Classifying Euclidean Spaces via Algebraic Topology

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What is topology?

Topology studies spaces, which are a pair (X, τ) of a set X and a topology τ on that set, a subset its power set elements of which are called open sets, which satisfies the following axioms:

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Then the maps in topology are those that preserve these open sets, which are continuous maps, formally defined as those where the pre-image of any open set is open.

Topology on $\mathbb{R}^n$

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In $\mathbb{R}^n$, the standard topology, called the Euclidean topology, is composed of unions of neighborhoods of points, given by open balls $\{x \in \mathbb{R}^n : |x - x_0| < r\}$, for some point x_0 and radius r.

Definition: homeomorphism

Definition

A homeomorphism between space X and space Y is denoted as $X \cong Y$, if there is a bijective function $f : X \to Y$ such that both f and f^{-1} are continuous.

Figure: Mug to Donut.[1]

[1] Picture from http://www.segerman.org/images/topology_joke.jpg

Main theorem

Theorem

$\mathbb{R}^m$ is homeomorphic to $\mathbb{R}^n$ if and only if $m = n$.

Sketch of Proof.

For $m = n$, the identity map $\mathrm{id} : \mathbb{R}^m \to \mathbb{R}^n$ is a homeomorphism. The rest of this presentation will thus be devoted to the case when $m \neq n$. □

How to tell spaces apart

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To prove two spaces not homeomorphic, examining all possible maps to prove none are homeomorphisms is infeasible.

There is a plethora of tools though called homeomorphism invariants—properties of spaces unchanged by homeomorphisms, which prove two spaces not homeomorphic if they differ for them.

Invariant 1: path-connectedness

Definition

A space X is path-connected if for any $u, v \in X$, there exists a continuous function $f : I \to X$, where I is the unit interval $[0, 1]$ such that $f(0) = u$ and $f(1) = v$.

[2] Images both created in Desmos.

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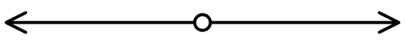
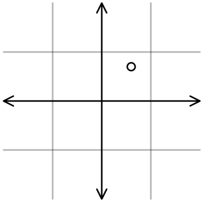


$\mathbb{R}$ with the origin removed.



$\mathbb{R}^2$ with a point removed.[2]

[2] Images both created in Desmos.

Proof of main theorem for $m = 1$, $n = 2$

Theorem

$\mathbb{R} \not\cong \mathbb{R}^2$.

Sketch of Proof.

Suppose there exists a homeomorphism $f : \mathbb{R} \to \mathbb{R}^2$ with $f(0) = (a, b)$.

Removing these points gives $\mathbb{R} - \{0\} \cong \mathbb{R}^2 - \{(a, b)\}$.

But $\mathbb{R} - \{0\}$ is not path-connected, while $\mathbb{R}^2 - \{(a, b)\}$ is. Contradiction! $\square$

Invariant 2: homotopy

Definition

Let f, g be continuous maps from space X to space Y. Denote the unit interval $[0, 1] \in \mathbb{R}$ with I. We say that f is homotopic to g, otherwise denoted as $f \simeq g$, if there exists a continuous function $F : X \times I \to Y$ such that for all $x \in X$, $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$. We call F a homotopy between f and g.

Invariant 2: homotopy

Definition

Spaces X and Y are homotopy equivalent if there is a pair of continuous maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f \simeq \mathrm{id}_X$, the identity map for X, and $f \circ g \simeq \mathrm{id}_Y$, the identity map for Y.

Theorem

If $X \cong Y$ (homeomorphic), then $X \simeq Y$ (homotopic).

How to use homotopy to tell spaces apart?

We use fundamental groups $\pi_1(X, x)$, groups with elements being path-homotopy classes of loops in space X based at point x and operation being path concatenation.

Examples:

Circle S^1: $\pi_1(S^1) \cong \mathbb{Z}$ (counts winding).

Sphere S^2: $\pi_1(S^2) \cong 0$ (all loops contractible).

Based loops

Definition

A path in space X is a continuous map $f : I \to X$.
A loop is a path where $f(0) = f(1)$.
A based loop at point x satisfies $f(0) = f(1) = x$.

Path-homotopy

Definition

Two paths f, f_1 in space X are path-homotopic if they share endpoints and there exists a homotopy for f and f_1 that preserves the endpoints' locations throughout.

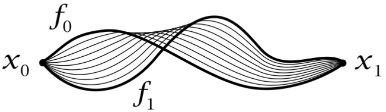


Figure: Path-Homotopy [2].

