DIVISIBILITY IN POWER MONOIDS

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Abstract. Let M be a commutative monoid. The (restricted) finitary power monoid of M is the monoid consisting of all finite nonempty subsets of M (containing a unit) under the so called sumset operation. We say that M possesses the MCD (resp., MCD-finite) property if every nonempty finite subset of M admits at least one (at most finitely many) maximal common divisors (MCD). In this paper we investigate divisibility conditions based on MCDs in the class of finitary power monoids. Our first goal is to study the ascent of the MCD and MCD-finite properties from the base monoid M to its finitary power monoid $\mathscr{P}_{\text{fin}}(M)$ and its restricted finitary power monoid $\mathscr{P}_{\text{fin},\mathscr{U}}(M)$. We prove that both properties ascend in full generality: if M is an MCD monoid (resp., an MCD-finite monoid), then so is $\mathscr{P}_{\text{fin}}(M)$ (resp., $\mathscr{P}_{\text{fin},\mathscr{U}}(M)$). Then we turn to the irreducible-divisor-finite (IDF) property, whose ascent to polynomial extensions has been considered over the past three decades by many authors, including Malcolmson, Okoh, and Zafrullah. In this direction, we prove that the IDF property ascends to finitary power monoids over the class of MCD-finite monoids, whose analogue for polynomial extensions was established in 2018 by Eftekhari and Khorsandi. In the final section we consider polynomial extensions: first, we prove that the MCD-finite property ascends to polynomial extensions, and then we prove that every primitive-super-primitive monoid (PSP monoid) possesses the MCD-finite property, which allows us to connect two recent results about the ascent of the IDF property to polynomial extensions.

1. Introduction

Let M be a commutative monoid, which is additively written, and let $\mathscr{U}(M)$ denote the group of invertible elements of M (i.e., the group of units). The *large power monoid* of M is the set consisting of all nonempty subsets of M under the following binary operation, often called the *sumset* or the *Minkowski sum*: for all nonempty subsets S and T of M,

$$(S,T) \mapsto S + T := \{s + t : (s,t) \in S \times T\}. \tag{1.1}$$

The finitary power monoid of M, denoted by $\mathscr{P}_{\text{fin}}(M)$, is the submonoid of the power monoid of M consisting of all nonempty finite subsets of M. A systematic study of power monoids and finitary power monoids was initiated by Tamura and Shafer [29] back in the sixties. Certain submonoids of $\mathscr{P}_{\text{fin}}(M)$ have been considered in recent literature, including the restricted finitary power monoid $\mathscr{P}_{\text{fin},\mathscr{U}}(M)$, which is the submonoid of the finitary power monoid of M consisting of all nonempty finite subsets of M containing at least an invertible element. These constructions have recently been used as a combinatorial framework for

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studying factorization and divisibility in additive monoids and in integral domains via their multiplicative monoids of nonzero elements.

An element of a commutative monoid M is called *atomic* if either it is a unit or it can be written as a finite sum of atoms (i.e., irreducible elements), and we say that M is atomic if every element of M is atomic. Also, M is said to satisfy the *ascending chain condition on principal ideals* (ACCP) if M does not contain any infinite strictly ascending chain of principal ideals. Another finiteness condition, intermediate between atomicity and ACCP, is the irreducible-divisor-finite (IDF) property, introduced and first studied by Grams and Warner [23] in 1975: M is called an IDF monoid if every element of M has only finitely many irreducible divisors up to associate. The IDF property has been studied extensively as it naturally interpolates between atomicity and stronger uniqueness conditions such as finite factorization and unique factorization.

In their influential paper [1], Anderson, Anderson, and Zafrullah introduced and studied the bounded and finite factorization properties (two natural relaxations of unique factorization property) in the setting of integral domains, thereby laying the foundation for a subsequent systematic investigation of factorization theory. In the same paper, the authors included exactly the following two open questions:

- [1, Question 1] Does the property of being an atomic domain ascend to polynomial rings?
- [1, Question 2] Does the property of being an IDF domain ascend to polynomial rings?

A commutative monoid is said to have the *MCD property* if every nonempty finite subset admits a maximal common divisor (MCD). It is well known that every commutative monoid that satisfies the ascending chain condition on principal ideals (ACCP) necessarily has the MCD property. In 1993, Roitman [28] gave a negative answer to Question 1 and, in the same paper, he proved that the property of being atomic does ascend to polynomial domains when restricted to the class of MCD domains. The ascent of atomicity to finitary power monoids parallels the ascent of atomicity to polynomial extensions (see [17, Section 3] and [11, Section 4]). The ascent of the MCD property to finitary power monoids over the class of linearly orderable monoids was established in [11, Proposition 4.1]. In Section 3, we generalize this result by replacing the class of linearly orderable monoids by the larger class of cancellative monoids (without assuming the torsion-free condition).

