M be a set. An atlas $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$ is a collection of pairwise compatible charts that cover M.

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Definition (Maximal Atlas)

We define an atlas as a maximal atlas if it contains all possible charts compatible with its existing charts.

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2. For any distinct points p and q on M, there exists charts $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) such that $p \in U_\alpha$, $q \in U_\beta$ and $U_\alpha \cap U_\beta = \emptyset$.

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Definition (Smooth Manifold)

An n-dimensional manifold M is **smooth** if for any two charts (U, φ) and (V, ψ), the map $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ and its inverse are smooth.

Smooth Functions

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Definition (Smooth Function on a Manifold)

A function $f : M \to \mathbb{R}^k$ is smooth if, for every chart (U, φ) on M, the function $f \circ \varphi^{-1}$ is smooth.

- $C^\infty(M)$ is the set of smooth functions on a manifold M.

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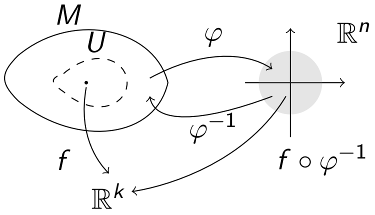

Diagram of $f \circ \varphi^{-1}$

Smooth Functions

Definition (Smooth Function from M to N)

A map $F : M \to N$ is smooth if, for every point $p \in M$, there exist charts (U, φ) around p and (V, ψ) around $F(p)$ with $F(U) \subset V$, such that $\psi \circ F \circ \varphi^{-1}$ is smooth.

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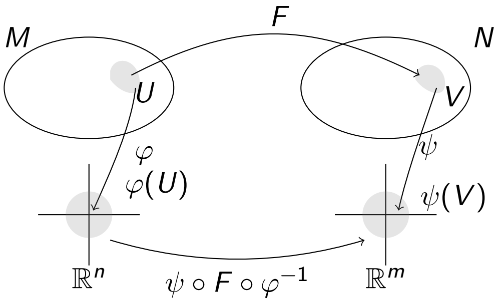

Diagram of $\psi \circ F \circ \varphi^{-1}$

Tangent Spaces

Definition (Tangent Vector)

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A map $v_\gamma : C^\infty(M) \to \mathbb{R}$ given by $v_\gamma f = \partial_t|_{t=0}(f \circ \gamma)$, where $\gamma(t)$ is a curve on M with $\gamma(0) = p$, is called a tangent vector at p.

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- We can think of v_γ as a directional derivative at p.
- v_γ obeys the product rule: $v_\gamma(fg) = v_\gamma(f)g + fv_\gamma(g)$.

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Definition (Tangent Space)

The tangent space at $p \in M$ denoted as T_pM is the set of all tangent vectors at a point p.

Tangent Spaces

Proposition

Let $(x_1, \ldots, x_n)$ be a local coordinate system on M near p. Then the tangent vectors $\partial_1, \ldots, \partial_n$, also denoted as $\partial_{x_1}, \ldots, \partial_{x_n}$, form a basis of the tangent space at p.

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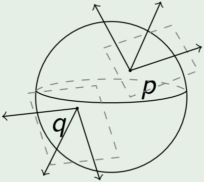
Example

Let S^2 denote the unit sphere in $\mathbb{R}^3$.



T_pS^2 and T_qS^2 are two tangent spaces on S^2.

Vector Fields

Definition

Vector Fields

Definition

A family $(X_p)_{p \in M}$ with $X_p \in T_pM$ is a vector field if for every $f \in C^\infty(M)$, the map $p \mapsto X_p(f)$ is smooth.

- Equivalently, around each point p on M, choose local coordinates $x_1, \ldots, x_n$, then the vector field X can be written in components as $X(x) = a_1(x)\partial_1 + \cdots + a_n(x)\partial_n$, where each coefficient function $a_j : \mathbb{R}^n \to \mathbb{R}$ is smooth.

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- Informally, a vector field smoothly assigns a tangent vector to each point.
- We denote the space of all smooth vector fields on M by $\mathfrak{X}(M)$.

Riemannian Metric

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$$g_p : T_pM \times T_pM \to \mathbb{R}$$

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- Define curve length using g: $L(\gamma) = \int_a^b \sqrt{\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle_g} dt$.

Riemannian Manifolds

Definition

A manifold M equipped with a Riemannian metric g is a Riemannian manifold.

Now, let (M, g) (or simply M) denote a Riemannian manifold.

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Theorem

Every smooth manifold can be equipped with a Riemannian metric.

