ON THE ADDITIVE MONOIDS OF FINITELY GENERATED RATIONAL SEMIRINGS

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Abstract. A subset S of the field of real numbers is called a positive semiring provided that $0, 1 \in S$ and S is closed under both addition and multiplication. A positive semiring is called finitely generated if it has the form $\mathbb{N}_0[r_1, \dots, r_n]$ for some real numbers $r_1, \dots, r_n$. In this paper, we study the underlying additive monoids of finitely generated subsemirings of the field of rationals under the lenses of atomicity and factorization theory. First, we characterize when these additive monoids are abelian groups and determine their Grothendieck groups. Then we characterize when they satisfy the bounded and the finite factorization properties: we find that these two factorization properties are equivalent on the class of monoids we study here. Finally, we show that the half-factorial and the factorial properties are also equivalent on the class of monoids of interest, and we determine when they hold. As an immediate consequence of this last result, we can verify that, among the positive semirings we consider, the only one having both additive and multiplicative monoids UF is $\mathbb{N}_0$, confirming the statement of the Bi-UF Positive Conjecture.

1. Introduction

A commutative semiring is a set endowed with two binary operations, addition and multiplication, such that it forms a commutative monoid under addition, a commutative semigroup under multiplication, and multiplication distributes over addition. A subset S of the nonnegative real line is called a positive semiring provided that $0, 1 \in S$ and S is closed under both operations. Unlike subrings of $\mathbb{C}$, the underlying additive monoids of a subsemiring S of $\mathbb{C}$ frequently exhibit highly complex, non-unique factorization behavior, even when S is simply generated or when S is a subsemiring of the field $\mathbb{Q}$.

A simply generated subsemiring $\mathbb{N}_0[\zeta] := \{p(\zeta) : p(x) \in \mathbb{N}_0[x]\}$, where $\zeta \in \mathbb{C}$, of the prototypical semiring $\mathbb{N}_0$ is called a simple semiring extension. The foundational step in the systematic investigation of the atomic and factorization properties of the additive monoids of simple semiring extensions was taken by S. T. Chapman, F. Gotti, and M. Gotti in their paper [8]. In that work, the authors explored the atomic structure

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and the sets of lengths of factorizations of the semiring $\mathbb{N}_0[q]$, where q is a positive rational number. Their results brought substantial attention to the rich arithmetic encoded in the additive monoids of rational semirings, inspiring a significant amount of subsequent research on the atomicity and factorization theory of these algebraic objects in just a couple of years (see the recent papers [1, 15, 17, 31] and references therein).

The current interest in the factorization theory of positive semirings has also been significantly stimulated by the investigation of bi-atomic properties, especially the bi-UF and the bi-HF properties, introduced by N. R. Baeth, S. T. Chapman, and F. Gotti [4] in their study of classes of positive semirings whose underlying additive and multiplicative monoids satisfy the same factorization property. This dual approach naturally motivated a deeper exploration into the arithmetic of positive semirings and gave rise to a central, still-unresolved problem known today as the Bi-UF Positive Conjecture. It states that the prototypical semiring $\mathbb{N}_0$ is the only positive semiring for which both the additive and the multiplicative monoids satisfy the unique factorization (UF) property. Understanding precisely when unique factorization holds additively is a crucial step toward a complete resolution of the problem.

Motivated by these early findings, researchers naturally sought to understand the behavior of more general semiring extensions. The first major generalization of the aforementioned paper was carried out by J. Correa-Morris and F. Gotti in [14]. In their work, the authors extended the positive rational parameter q to any positive algebraic parameter α (the arithmetic of additive monoids of $\mathbb{N}_0[\alpha]$ when α is transcendental or has no positive conjugate is rather trivial as such monoid is either free or an abelian group). In the mentioned paper, the authors offer a comprehensive study of the atomic and factorization theory of $(\mathbb{N}_0[\alpha], +)$, characterizing the atomic property, the bounded and the finite factorization properties, and both the factorial and the half-factorial properties in terms of the generating parameter α.