In 2009, Malcolmson and Okoh [27, Theorem 6.5] gave a negative answer to Question 2 above, proving that every countable domain embeds into a countable antimatter domain whose polynomial extension does not have the IDF property (an integral domain is called antimatter if it has no atoms). In the same paper, they prove that the IDF property ascends from any GCD domain to its polynomial extension. In 2018, Eftekhari and Khorsandi [12] introduced and studied the MCD-finite property and, in the same paper they prove that the IDF property ascends to polynomial extensions over the class of MCD-finite domains, generalizing the ascent of the IDF property previously proved by Malcolmson and Okoh. In Section 4 we introduced a class of rank-one monoids that are MCD-finite. Then we prove that the MCD-finite property ascends to power monoids, generalizing a result by Dani et al. [11]). We conclude the same section investigating the ascent of the IDF property to power monoids. First, we prove that the IDF property ascends to power monoids over the class of linearly orderable MCD-finite monoids, mirroring the ascent of the IDF to polynomial extensions over the class of MCD-finite domains. This complements a construction in [11, Section 6] of an Archimedean linearly orderable monoid M that is an IDF monoid but its power monoid is not an IDF monoid.

Let R be an integral domain. A nonzero polynomial $f(x) \in R[x]$ is called primitive if the ideal I_f of R generated by the coefficients of $f(x)$ is not contained in any proper principal ideal of R, while we say that $f(x)$ is super-primitive if the inverse of the ideal I_f is the whole ring R. It follows from the given definitions that every super primitive polynomial is primitive. Following Arnold and Sheldon [5], we say that R is a primitive-super-primitive (PSP) domain if every primitive polynomial of R is super primitive. PSP domains were introduced and first investigated in [5] back in 1975 in the setting of integral domains. To generalize the PSP property to a cancellative commutative monoid M, we only need to define the notions of primitive and super primitive ideals mimicking the corresponding standard definition used for polynomials: a finitely generated ideal I of M is called

- primitive if I is not contained in any proper principal ideal of M and
- super primitive if the inverse ideal I^{-1} equals M.

Then we say that M is a PSP monoid if every primitive ideal of M is super primitive. We dedicate the last section to the ascent of the MCD and IDF to polynomial extensions. We prove that every nonempty finite subset of a PSP monoid has at most one MCD up to associate, which implies that every PSP monoid is an MCD-finite monoid. Thus, the class of PSP monoids contains the class of GCD monoids and is contained in the class of MCD-finite monoids, as the diagram in Figure 2 illustrates. As a consequence, we obtain that the ascent of the IDF property to polynomial extensions over the class of PSP domains (as proved in [22]) is a special case of the ascent of the IDF property to polynomial extensions over the class of MCD-finite monoids (as proved in [12]). In Section 5, we also establish the ascent of the MCD-finite property to polynomial extensions.

$$\textbf{GCD monoids} \ \underset{\Longrightarrow}{\not\Longleftarrow}\ \textbf{PSP} \ \underset{\Longrightarrow}{\not\Longleftarrow}\ \textbf{MCD-finite}$$

FIGURE 1. Nested classes of generalized GCD monoids.

2. Background

In this section, we collect the basic terminology and notation used throughout the paper. All monoids are assumed to be commutative and written multiplicatively with identity element 1. We will denote the group of units of M by $M^\times$ (or $\mathscr{U}(M)$ when M is written additively). The set consisting of all the left cosets of M,

$$M/M^\times := \{aM^\times : a \in M\},$$

is the monoid whose elements are the associate classes of M, and M is called *reduced* when $M/M^\times$ is trivial. For any nonempty subset $S \subseteq M$, the quotient $S/\!\sim$ denotes the family of associate classes in M having a representative in S:

$$S/\!\sim \,:= \{\, sM^\times : s \in S \,\} \subseteq M/M^\times.$$

2.1. **Cancellativity and Divisibility.** For the rest of this section, we assume that S is a nonempty finite subset of M. An element $a \in M$ is called *cancellative with respect to* S if for all $s, t \in M$, the equality $as = at$ implies that $s = t$, while a is called *cancellative* if a is cancellative with respect to M. The monoid M is called *cancellative* if it consists of cancellative elements. Cancellativity is not assumed globally in this paper; instead, it is invoked only when required by a specific argument. It is well known that the monoid M is cancellative if and only if it is isomorphic to a submonoid of an abelian group, and in such a case, the smallest abelian group having a submonoid isomorphic to M is denoted by $\mathscr{G}(M)$ and called the *Grothendieck group* of M.