Example

In $\mathbb{R}^2$, the standard metric is $g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Riemannian Manifolds

Definition (Hyperbolic Plane)

The hyperbolic plane is given by $\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ with the metric $g = \begin{pmatrix} \frac{1}{y^2} & 0 \\ 0 & \frac{1}{y^2} \end{pmatrix}$.

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Note that this metric goes to zero as $y \to +\infty$ and diverges as $y \to 0^+$.

Connections

Definition (Connection)

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A connection allows us to compare tangent vectors at different points on a manifold.

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Definition (Levi-Civita Connection)

Given a Riemannian manifold (M, g) and vector fields X, Y and Z on M, the Levi-Civita connection is a connection ∇ that satisfies the following additional properties:

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Theorem (Levi-Civita Connection)

Every Riemannian manifold has a unique Levi-Civita connection.

The Levi-Civita connection is the most natural connection for any Riemannian manifold.

Christoffel Symbols

The Christoffel symbols give us the concrete terms for any connection.

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Definition (Christoffel Symbols)

Given local coordinates $x_1, x_2, \ldots, x_n$ and coordinate basis vector fields $\partial_1, \partial_2, \ldots \partial_n$, the Christoffel symbols Γ are defined as

$$\nabla_{\partial_j} \partial_k = \sum_{i=1}^{n} \Gamma^i_{jk} \partial_i$$

where ∇ is the Levi-Civita connection.

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The Christoffel symbols describe how coordinate basis vectors ∂_i change as we compare two different tangent vectors on a manifold.

Christoffel Symbols

Lemma (Christoffel Symbols)

If ∇ is the Levi-Civita connection of a Riemannian manifold (M, g), then the Christoffel symbols are given by

$$\Gamma_{ij}^{k} = \sum_{\ell=1}^{n} \left(\frac{1}{2} g^{k\ell} (\partial_i g_{\ell j} + \partial_j g_{i\ell} - \partial_l g_{ji}) \right)$$

where g^{ij} is the inverse matrix element of g_{ij}.

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This shows how Christoffel symbols can be written directly in terms of the metric.

Christoffel Symbols on the Hyperbolic Plane

Example

On the hyperbolic plane $\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ with metric

$$g = \begin{pmatrix} \frac{1}{y^2} & 0 \\ 0 & \frac{1}{y^2} \end{pmatrix},$$

the Christoffel symbols are

$$\Gamma^x_{xy} = \Gamma^x_{yx} = -\frac{1}{y}, \quad \Gamma^y_{xx} = \frac{1}{y}, \quad \Gamma^y_{yy} = -\frac{1}{y}$$

with the rest $\Gamma^x_{xx}, \Gamma^x_{yy}, \Gamma^y_{xy}, \Gamma^y_{yx}$ equal 0.

Geodesics

From here on let ∇ denote the Levi-Civita connection.

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Definition (Geodesic)

A curve $\gamma : (a, b) \to M$ is a geodesic if $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$.

- If γ is a geodesic, then $\langle \dot{\gamma}, \dot{\gamma} \rangle_g = C$ where C is a constant. In other words, a geodesic has constant speed.

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Proposition (Geodesics)

If $p, q \in M$ are close enough, then a geodesic between p and q represents the path of minimal length.

Recall that the length of a curve $L(\gamma) = \int_a^b \sqrt{\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle_{g_{\gamma(t)}}} dt$.

Geodesics on the Hyperbolic Plane

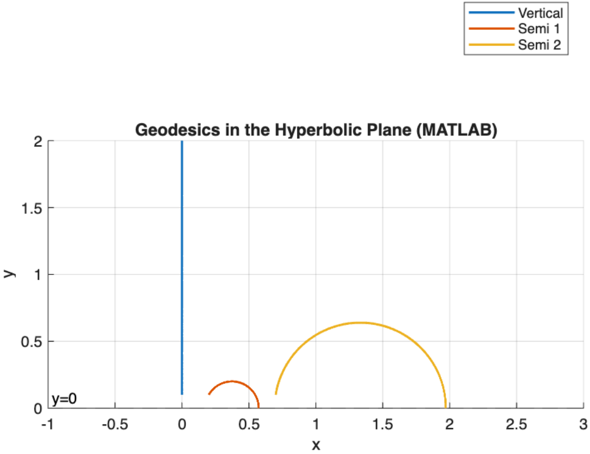
Example

In the hyperbolic plane, the geodesic equation is given by

$$\begin{cases} \ddot{x} - \frac{2}{y}\dot{x}\dot{y} = 0 \\ \ddot{y} + \frac{1}{y}\dot{x}^2 - \frac{1}{y}\dot{y}^2 = 0 \end{cases}.$$

Geodesics on the hyperbolic plane are either semicircles that are perpendicular to the x-axis or vertical rays with an open end.