In this paper, we explore the atomic and factorization properties of another natural and significant generalization of the semirings $\mathbb{N}_0[q]$: the finitely generated rational subsemiring $\mathbb{N}_0[Q] = \mathbb{N}_0[q_1, \dots, q_n]$ of the field of rationals, which is generated by finitely many rationals $q_1, \dots, q_n$ (here $Q := \{q_1, \dots, q_n\}$). Specifically, we study the atomicity and factorization of the additive monoid of $\mathbb{N}_0[q_1, \dots, q_n]$ by employing a methodology parallel to the one practiced by J. Correa-Morris and F. Gotti [14] in their study of positive algebraic parameters α, which is based on the atomic diagram in Figure 1, introduced by D. D. Anderson, D. F. Anderson, and M. Zafrullah in their landmark paper [2].

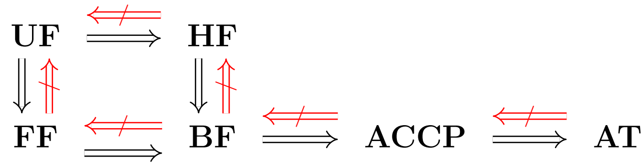

FIGURE 1. The implications in the diagram show the inclusions among subclasses of atomic monoids (AT stands for the class of atomic monoids). The (red) marked arrows emphasize that none of the shown implications are reversible.

Moving from a single generator to a finite set of rational generators introduces new layers of arithmetic complexity, as the interplay between the numerators and denominators of the various generators dictates the underlying structure of the factorizations.

Despite these challenges, our investigation yields a comprehensive picture of the factorization landscape for finitely generated rational semirings, revealing a remarkable structural rigidity. In the broader context of general commutative monoids, factorization properties, such as the unique factorization property (UFM), the half-factorial property (HFM), the finite factorization property (FFM), the bounded factorization property (BFM), and the ascending chain condition on principal ideals (ACCP), form a strict hierarchy often separated by pathological counterexamples. However, a major theme of our findings is that within the class of finitely generated rational semirings, this traditional hierarchy completely collapses into exact and elegant equivalences based purely on the magnitude and integrality of the generating set.

We begin our study in Section 3 by determining the precise conditions under which the additive monoid M_Q of $\mathbb{N}_0[Q]$ forms an abelian group, which occurs if and only if $\min Q < 0$ (Proposition 3.1), and then we find the Grothendieck group of M_Q for every subset Q of rationals (Proposition 3.4). Then we restrict our focus to the reduced cases: $\min Q \geq 0$. Following this, we prove that the finite factorization property, the bounded factorization property, and the ACCP conditions are equivalent on the class of monoids M_Q, and we characterize this equivalence exactly by the simple lower bound condition $\min Q \geq 1$ (Theorem 4.1). Finally, we demonstrate that the HF property and the UF property are also equivalent in this context, both holding if and only if the generating set consists entirely of nonnegative integers, i.e., $Q \subseteq \mathbb{N}_0$ (Proposition 4.2). Finally, we identify pair of positive rationals (q, r) such that $M_{q,r}$ is an atomic monoid that does not satisfy the ACCP.

Ultimately, these findings emphasize that while arbitrary submonoids of the nonnegative rationals (such as general, dense Puiseux monoids) can exhibit notoriously wild and pathological factorization invariants, the additive monoids arising from finitely generated rational semirings are highly structured. Rather than producing scattered factorization behaviors, their arithmetic properties are elegantly and perfectly classifiable, strictly governed by the simple boundaries and integrality of their discrete generators.

2. Preliminary

In this section, we briefly introduce a few concepts related to our exposition as an excuse to establish the notation we shall be using throughout this paper.

2.1. **General Notation.** As it is customary, we let $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the fields of rational numbers, real numbers, and complex numbers, respectively. In addition, we let $\mathbb{P}$, $\mathbb{N}$, and $\mathbb{N}_0$ denote the sets of standard primes, positive integers, and nonnegative integers, respectively. For any real number r and a subset S of the real line, we set $S_{\ge r} := \{s \in S : s \ge r\}$. For any pair $(m, n) \in \mathbb{Z}^2$ with $m \le n$, we set