Divisibility is taken in the standard monoid-theoretic sense: for each $c \in M$, we say that an element $d \in M$ *divides* c if $c \in dM$, in which case we write $d \mid_M c$. For each $c \in M$, we set

$$D_c := \{d \in M : d \mid_M c\}.$$

For the rest of this section, we let S be a nonempty finite subset of M. An element $d \in M$ is called a *common divisor* of S if $d \mid_M s$ for every $s \in S$. A common divisor d of S is called a *greatest common divisor* (GCD) if every common divisor of S in M divides d.

For $m, d \in M$, we set $m/d := \{c \in M : cd = m\}$, and we observe that m/d is a nonempty set if and only if $d \mid_M m$, in which case, m/d is a singleton if d is cancellative with respect to m/d. It is also convenient to set

$$S/d := \bigcup_{s \in S} s/d.$$

A common divisor $m \in M$ of S is called a *maximal common divisor* (MCD) of S in M if the set of common divisors of S/m is $M^\times$. It follows from the definitions that every GCD is an MCD. We let $\text{mcd}_M(S)$ denote the set consisting of all MCDs of S in M. We proceed to introduce some of the most relevant classes of monoids whose (restricted) power monoids we study in this paper.

Definition 2.1. Let M be a commutative monoid.

- M is called a *quasi*-GCD (q-GCD) *monoid* if $|\text{mcd}_M(S)/\!\sim| \le 1$ for every nonempty finite subset $S \subset M$.
- M is called an *MCD monoid* if $\text{mcd}_M(S)/\!\sim$ is nonempty for every nonempty finite subset $S \subset M$.
- M is called an *MCD-finite monoid* if $|\text{mcd}_M(S)/\!\sim| < \infty$ for every nonempty finite subset $S \subset M$.

It is clear that every GCD monoid is a q-GCD monoid, while every q-GCD monoid is an MCD-finite monoid. Observe that if d is a GCD of S in M, then d is also an MCD of S in M. Hence every GCD monoid is both an MCD monoid and an MCD-finite monoid. Thus, we obtain nested classes of generalized GCD monoids determined by the black-arrow implications:

$$\textbf{GCD} \overset{\not\Longleftarrow}{\Longrightarrow} \textbf{PSP} \overset{\not\Longleftarrow}{\Longrightarrow} \textbf{q-GCD} \overset{\not\Longleftarrow}{\Longrightarrow} \textbf{MCD-finite}\,.$$

FIGURE 2. Nested classes of generalized GCD monoids.

2.2. **Ideals and PSP Domain.** Let $\mathscr{G}(M)$ be the Grothendieck group of the monoid M. A subset I of M is called an ideal if $IM \subseteq I$ or, equivalently, $IM \subseteq I$. For subsets S and T of $\mathscr{G}(M)$, we write

$$(S : T) := \{g \in \mathscr{G}(M) : gT \subseteq S\},$$

Let I be an ideal of M and set $I^{-1} := (M : I)$ and $I_v = (M : (M : I))$. If an ideal I of M satisfies that $I_v = I$, then I is called *divisorial.* We say that

- I is a *primitive ideal* of M if I is finitely generated and is not contained in any proper principal ideal of M, and
- I is a *super-primitive ideal* if I is a finitely generated ideal of M and $I^{-1} = M$.

It follows directly from the definitions that every super-primitive ideal is a primitive ideal. However, the converse does not hold in general.

Definition 2.2. Let M be a cancellative commutative monoid. If every primitive ideal of M is super-primitive, then M is called a *PSP monoid.*

The PSP property was first studied by Tang [30], who proved that, for any integral domain R, the product of a super-primitive polynomial and a primitive (super-primitive) polynomial is a primitive (resp., super-primitive) polynomial, which implies that every GCD domain is a PSP and also that every PSP satisfies the Gauss's lemma. Generalized GCD has been a recurrent subject of study in the literature of commutative semigroup [3] and commutative rings [2].

Irreducibles and Factorizations. An element $a \in M \setminus M^{\times}$ is called an *irreducible* (or an *atom*) if whenever $a = uv$ for some $u, v \in M$, one of the elements u or v belongs to $M^{\times}$. The set consisting of all irreducibles of M is denoted by $\mathscr{A}(M)$. For $b \in M$, we let $\mathscr{D}(b)$ denote the set consisting of all irreducibles dividing b in M:

$$\mathscr{D}(b) := \{\, a \in \mathscr{A}(M) : a \mid_M b \,\}.$$

We now introduce the IDF property, which is central in this paper and plays a fundamental role in [22, 12], the papers motivating Sections 4 and 5. An element $b \in M$ has the *irreducible-divisor-finite* (IDF) property (or is an *IDF element*) if the sets $\mathscr{D}(b)/\!\sim$ is finite, which means that b is only divisible by finitely many irreducibles up to associates. Notice that every unit is an IDF element.