Geodesics on the Hyperbolic Plane



Vertical and semicircle geodesics

Riemann Curvature Tensor

Definition (Riemann Curvature Tensor)

The Riemann curvature tensor

$$R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$$

is defined by

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In essence, $R(X, Y)$ measures how much ∇_X and ∇_Y fail to commute.

The third term ensures that $R(X, Y)Z$ has certain nice properties.

Riemann Curvature Tensor (contd.)

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- $R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0$,
- and R is $C^\infty(M)$ linear in each argument.

Remark: From the fourth property, R is uniquely determined by its values on the basis vectors $R(\partial_i, \partial_j)\partial_k$.

Riemann Curvature Tensor (contd.)

Example

Riemann Curvature Tensor (contd.)

Example

In the hyperbolic plane $\mathbb{H}^2$ we can compute $R(\partial_i, \partial_j)\partial_k$ for each i, j, k:

$$R(\partial_1, \partial_2)\partial_1 = \frac{\partial_2}{y^2}, \quad R(\partial_2, \partial_1)\partial_1 = -\frac{\partial_2}{y^2},$$

$$R(\partial_1, \partial_2)\partial_2 = -\frac{\partial_1}{y^2}, \quad R(\partial_2, \partial_1)\partial_2 = \frac{\partial_1}{y^2},$$

and the rest are 0.

Sectional Curvature

Definition (Sectional Curvature)

Let $p \in M$. For two linearly independent $u, v \in T_pM$, the sectional curvature is

$$K(u, v) = \frac{\langle R(u, v)v, u\rangle_g}{N},$$

where $N := \langle u, u\rangle_g \langle v, v\rangle_g - \langle u, v\rangle_g^2$.

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We divide by N so that $K(u, v)$ depends only on the directions of u and v.

Sectional Curvature (contd.)

Lemma

The sectional curvature $K(u, v) = \frac{\langle R(u,v)v,u \rangle}{N}$ depends only on the plane spanned by u, v.

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In hyperbolic space, all tangent spaces have dimension 2, so any two vectors have the same span. Thus, $K(u, v) = K(\partial_1, \partial_2)$ at each point.

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Theorem (Sectional Curvature of $\mathbb{H}^2$)

Hyperbolic space $\mathbb{H}^2$ has sectional curvature $K(\partial_1, \partial_2) = -1$ at each point.

Jacobi Fields

Jacobi fields bridge the gap between geodesics and curvature.

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Definition (Jacobi Field)

Let I be an interval and $\gamma : I \to M$ a geodesic. A vector field $J(t)$ on $\gamma(t)$ (meaning $J(t) \in T_{\gamma(t)}M$) satisfying

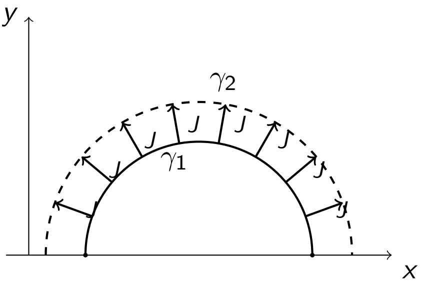
$$\nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}}J(t) + R(J(t), \dot{\gamma}(t))\dot{\gamma}(t) = 0$$

is called a Jacobi field along γ.

Jacobi Fields (contd.)

Lemma (Jacobi Fields)

Let $\gamma(s,t)$ be a smooth family of geodesics dependent on the parameter s. Then $J(t) := (\partial_s \gamma(s,t))\Big|_{s=0}$ is a Jacobi field along $\gamma(0,t)$.



A Jacobi field deforming a geodesic in $\mathbb{H}^2$

Jacobi Fields (contd.)

Let $\gamma(t)$ be a periodic geodesic. That is, there exists $T > 0$ such that $\gamma(t) = \gamma(t + T)$ for all t.

Jacobi Fields (contd.)

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A Jacobi Field $J(t)$ along $\gamma(t)$ is called periodic if $J(t) = J(t + T)$ and $(\nabla_{\dot{\gamma}} J)(t) = (\nabla_{\dot{\gamma}} J)(t + T)$ for all t.

Jacobi Fields (contd.)

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Theorem (Periodic Jacobi Fields with $K < 0$)

Suppose (M, g) has negative sectional curvature. Then any periodic Jacobi field is 0.

Acknowledgements

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Thank you all for listening!