$$[\![m, n]\!] := \{k \in \mathbb{Z} : m \le k \le n\}.$$

For any positive rational q, call the unique relatively primes $\mathsf{n}(q)$ and $\mathsf{d}(q)$ positive integers such that $q = \mathsf{n}(q)/\mathsf{d}(q)$ the *numerator* and the *denominator* of q, respectively. Recall that a positive rational is a *unit fraction* if its numerator is 1. For a prime p, the *p-adic valuation* on $\mathbb{Q}$ is the map $v_p \colon \mathbb{Q} \to \mathbb{Z} \cup \{\infty\}$ defined as follows: $v_p(0) = \infty$ and $v_p(q) = v_p(\mathsf{n}(q)) - v_p(\mathsf{d}(q))$ for any $q \ne 0$, where for $n \in \mathbb{N}$ the value $v_p(n)$ is the exponent of the maximal power of p dividing n. Thus, for all $q_1, \dots, q_n \in \mathbb{Q}_{>0}$,

$$v_p(q_1 + \cdots + q_n) \ge \min\{v_p(q_1), \dots, v_p(q_n)\}.$$

2.2. **Commutative Monoids.** A pair $(S, *)$, where S is a set and $*$ is a binary operation on S is called a *semigroup* if $*$ is *associative*: $(a * b) * c = a * (b * c)$ for all $a, b, c \in S$. A semigroup $(S, *)$ is called a monoid provided that there exists an element $e \in S$ such that $e * s = s * e = s$ for all $s \in S$, in which case, e is called the *identity element* of S. We often abuse notation, and write a monoid $(M, *)$ simply as M when there is no risk of ambiguity or the binary operation is clear from the context. Let M be a monoid. We say that M is *commutative* if $a * b = b * a$ for all $a, b \in M$, while we say that M is cancellative if for all $a, b, c \in M$, the equality $a*b = a*c$ (or $b*a = c*a$) implies that $b = c$.

From now on, we tacitly assume that every monoid we deal with in the scope of this paper is cancellative and commutative. Let M be a monoid. A subset of M is called a *submonoid* of M if it contains 0 and is closed under addition. For any subset S of M, we let $\langle S \rangle$ denote the submonoid of M generated by S, and we say that M is *finitely generated* if $M = \langle S \rangle$ for some finite subset S of M. A submonoid of the additive monoid $\mathbb{R}_{\ge 0}$ is called a *positive monoid*. For recent studies of positive monoids, the interested reader can check [21, 15].

Since M is cancellative and commutative, it can be embedded into an abelian group $\mathscr{G}(M)$ that is minimal in the following sense: every abelian group containing an isomorphic copy of M as a submonoid also contains an isomorphic copy of $\mathscr{G}(M)$ as a subgroup. The group $\mathscr{G}(M)$ is unique up to isomorphism and it is called the Grothendieck group of M. The *rank* of M is defined to be the rank of its Grothendieck group as a $\mathbb{Z}$-module. Following Gotti [23], we refer to a rank-1 positive monoid as a *Puiseux monoid*. Puiseux monoids have been actively investigated recently from the factorization viewpoint [6, 10, 25, 33] and in connection to monoid algebras [16, 18, 22, 28] (see [9] for a survey).

We let $\mathscr{U}(M)$ be the group consisting of all the invertible elements of M (i.e., $u \in M$ such that $u * v = e$ for some $v \in M$). The monoid M is called *reduced* if $\mathscr{U}(M) = \{e\}$. Note that positive monoids and Puiseux monoids are reduced. For any $a, b \in M$, we say that b *divides* a in M and write $b \mid_M a$ provided that $a = b * c$ for some $c \in M$. An element $a \in M \setminus \mathscr{U}(M)$ is called an *atom* if whenever $a = b * c$ for some $b, c \in M$, either b or c belongs to $\mathscr{U}(M)$. The set of atoms of M is denoted by $\mathscr{A}(M)$. Observe that $\mathscr{A}(M)$ is a subset of any generating set of M when M is reduced. We say that $a \in M$ is *atomic* if either a is invertible or a factors into finitely many atoms (allowing repetitions). Following [13], we say that the monoid M is atomic if it consists of atomic elements. See [15] for a recent survey on atomicity in both monoids and domains.