Definition 2.3. A commutative monoid M is called an *IDF monoid* if every element $b \in M$ has the IDF property.

The IDF property has been systematically considered by many authors since then (see the recent paper [22] and reference therein).

Linearly Orderable Monoids. Linearly orderable monoids provide an important class where divisibility enjoys additional structure.

Definition 2.4 (Linearly orderable monoid)**.** A commutative monoid M is *linearly orderable* if there exists a total order $\preceq$ on M such that, for all $a, b \in M$, the inequality $a \prec b$ implies $ac \prec bc$ for all $c \in M$. In this case, M is said to be *linearly ordered* with respect to $\preceq$.

Linearly orderable monoids are cancellative. Indeed, it is due to Levi [24] that an abelian group is linearly orderable if and only if it is torsion-free and, as a consequence, a commutative monoid is linearly orderable if and only if it is cancellative and torsion-free. The relevance of linearly orderable monoids in this paper stems from their compatibility with finitary power monoid constructions and the IDF property.

Finitary Power Monoids. We proceed to introduce the most central algebraic structure in the scope of this paper: the monoid consisting of all nonempty finite subsets of M. As we did in the introduction, we let $\mathscr{P}(M)$ denote the large power monoid of M whose product, defined additively in (1.1), is written multiplicatively throughout this section: for any two nonempty subsets S and T of M,

$$ST := \{\, st : (s,t) \in S \times T \,\}. \tag{2.1}$$

This set-wise product induces a monoid structure on $\mathscr{P}(M)$ with identity element $\{1\}$.

Definition 2.5. The *finitary power monoid* of a commutative monoid M is the submonoid $\mathscr{P}_{\text{fin}}(M)$ of $\mathscr{P}(M)$ consisting of all nonempty finite subsets:

$$\mathscr{P}_{\text{fin}}(M) = \{\, S \subseteq M : 1 \le |S| < \infty \,\}.$$

As $\mathscr{P}_{\text{fin}}(M)$ is the fundamental algebraic object inside the scope of this paper, from now on we will refer to $\mathscr{P}_{\text{fin}}(M)$ simply as the *power monoid* of M. Along the paper, the relevance of the submonoid $\mathscr{P}_{\text{fin},\mathscr{U}}(M)$ of $\mathscr{P}_{\text{fin}}(M)$ consisting of all the subsets of M intersecting $M^\times$ is also significant. We call the monoid $\mathscr{P}_{\text{fin},\mathscr{U}}(M)$ the *restricted power monoid* of M. is a submonoid of M consisting of all nonempty subsets of M containing at least a unit:

$$\mathscr{P}_{\text{fin},\mathscr{U}}(M) := \{\, S \in \mathscr{P}_{\text{fin}}(M) : S \cap M^\times \ne \varnothing \,\}.$$

To simplify terminology, in the scope of this paper we call the monoids $\mathscr{P}_{\text{fin}}(M)$ and $\mathscr{P}_{\text{fin},\mathscr{U}}(M)$ the *power monoid* and the *restricted power monoid* of M, respectively.

3. The MCD Property

The main purpose of this section is to understand the potential ascent of the MCD property and the MCD-finite property in the setting of finitary power monoids. We also prove that the IDF property ascends to restricted power monoids.

The existence of MCDs in the setting of power monoids was first studied by Dany et al. in [11], where they proved that the MCD property ascends to finitary power monoids over the class of linearly orderable monoids. We will generalize this result in the next theorem, proving that the MCD property ascends to finitary power monoids over the class of cancellative commutative monoids (without assuming torsion-freeness). First, let us argue the following lemma.

Lemma 3.1. *Let M be a cancellative and commutative monoid. Let A and B be elements of $\mathscr{P}_{\text{fin}}(M)$. Then $|A+B| \geq \max(|A|, |B|)$.*

Proof. Without loss of generality, let $|A| \geq |B|$, let $A := \{a_1, \dots, a_k\}$ and let $b \in B$. Then $a_1 + b, a_2 + b, \dots, a_k + b$ are all elements of $A + B$. They must be distinct as if $a_i + b = a_j + b$ for $i \neq j$, then by cancellativity, we have that $a_i = a_j$, which is a contradiction. Therefore, $A + B$ has at least $|A|$ distinct elements, so $|A+B| \geq \max(|A|, |B|)$. □

The following proposition will be helpful to establish the main results of this section.