Path concatenation

Definition

Path concatenation, denoted as $*$, merges two paths f, g into one if $f(1) = g(0)$. Formally,

$$(f * g)(t) = \begin{cases} f(2t), & t \in [0, \frac{1}{2}] \\ g(2t-1), & t \in (\frac{1}{2}, 1]. \end{cases}$$

Definition

We may analogously operate on equivalence classes. Define $[f] * [g] = [f * g]$.

Why is fundamental group a group?

Proposition

(Well-definedness) *Let paths* f_0, f_1, g_0, g_1 *satisfy* $f_0 \simeq f_1$ *and* $g_0 \simeq g_1$*, then* $[f_0 * g_0] = [f_1 * g_1]$.

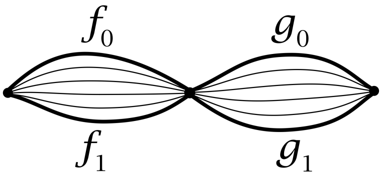


Figure: Well-Definedness [2].

Why is fundamental group a group?

Proposition

(**Associativity**) *For paths f, g, h such that $f(1) = g(0)$, $g(1) = h(0)$, we have $([f] * [g]) * [h] = [f] * ([g] * [h])$.*

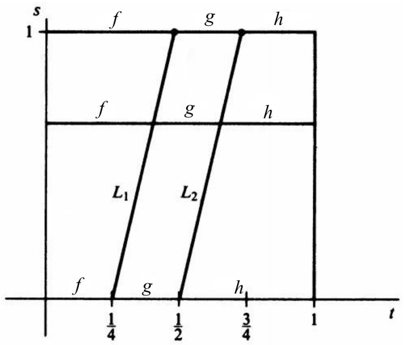


Figure: Associativity [1].

Why is fundamental group a group?

Proposition

*(**Identity**) The constant loop acts as identity to the group.*

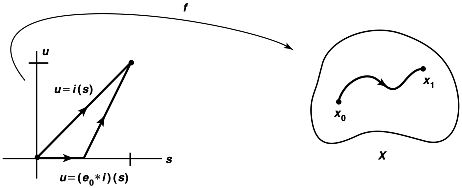


Figure: A path composed with the identity element [3].

Why is fundamental group a group?

Proposition

*(**Inverse**) For all loops $f(t)$, the reverse loop $f(1-t)$ provides inverse.*

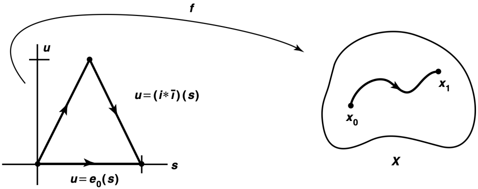


Figure: A path composed with its inverse [3].

Fundamental group of a circle

Denote S^1 as the circle and S^2 as the sphere.

1. $\pi_1(S^1) = \mathbb{Z}$.

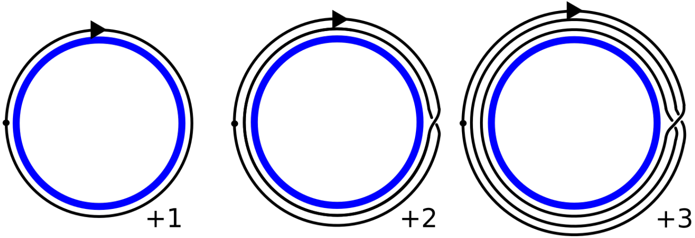


Figure: Loops in a circle.[3]

[3] Picture from https://commons.wikimedia.org/wiki/File:Fundamental_group_of_the_circle.svg

Fundamental group of a sphere

2. $\pi_1(S^2) = 0$.

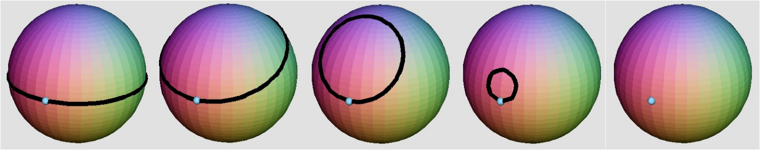


Figure: Loops in a 2-sphere.[4]

[4] Picture from https://en.wikipedia.org/wiki/File:P1S2all.jpg

Fundamental group facts

Proposition

Homotopy equivalent spaces have isomorphic fundamental groups.