A subset I of M is called an *ideal* of M if $I * M := \{ab : (a, b) \in I \times M\} \subseteq I$, and an ideal of the form $a * M$ for some $a \in M$ is called a *principal ideal*. We say that the monoid M satisfies the *ascending chain condition on principal ideals* (ACCP) provided that, for any sequence $(I_n)_{n \geq 1}$ of principal ideals of M such that $I_n \subseteq I_{n+1}$ for every $n \in \mathbb{N}$, we can pick $m \in \mathbb{N}$ such that $I_n = I_m$ for all $n \in \mathbb{N}$ with $n \geq m$. It is well known and not hard to show that every monoid that satisfies the ACCP is atomic. Atomic domains not satisfying the ACCP have been recently constructed in [5, 27, 7] (the first one was constructed by Grams [29] back in 1974).

2.3. **Factorizations.** For the rest of this section, assume that M is atomic. Let $\mathsf{Z}(M)$ denote the free commutative monoid on the set of atoms of the reduced monoid $M/\mathscr{U}(M)$. Let $\phi \colon \mathsf{Z}(M) \to M/\mathscr{U}(M)$ be the unique monoid homomorphism fixing every atom of $M/\mathscr{U}(M)$. For $b \in M$ and atoms $a_1, \dots, a_\ell$ in the reduced monoid $M/\mathscr{U}(M)$, we call $z := a_1 \cdots a_\ell \in \mathsf{Z}(M)$ a *factorization* of $b \in M$ of *length* ℓ if $\phi(z) = bM^\times$. For any $b \in M$, set

$$\mathsf{Z}(b) := \phi^{-1}(b * \mathscr{U}(M)) \subseteq \mathsf{Z}(M).$$

If $\mathsf{Z}(b)$ is a singleton for all $b \in M$, then M is called a *unique factorization monoid* (UFM). More generally, we say that M is a *finite factorization monoid* (FFM) if $\mathsf{Z}(b)$ is (nonempty and) finite for all $b \in M$. It is clear that every UFM is an FFM. Finitely generated monoids are examples of FFMs (see [19, Proposition 2.7.8]). The interested reader can find surveys on the unique and finite factorization properties in [34] and [3], respectively. For any $b \in M$, set

$$\mathsf{L}(b) := \{|z| : z \in \mathsf{Z}(b)\}.$$

If $\mathsf{L}(b)$ is a singleton for each $b \in M$, then we say that M is a *half-factorial monoid* (HFM). Observe that every UFM is an HFM. Also, if $\mathsf{L}(b)$ is (nonempty and) finite for all $b \in M$, then M is called a *bounded factorization monoid* (BFM). It follows from the corresponding definitions that if a monoid is either an FFM or an HFM, then it is a BFM. In addition, it is well known that every BFM satisfies the ACCP (see [30, Corollary 1]).

2.4. **Rational and Positive Semirings.** A *commutative semiring* is a set S along with an additive operation "+" and a multiplicative operation "$\cdot$" such that the pairs $(S, +)$ and $(S, \cdot)$ are commutative monoids with identity elements 0 and 1, respectively, and the multiplicative operation distributes over the additive operation. Every semiring we deal with in this paper is tacitly assumed to be commutative. Let S be a semiring. A submonoid of S is said to be a *subsemiring* of S, if it is closed under multiplication and contains 1. A semiring is called a *semidomain* provided that it is isomorphic to a subsemiring of a field.

Let S be a semidomain. In this case, the set consisting of all nonzero elements of S is a commutative monoid, which we denote by S^* and call the multiplicative monoid of S. We call $(S, +)$ and S^* the *additive monoid* and the *multiplicative monoid* of S, respectively. The group of units of S^* is denoted by $S^\times$. In the same way that a cancellative commutative monoid can be minimally embedded into an abelian group, a semidomain can be minimally embedded into an integral domain, which we call its *Grothendieck domain*.