Proposition 3.2. *Let M be an MCD monoid, and let $\mathscr{S}$ be a finite nonempty subset of $\mathscr{P}_{\text{fin}}(M)$. If $D \in \mathscr{P}_{\text{fin}}(M)$ is a common divisor of $\mathscr{S}$ in $\mathscr{P}_{\text{fin}}(M)$, then there exists an element $r \in M$ that satisfies the following conditions:*

(a) *$D + \{r\}$ is also a common divisor of $\mathscr{S}$, and*

(b) *for any $r' \in M$ such that $r \mid_M r'$, the fact that $D + \{r'\}$ is a common divisor of $\mathscr{S}$ implies that $r' \sim_M r$.*

Proof. Assume that M is written additively, and set $\mathscr{P} := \mathscr{P}_{\text{fin}}(M)$. Suppose, by way of contradiction, that there exists a common divisor D of $\mathscr{S}$ in $\mathscr{P} := \mathscr{P}_{\text{fin}}(M)$ such that none of the elements $r \in M$ simultaneously satisfy conditions (a) and (b) in the statement of the lemma.

Set $n := |\mathscr{S}|$ and take $S_1, \dots, S_n$ in $\mathscr{P}_{\text{fin}}(M)$ such that $\mathscr{S} = \{S_1, \dots, S_n\}$. For each index $i \in [\![1, n]\!]$, the fact that D divides S_i, guarantees that the set

$$\mathscr{R}_i := S_i - D = \{R \in \mathscr{P}_{\text{fin}}(M) : R + D = S_i\}$$

is not empty. Then the product $\mathscr{R}_1 \times \cdots \times \mathscr{R}_n$ is nonempty, and so we can take an n-tuple $(R_{1,i}, \dots, R_{1,i})$ in $\mathscr{R}_1 \times \cdots \times \mathscr{R}_n$. As $\bigcup_{i=1}^n R_{1,i}$ is a nonempty finite subset of the MCD monoid M, the former must contain an MCD $m_1 \in M$. Observe that

$$S_i = R_{1,i} + D = (R_{1,i} - \{m_1\}) + (D + \{m_1\})$$

for every $i \in [\![1, n]\!]$, and so $D + \{m_1\}$ is a common divisor of $\mathscr{S}$ in the power monoid $\mathscr{P}$, whence m_1 satisfies condition (a) in the lemma. Therefore m_1 cannot guarantee condition (b), which ensures the existence of an element $m_2 \in M$ such that $m_2 - m_1 \in M \setminus \mathscr{U}(M)$ such that $D + \{m_2\}$ is a common divisor of $\mathscr{S}$. Therefore, for each $i \in [\![1, n]\!]$, we can take a nonempty finite subset $R_{2,i}$ of M such that $R_{2,i} + (D + \{m_2\}) = S_i$. Observe that m_2 is an MCD of $\bigcup_{i=1}^n R_{2,i}$. Continue in this fashion, we can obtain an n-tuple $(R_{j,1}, \dots, R_{j,n})$ in $\mathscr{R}_1 \times \cdots \times \mathscr{R}_n$ and an element $m_j \in M$ for every index $j \in \mathbb{N}$ such that the following two conditions hold:

- $D + \{m_j\}$ is a common divisor of $\mathscr{S}$,
- m_j properly divides m_{j+1} in M,
- $R_{j,i} + (D + \{m_j\}) = S_i$ for every $i \in [\![1, n]\!]$, and
- m_2 is an MCD of $\bigcup_{i=1}^n R_{2,i}$.

Since D is a common divisor of $\mathscr{S}$, the set $\mathscr{R}_i$ is nonempty for every $i \in [\![1, n]\!]$ and, therefore, $\mathscr{R}_1 \times \cdots \times \mathscr{R}_n$ is a finite set and, as a result, there exist indices $k, \ell \in \mathbb{N}$ with $k < \ell$ such that $R_{k,i} = R_{\ell,i}$ for every $i \in [\![1, n]\!]$. This implies that m_k and m_ℓ are both MCDs of $\bigcup_{i=1}^n R_{\ell,i}$, which contradicts the fact that m_k properly divides m_ℓ in M. □

We are in a position to generalize the ascent of the MCD property to power monoids and restricted power monoids over the setting of cancellative commutative monoids.

Theorem 3.3. *For a cancellative and commutative monoid M, the following statements hold.*

(1) *If M is an MCD monoid, then $\mathscr{P}_{\text{fin}}(M)$ is also an MCD monoid.*

(2) *If M is an MCD monoid, then $\mathscr{P}_{\text{fin},\mathscr{U}}(M)$ is also an MCD monoid.*

Proof. In order to ease notation, let us set $\mathscr{P} := \mathscr{P}_{\text{fin}}(M)$.