Proposition

If two spaces have different fundamental groups, then they are not homeomorphic.

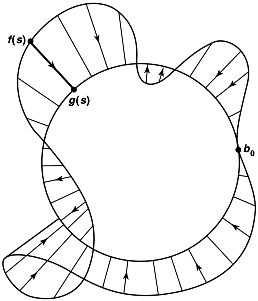


Figure: Correspondence of equivalence classes for $\mathbb{R}^2 - \{(0,0)\}$ and S^1[3].

Proof of main theorem for $m = 2$, $n = 3$

Theorem

$\mathbb{R}^2 \not\cong \mathbb{R}^3$

Sketch of Proof.

Proving by contradiction, suppose that there exists some homeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^3$.

Then, let $f((0,0)) = (a,b,c)$. Thus, $\mathbb{R}^2 - \{(0,0)\} \cong \mathbb{R}^3 - \{(a,b,c)\}$.

Note $\mathbb{R}^2 - \{(0,0)\} \simeq S^1$ and $\mathbb{R}^3 - \{(a,b,c)\} \simeq S^2$. Thus, $S^1 \simeq S^2$.

However, Since $\pi_1(S^1) \cong \mathbb{Z} \not\cong \{e\} \cong \pi_1(S^2)$, $S^1 \not\simeq S^2$, contradiction!

□

Generalizing our results

Intuitively, we need a way to detect higher dimensional holes, so we need stronger tools.

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Definition

Define an *n-sphere*, denoted as S^n, to be the set

$$S^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}.$$

Higher homotopy groups

Instead of using loops, or S^1's, we may compute higher homotopy groups, $\pi_n(X)$, and use S^n to measure differences of spaces.

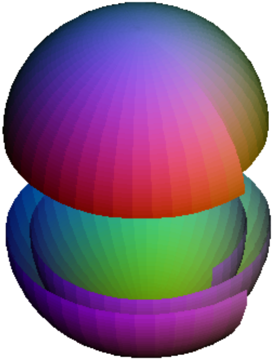

Figure: $\pi_2(S^2) = \mathbb{Z}$.[5]

[5] Picture from https://upload.wikimedia.org/wikipedia/commons/5/50/Sphere_wrapped_round_itself.png

Homology groups

Another way to detect higher dimensional holes is to use homology, a weaker homotopy invariant.

Homology groups

Another way to detect higher dimensional holes is to use homology, a weaker homotopy invariant.

Similar to higher homotopy groups, each space has their higher **homology groups** $H_i(X)$ for $i \in \mathbb{Z}_0^+$.

Homology is way easier to compute.

	S^0	S^1	S^2	S^3	S^4	S^5	S^6	S^7	S^8
π_1	0	$\mathbb{Z}$	0	0	0	0	0	0	0
π_2	0	0	$\mathbb{Z}$	0	0	0	0	0	0
π_3	0	0	$\mathbb{Z}$	$\mathbb{Z}$	0	0	0	0	0
π_4	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	0
π_5	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
π_6	0	0	$\mathbb{Z}_{12}$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
π_7	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
π_8	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
π_9	0	0	$\mathbb{Z}_3$	$\mathbb{Z}_3$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
π_{10}	0	0	$\mathbb{Z}_{15}$	$\mathbb{Z}_{15}$	$\mathbb{Z}_{24}\times\mathbb{Z}_3$	$\mathbb{Z}_2$	0	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$
π_{11}	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	$\mathbb{Z}_{24}$
π_{12}	0	0	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_{30}$	$\mathbb{Z}_2$	0	0
π_{13}	0	0	$\mathbb{Z}_{12}\times\mathbb{Z}_2$	$\mathbb{Z}_{12}\times\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2$	$\mathbb{Z}_{60}$	$\mathbb{Z}_2$	0

Figure: Homotopy groups $\pi_i(S^n)$ of spheres.[6]

Table: Homology groups $H_i(S^n)$ of spheres.

	S^1	S^2	S^3	S^4	$\cdots$
H_0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\cdots$
H_1	$\mathbb{Z}$	0	0	0	$\cdots$
H_2	0	$\mathbb{Z}$	0	0	$\cdots$
H_3	0	0	$\mathbb{Z}$	0	$\cdots$
H_4	0	0	0	$\mathbb{Z}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

[6] Figure from https://www.semanticscholar.org/paper/Homotopy-Type-Theory:-Univalent-Foundations-of-Program/bdd73e7047e4dddcec4757146d014b45566c251b

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Thank you for your attention!!!