For pairwise distinct indeterminates $x_1, \dots, x_n$, we let $\mathbb{N}_0[x_1, \dots, x_n]$ denote the subsemiring of the polynomial ring $\mathbb{Z}[x_1, \dots, x_n]$ consisting of all polynomials with nonnegative integer coefficients. A *positive semiring* (resp., *rational semiring*) is a subsemiring of $\mathbb{R}_{\ge 0}$ (resp., $\mathbb{Q}_{\ge 0}$). The prototypical example of positive/rational semiring is $\mathbb{N}_0$. Observe that every subsemiring S of $\mathbb{C}$ contains $\mathbb{N}_0$. A subsemiring of $\mathbb{C}$ of the form

$$\mathbb{N}_0[r] := \{f(r) : f(x) \in \mathbb{N}_0[x]\}$$

is called a *simple semiring extension* of $\mathbb{N}_0$. For a subset S of $\mathbb{C}$, we let $\mathbb{N}_0[S]$ denote the smallest subsemiring of $\mathbb{C}$ containing S. A subsemiring of $\mathbb{C}$ is called *finitely generated* if it has the form $\mathbb{N}_0[S]$ for some finite subset S of $\mathbb{C}$. For elements $s_1, \dots, s_n \in \mathbb{C}$, we write $\mathbb{N}_0[s_1, \dots, s_n]$ instead of $\mathbb{N}_0[\{s_1, \dots, s_n\}]$, and we notice that

$$\mathbb{N}_0[s_1, \dots, s_n] := \big\{f(s_1, \dots, s_n) : f \in \mathbb{N}_0[x_1, \dots, x_n]\big\}.$$

In the scope of this paper, our algebraic objects of interest are the finitely generated subsemirings of $\mathbb{Q}$, the field of rational numbers. Therefore for any nonempty finite subset Q consisting of rational numbers, we let M_Q denote the underlying additive monoid of $\mathbb{N}_0[Q]$. Therefore if Q is the set $\{q_1, \dots, q_n\}$,

$$M_Q := \mathbb{N}_0[q_1, \dots, q_n]. \tag{2.1}$$

3. Algebraic Considerations

In this first section, we briefly determine the subsets Q consisting of rationals such that M_Q is an abelian group. As a consequence, we will identify the finite nonempty sets Q of rationals such that $\mathbb{N}_0[Q]$ is a commutative ring.

Proposition 3.1. *Let Q be a finite nonempty subset of $\mathbb{Q}$, and let M_Q be the additive monoid of the semiring $\mathbb{N}_0[Q]$. Then M_Q is a group if and only if $\min Q < 0$.*

Proof. For the direct implication, assume that M_Q is a group. The fact that $\mathbb{N}_0[Q]$ is a semiring implies that $1 \in \mathbb{N}_0[Q] = M_Q$, and so $-1 \in M_Q$ because M_Q is a group and so 1 must be invertible. Therefore the semiring $\mathbb{N}_0[Q]$ has a negative number, which implies that there is at least a negative number in Q. Hence $\min Q < 0$, as desired.

Conversely, assume that $q := \min Q < 0$. This implies that $-n := -\mathsf{d}(q)q \in \mathbb{N}$. Therefore $n \in -\mathbb{N} \cap M_Q$, which means that M_Q contains the negative integer n. As $-n - 1 \in \mathbb{N}_0$, we can add $-n - 1$ copies of 1 to n to obtain -1, that is, $-1 = n + (-n - 1) \cdot 1 \in M_Q$. Then for every $r \in M_Q$, the fact that $\mathbb{N}_0[Q]$ is closed under multiplication and contains -1 ensures that $-r = (-1)r \in M_Q$. Hence M_Q is a group. □

As a consequence, we obtain the following corollary.

Corollary 3.2. *For any nonempty subset Q of rational numbers, $\mathbb{N}_0[Q]$ is a commutative ring if and only if $\min Q < 0$.*

Proof. It follows immediately from Proposition 3.1 as, for each nonempty finite subset Q of rational numbers, the semiring $\mathbb{N}_0[Q]$ is a ring if and only if M_Q is an abelian group. □

Next, we find the Grothendieck group of the monoid M_Q. First, let us argue the following lemma.

Lemma 3.3. *Let F be a field, and let S be a subsemiring of F. Then the Grothendieck group of the underlying additive monoid of S inside the underlying additive monoid of F is the smallest subring of F containing S.*

Proof. Let D be the Grothendieck group of the underlying additive monoid of S inside the underlying additive monoid of F. For any $s_1, s_2, t_1, t_2 \in S$, we notice that

$$(s_1 - s_2)(t_1 - t_2) = (s_1 t_1 + s_2 t_2) - (s_1 t_2 + s_2 t_1) \in D,$$

which means that D is closed under the multiplication of F. This, along with the fact that $1 = 1 - 0 \in D$, guarantees that D is a subring of F. It is clear that every subring of F that contains S must contain $s - t$ for all $s, t \in S$ and, therefore, must contain S. Thus, we conclude that D is the smallest subring of F containing S. □

In light of Lemma 3.3, we call the Grothendieck group of the underlying additive monoid of a subsemiring S of a field F, the *Grothendieck domain* of S.