(1) It suffices to fix a nonempty finite subset $\mathscr{S} := \{S_1, \dots, S_n\}$ of the power monoid $\mathscr{P}$ and argue that $\mathscr{S}$ has an MCD in $\mathscr{P}$. Since every divisor has cardinality at most

$$\min\{|S_i| \colon i \in [\![1, n]\!]\},$$

there must be a common divisor of $\mathscr{S}$ in the power monoid $\mathscr{P}$ having maximum size: D be a divisor whose size is as large as it can possibly be. In light of Proposition 3.2, we can take $m \in M$ such that the following two conditions hold:

- $D + \{m\}$ is also a common divisor of $\mathscr{S}$, and
- for each $d \in M$, the fact that $D + \{m + d\}$ is a common divisor of $\mathscr{S}$ implies that $d \in \mathscr{U}(M)$.

Set $D_1 := D + \{m\}$ and observe that D_1 is also a common divisor of $\mathscr{S}$ having maximum size possible. We proceed to argue that D_1 is an MCD of $\mathscr{S}$ in $\mathscr{P}$. To do so, let $T := \{t_1, \dots, t_m\}$ be a common divisor of $\mathscr{S} - D_1$ in $\mathscr{P}$. It follows from the maximality of D_1 that $|D_1 + T| = |D_1|$. Hence, for every $i \in [\![1, m]\!]$, the equality $D_1 + T = D_1 + \{t_i\}$ holds, and so $|D_1 + \{t_i\}| = |D_1|$ and $D_1 + \{t_i\} \subseteq D + T$. Since $m + t_1$ is a common divisor of $\mathscr{S} - D$, it follows that $t_1 \in \mathscr{U}(M)$. Therefore we can deduce from the equality $D_1 + T = D_1 + \{t_1\}$ that $D + T$ is associate of D_1 in $\mathscr{P}$, which means that D_1 is an MCD of $\mathscr{S}$ in $\mathscr{P}$.

(2) We show that $\mathscr{P}_{\text{fin},\mathscr{U}}(M)$ is a divisor-closed submonoid of $\mathscr{P}_{\text{fin}}(M)$. Let A and B be elements of $\mathscr{P}_{\text{fin}}(M)$ such that $A \notin \mathscr{P}_{\text{fin},\mathscr{U}}(M)$, $B \in \mathscr{P}_{\text{fin},\mathscr{U}}(M)$, and $A \mid_{\mathscr{P}} B$. Write $A + A' = B$ for some A' in the power monoid $\mathscr{P}_{\text{fin},\mathscr{U}}(M)$, and fix $u \in \mathscr{U}(M)$ that belongs to B. Then there exist $a \in A$ and $a' \in A'$ such that $a + a' = u$, from which we obtain that a also belongs to $\mathscr{U}(M)$ so $A \in \mathscr{P}_{\text{fin},\mathscr{U}}(M)$, giving a contradiction. Hence all divisors of elements of $\mathscr{P}_{\text{fin},\mathscr{U}}(M)$ must also be in $\mathscr{P}_{\text{fin},\mathscr{U}}(M)$.

Therefore if we let T be the MCD of a subset $\mathscr{S}$ of $\mathscr{P}_{\text{fin},\mathscr{U}}(M)$, then it must be an element of $\mathscr{P}_{\text{fin},\mathscr{U}}(M)$. Furthermore, it still must be an MCD as if there is a common divisor D such that $T \mid_{\mathscr{P}} D$ and T and D are not associates, then D is a common divisor of $\mathscr{S}$ in $\mathscr{P}_{\text{fin}}(M)$ and so it lies in $\mathscr{P}_{\text{fin},\mathscr{U}}(M)$ because $\mathscr{P}_{\text{fin},\mathscr{U}}(M)$ is a divisor-closed submonoid. However, this contradicts the maximality of T. □

Even for a rank-one linearly orderable monoid M, it may exist a nonempty finite subset S of M whose only common divisors are the invertible elements of M for which we can pick two finite nonempty subsets A and B of M such that the following conditions hold:

(a) $A = B + S$, and

(b) $B + \{d\}$ divides A in $\mathscr{P}_{\text{fin}}(M)$.

Let us take a look at an example.

Example 3.4. Consider the numerical monoid $M := \{0\} \cup \mathbb{N}_{\ge 2}$, and then let $\mathscr{P}$ be the power monoid of M. Set $S := \{4, 5, 6\}$ and observe that the only common divisor of S in M is the zero element. In addition, after setting $A := \{6, 7, 8, 9\}$ and $B := \{2, 3\}$, we obtain the following:

$$B + S = \{2, 3\} + \{4, 5, 6\} = \{6, 7, 8, 9\} = A,$$

and so A, B, and S satisfy the condition (a). In addition, with the choice of $d = 4$ we guarantee that the condition (b) is also satisfied: indeed,

$$B + \{d\} = \{2, 3\} + \{4\} = \{6, 7\},$$

which divides A in the power monoid $\mathscr{P}_{\text{fin}}(M)$ because $\{6, 7\} + \{0, 2\} = \{6, 7, 8, 9\} = A$.

4. The MCD-Finite Property

In this section we study the MCD-finite property and, as our primary results, we establish the ascent of the MCD-finite property in both the setting of power monoids and the setting of polynomial extensions.

4.1. A Class of Rank-One MCD-finite Monoids.

We start by taking a deeper look to a class of atomic MCD rank-one monoids introduced in [19] and recently considered in [26]. Let P be an infinite set of primes, and let $(p_n)_{n\ge 1}$ be the strictly increasing sequence with underlying set P. Then consider the Puiseux monoid M_P defined as follows:

$$M_P := \Big\langle \frac{1}{p_n p_{n+2}} : n \in \mathbb{N} \Big\rangle. \tag{4.1}$$

Following [26], we call the monoid M_P the 2-*prime reciprocal monoid* induced by the set of primes P. It is not hard to verify that the monoid M_P is atomic. Indeed, it was proved in [19, Proposition 3.10] that M is strongly atomic, and it was proved in [26, Theorem 5.3] that M_P is an MCD monoid. One can readily verify that

$$\mathscr{A}(M_P) = \Big\{ \frac{1}{p_n p_{n+2}} : n \in \mathbb{N} \Big\}. \tag{4.2}$$

On the other hand, the monoid M_P does not satisfy the ACCP. Indeed, as shown in [19, Example 3.9], for each $n \in \mathbb{N}$, the following identity

$$\frac{1}{p_n} = \frac{1}{p_{n+2}} + (p_{n+2} - p_n)\frac{1}{p_n p_{n+2}}$$

holds, which implies that $\frac{1}{p_n} + M_P \subseteq \frac{1}{p_{n+2}} + M_P$, whence $(\frac{1}{p_{2n+1}} + M_P)_{n\ge 0}$ and $(\frac{1}{p_{2n}} + M_P)_{n\ge 1}$ are ascending chains of principal ideals that do not stabilize in M_P.

Next, we find a canonical sum decomposition for elements inside M_P. Although the sum decomposition we proposed is similar to that given in the proof of [26, Theorem 5.2], the one we describe here is slightly more refined and so we can prove that is unique.

Proposition 4.1. *Let P be an infinite set of primes, and let M_P be the 2-prime reciprocal monoid induced by P. Each $r \in M$ can be written as follows:*

$$r = c_0(r) + \frac{n_1(r)}{p_1} + \frac{n_2(r)}{p_2} + \sum_{i=1}^{N} c_i(r) \frac{1}{p_i p_{i+2}}, \tag{4.3}$$

where $N \in \mathbb{N}$, $c_0(r) \in \mathbb{N}_0$, $n_1(r) \in [\![0, p_1 - 1]\!]$, $n_2(r) \in [\![0, p_2 - 1]\!]$, $c_N(r) \in \mathbb{N}$, and $c_i(r) \in [\![0, p_{i+2} - 1]\!]$ for every $i \in [\![1, N]\!]$.

Proof. Fix $r \in M_P$. It is clear that if r is a positive integer, then we can decompose r as in (4.3) by setting $N = 0$ and $n_1(r) = n_2(r) = 0$. Therefore we assume that $r \notin \mathbb{N}_0$. Observe that if a prime p divides the denominator of r, then $p \in P$. Let p_n be the largest prime that divides the denominator of r, and set $N = n - 2$. Now write

$$r = d_0 + \frac{n_1}{p_1} + \frac{n_2}{p_2} + \sum_{i=1}^{N'} d_i \frac{1}{p_i p_{i+2}}$$

for some initial triple $(d_0, n_1, n_2) \in \mathbb{N}_0 \times [\![0, p_1 - 1]\!] \times [\![0, p_2 - 1]\!]$ and some index $N' \in \mathbb{N}_0$ such that each of the coefficients $d_i \in [\![0, p_{i+2} - 1]\!]$ for $i \in [\![1, N']\!]$, with $d_1, \dots, d_{N'} \in \mathbb{N}_0$ and $d_{N'} > 0$. We can further assume that the index N' has been taken minimum such that $d_{N'} > 0$. Let us argue the following claim.

CLAIM. $N' \le N$.