Proposition 3.4. *For a nonempty finite subset Q of rationals, the Grothendieck group of M_Q is $\mathbb{Z}[Q]$.*

Proof. Let Q be a finite nonempty set consisting of rationals. Because Q consists of rationals, the field of fractions of the integral domain $\mathbb{Z}[Q]$ is $\mathbb{Q}$. Let D be the Grothendieck domain of $\mathbb{N}_0[Q]$ inside $\mathbb{Q}$. As the underlying additive monoid of D is the Grothendieck group of $\mathbb{N}_0[Q]$, we are done once we argue that the integral domains D and $\mathbb{Z}[Q]$ are the same. Since $\mathbb{Z}[Q]$ is a subring of F containing $\mathbb{N}_0[Q]$, it follows from the minimality of D that $D \subseteq \mathbb{Z}[Q]$. On the other hand, as D is a subring of $\mathbb{Q}$, it must contain the set $\{\pm 1\}$ and, therefore, $\mathbb{Z} \subseteq D$. This, along with the fact that $Q \subseteq \mathbb{N}_0[Q] \subseteq D$, ensures that $\mathbb{Z}[Q] \subseteq D$. Hence $D = \mathbb{Z}[Q]$, which concludes the proof. □

As our primary purpose in this paper is to investigate the atomic and factorization behavior of finitely generated rational semirings, we can reduce our study to those semirings $\mathbb{N}_0[Q]$ that are not abelian groups. Hence, in light of Proposition 3.1, from now on we assume that

$$\min Q \geq 0.$$

4. Factorization Properties

The primary purpose of this section is to determine which monoids M_Q have the BF property and do the same for the FF property. As our main result, we will prove that these two properties and the ACCP condition are equivalent and any of them holds precisely when $\min Q \geq 1$. It was proved in [14, Theorem 4.11] that for any $\alpha \in \mathbb{C}$ the following conditions are equivalent for the additive monoid of the simple extension semiring $\mathbb{N}_0[\alpha]$:

- $\mathbb{N}_0[\alpha]$ is an FFM;
- $\mathbb{N}_0[\alpha]$ is a BFM;
- $\mathbb{N}_0[\alpha]$ satisfies the ACCP.

It turns out that three mentioned conditions also hold after replacing $\mathbb{N}_0[\alpha]$ by $\mathbb{N}_0[Q]$ for any nonempty finite subset of $\mathbb{Q}_{>0}$. We proceed to establish this result.

Theorem 4.1. *Let Q be a nonempty finite subset of $\mathbb{Q}_{>0}$ and let M_Q be the additive monoid of the semiring $\mathbb{N}_0[Q]$. Then the following conditions are equivalent.*

(a) $\min Q \ge 1$.

(b) *M_Q is an FFM.*

(c) *M_Q is a BFM.*

(d) *M_Q satisfies the ACCP.*

Proof. Set $Q := \{q_1, \dots, q_n\}$. For each index $i \in [\![1, n]\!]$, we can assume that $q_i \notin \mathbb{N}_0$ as otherwise we can replace Q by $Q \setminus \{q_i\}$.

(a) ⇒ (b): Because $\min Q \ge 1$, all the generators $q_1, \dots, q_n$ are at least 1. Thus, for any $B \in \mathbb{N}$, we can take $N \in \mathbb{N}$ with $q_1^N > B$ and, therefore, the fact that $\prod_{i=1}^n q_i^{e_i} \ge \prod_{i=1}^n q_1^N > B$ for every $(e_1, \dots, e_n) \in \mathbb{N}_0^n$ guarantees that

$$M_Q \cap [0, B] \subseteq \{f(q_1, \dots, q_n) : \deg f < n\}.$$

As a result, the set $M_Q \cap [0, B]$ is finite for every $B \in \mathbb{R}$, and this implies that we can increasingly enumerate the elements of M_Q. As a result, the additive monoid M_Q is an increasing positive monoid. Then it follows from [21, Theorem 5.6] that M_Q is an FFM.

(b) ⇒ (c) ⇒ (d): These implications are special cases of the known facts that every FFM is a BFM and that every BFM satisfies the ACCP.

(d) ⇒ (a): Assume, towards a contradiction, that $q_1 < 1$. In this case, $\mathsf{n}(q_1) < \mathsf{d}(q_1)$. Observe that, for every $k \in \mathbb{N}_0$,

$$\mathsf{n}(q_1)q_1^k = \mathsf{d}(q_1)q_1^{k+1} = \mathsf{n}(q_1)q_1^{k+1} + (\mathsf{d}(q_1) - \mathsf{n}(q_1))q_1^{k+1},$$

from which one deduces that $\mathsf{n}(q_1)q_1^k - \mathsf{n}(q_1)q_1^{k+1} = (\mathsf{d}(q_1) - \mathsf{n}(q_1))q_1^{k+1} \in M_Q$ (this is because $\mathsf{d}(q_1) - \mathsf{n}(q_1) \in \mathbb{N}_0$). As a result, $\mathsf{n}(q_1)q_1^k + M_Q \subseteq \mathsf{n}(q_1)q_1^{k+1} + M_Q$, which means that $(\mathsf{n}(q_1)q_1^k + M_Q)_{k \ge 0}$ is an ascending chain of principal ideals in M_Q. Finally, notice that as M_Q is reduced, the fact that the sequence $(\mathsf{n}(q_1)q_1^k)_{k \ge 0}$ is strictly decreasing ensures that the ascending chain $(\mathsf{n}(q_1)q_1^k + M_Q)_{k \ge 0}$ does not stabilize. □

Next we characterize the subsets Q of $\mathbb{Q}_{\ge 0}$ for which the additive monoid M_Q satisfies the unique factorization property.

Proposition 4.2. *Let Q be a nonempty finite subset of $\mathbb{Q}_{\ge 0}$ and let M_Q be the additive monoid of the semiring $\mathbb{N}_0[Q]$. Then the following conditions are equivalent.*

(a) $Q \subseteq \mathbb{N}_0$.

(b) *M_Q is a UFM.*

(c) *M_Q is an HFM.*

Proof. (a) $\Rightarrow$ (b): If Q consists of nonnegative integer, then $M_Q = \mathbb{N}_0[Q] = \mathbb{N}_0$ and, therefore, M_Q is the free commutative monoid of rank 1, which is a UFM.

(b) $\Rightarrow$ (c): This follows immediately from the more general fact that every UFM is an HFM.

(c) $\Rightarrow$ (a): Finally, assume that M_Q is an HFM. Suppose, towards a contradiction, that Q contains an element that is not an integer. Observe that $Q \cap (0,1)$ must be empty as otherwise we can take $q \in Q$ with $0 < q < 1$ and proceed as we did in the last paragraph of the proof of Theorem 4.1 to show that $(\mathsf{n}(q)q^n + M_Q)_{n \ge 0}$ is an ascending chain of principal ideals of M_Q that does not stabilize, which is not possible because M_Q is an HFM (and every HFM satisfies the ACCP). As Q does not contain any positive rational less than 1, the inequality $\min M_Q \setminus \{0\} \ge 1$, which implies that $\{1, q_0\} \subseteq \mathscr{A}(M_Q)$, where $q_0 := \min Q \setminus \mathbb{N}_0$. This fact allows us to find two distinct factorizations of the element $\mathsf{n}(q_0)$, namely, the formal sum z_1 of $\mathsf{n}(q_0)$ copies of 1 and the formal sum z_2 of $\mathsf{d}(q_0)$ copies of q_0. As q_0 is not an integer, the lengths $\mathsf{n}(q_0)$ and $\mathsf{d}(q_0)$ of the factorizations z_1 and z_2, respectively, are different, which contradicts the fact that M_Q is an HFM. □

The Bi-UF Positive Conjecture, as posed by N. Baeth, S. T. Chapman, and F. Gotti in [4, Conjecture 7.7], states that $\mathbb{N}_0$ is the only positive semiring having both its additive and its multiplicative monoids satisfying the unique factorization property. Observe that as immediate consequences of 4.2, we can confirm the statement of this conjecture over the class consisting of all additive monoids of finitely generated rational semirings. We record this as the following remark.

Remark 4.3. Let Q be a nonempty set consisting of positive rationals. If both the additive and the multiplicative monoids of the rational semiring $\mathbb{N}_0[Q]$ are UFMs, then $Q \subseteq \mathbb{N}$ and so $\mathbb{N}_0[Q] = \mathbb{N}_0$.

5. Atomicity

In this last section we briefly consider the atomicity of the additive monoids M_Q, where Q as before is a finite nonempty subset of $\mathbb{Q}_{>0}$. It is well known that for every $q \in (0,1) \cap \mathbb{Q}$, the additive monoid of the simple semiring extension $\mathbb{N}_0[q]$ is atomic but does not satisfy the ACCP provided that the generator is not a unit fraction. Let us identify certain finite sets Q of positive rationals such that M_Q is atomic but does not satisfy the ACCP.

Proposition 5.1. *For $q \in \mathbb{Q}$ with $q > 0$, let $Q = \{q^{\pm 1}\}$. Then the following statements hold.*

1. *M_Q is atomic if and only if $q \notin \mathbb{N}_{\ge 2}^{\pm 1}$.*
2. *M_Q satisfies the ACCP if and only if $q = 1$.*

Proof. (1) It is clear that if $q \in \mathbb{N}^{\pm 1}_{\geq 2}$, after replacing q by q^{-1} if needed, we can assume that $q = 1/d$ for some $d \in \mathbb{N}$ with $d \geq 2$, in which case, $M_Q = (\mathbb{N}_0[1/d], +)$, which is not atomic (indeed, it contains no atoms). Conversely, suppose that $q \notin \mathbb{N}^{\pm 1}_{\geq 2}$, and set

$$A = \{q^n : n \in \mathbb{Z}\}.$$

As $q \neq 1$, after replacing q by q^{-1} if needed, we can assume, without loss of generality, that $q < 1$. Since M_Q is a reduced monoid generated by the set $A := \{q^n : n \in \mathbb{N}\}$, we obtain that $\mathscr{A}(M_Q) \subseteq A$. Therefore in order to prove that the set of atoms of M_Q is A, it is enough to fix $k \in \mathbb{Z}$ and show that $q^k \in \mathscr{A}(M_Q)$, as we proceed to do.

Observe that every divisor of q^k in M_Q must be contained in $M_k := \langle q^j : j \geq k \rangle$ is isomorphic to the underlying additive monoid M_q of $\mathbb{N}_0[q]$ under the isomorphism defined as multiplication by the rational q^{-k}. It is well known that

$$\mathscr{A}(M_q) = \{q^n : n \in \mathbb{N}_0\},$$

and this ensures that $1 \notin \langle q^n : n \in \mathbb{N} \rangle$. As a consequence, $q^k \notin \langle q^n : n > k \rangle$. On the other hand, the inequality $q < 1$ guarantees that no atom in $\{q^j : j < k\}$ divides q^k in M_Q. Hence $q^k \notin \langle A \setminus \{q^k\} \rangle$ and, therefore, $q^k \in \mathscr{A}(M_Q)$. Hence $\mathscr{A}(M) = A$, which implies that M_Q is atomic.

(2) If $q = 1$, then $M_Q = \mathbb{N}_0$ and so it satisfies the ACCP (indeed, in this case it is a UFM). Now suppose that $q \neq 1$ and assume, as we did before, that $q < 1$. Set $d := \mathsf{d}(q)$. In order to argue that M_Q does not satisfy the ACCP we can now proceed as we did in the last part of the proof of Theorem 4.1 to argue that the sequence of principal ideals $(\mathsf{n}(q)q^n + M_Q)_{n \geq 1}$ is ascending but does not stabilize. This concludes the proof. ∎

Characterizing atomicity inside the general class of additive monoids of finitely generated rational semirings does not seem to be a simple task. We conclude the paper with the following questions.

Question 5.2. For which nonempty finite sets Q consisting of positive rationals is the additive monoid of the finitely generated rational semiring $\mathbb{N}_0[Q]$ atomic?

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