PROOF OF CLAIM. Assume, by way of contradiction, that $N' > N$. Then $p_{N'+2}$ does not divide the denominator of r, and so $d_{N'}$ must be divisible by $p_{N'+2}$: in this case, we can write $d_{N'} = dp_{N'+2}$, decreasing $d_{N'}$ to 0 and increasing $d_{N'-2}$ by $dp_{N'-2}$, we get another representation of r with largest term decreased, contradicting the minimality of N'. Thus, the claim is established.

Hence we have argued that every element $r \in M_P$ that is not an integer can be decomposed as in (4.3). To show uniqueness, suppose that r has a decomposition as that in (4.3), and let

$$r = c_0 + \frac{k_1}{p_1} + \frac{k_2}{p_2} + \sum_{i=1}^{N} c_i \frac{1}{p_i p_{i+2}} \tag{4.4}$$

be another such decomposition of r, which means that $(c_0, k_1, k_2) \in \mathbb{N}_0 \times [\![0, p_1 - 1]\!] \times [\![0, p_2 - 1]\!]$ and $c_1, \dots, c_N \in \mathbb{N}_0$ with $c_N > 0$. After subtracting both sum decompositions, we obtain the following equality:

$$0 = (c_0 - d_0) + \frac{n_1 - k_1}{2} + \frac{n_2 - k_2}{2} + \sum_{i=1}^{N} (c_i - d_i) \frac{1}{p_i p_{i+2}}.$$

Taking p_{N+2}-valuation on both sides of the resulting equality, we find that $c_N - d_N = 0$. Repeating this by taking p_i-valuations for each $i \in [\![1, N + 2]\!]$, we obtain that $n_1 = k_1$, $n_2 = k_2$, and $c_i = d_i$ for every $i \in [\![0, N]\!]$, implying the uniqueness of the desired sum decomposition. □

We begin by proving that the 2-prime reciprocal monoids introduced in (4.1) have the MCD-finite property. Before doing so, it is convenient to introduce the notion of a maximum term. Let P be an infinite set of primes, and let M_P be the 2-prime reciprocal monoid induced by P. Choose an element $r \in M_P$. If r has canonical sum decomposition

$$r = c_0(r) + \frac{n_1(r)}{p_1} + \frac{n_2(r)}{p_2} + \sum_{i=1}^{N} c_i(r) \frac{1}{p_i p_{i+2}}$$

as in (4.3), then the integer N is called the *maximum term* of r in M_P. We are in a position to prove that every 2-prime reciprocal monoid has the MCD-finite property.

Theorem 4.2. *Let P be an infinite set of primes, and let M_P be the 2-prime reciprocal monoid induced by P. Then M_P is an MCD-finite monoid.*

Proof. Let S be a nonempty finite subset of M_P. Let N be an integer such that for every $n \in \mathbb{N}$ with $n > N$, the prime p_n does not divide the denominator of any element in S. Thus, let $s_1, \dots, s_n$ be the elements of S, and assume that $s_1 < \cdots < s_n$. For each $i \in [\![1, n]\!]$, let

$$s_i = c_0(s_i) + \frac{n_1(s_i)}{p_1} + \frac{n_2(s_i)}{p_2} + \sum_{j=1}^{N} c_j(s_i) \frac{1}{p_j p_{j+2}}$$

be the canonical sum decomposition of the element s_i in M_P. Also, let d be the largest common divisor of S whose canonical sum decomposition has maximum term at most N. Suppose that $S - d$ had a positive common divisor in M_P, and let $\frac{1}{p_\ell p_{\ell+2}}$ be the largest atom that is a common divisor of $S - d$. First, if $\ell \le n$, then $d + \ell$ is a common divisor of S with maximum term at most ℓ, contradicting the maximality of d. Next, suppose $\ell > n$, and fix $s \in S$. Let m be the maximum term in the canonical sum decomposition of $s - d$, and set $a := \frac{1}{p_m p_{m+2}}$. Then, if $m > \ell$, the denominator of $s - d$ is not divisible by p_{m+2}, implying that $c_m(s)$ is a multiple of p_m, and it could be replaced by $\frac{1}{p_m p_{m-2}}$ like the process described above. This contradicts maximality, so $m = \ell$. Thus, since the denominator of $s - d$ is not divisible by $p_{\ell+2}$, we again can instead write $s - d$ with the atom $\frac{1}{p_\ell p_{\ell-2}}$ in its representation, contradicting the maximality of $\frac{1}{p_\ell p_{\ell+2}}$. Thus $S - d$ has no common divisor and so we conclude that M_P is an MCD monoid.

As in the previous paragraph, let S be a nonempty finite subset of M_P and let d be an MCD of S in M_P. Let N be the maximum of the maximum term of each element in S. We proceed to establish the following claim.

Claim 2. The maximum term of d is at most N.

Proof of Claim 2. Suppose that the maximum term of d was an integer $N' > N$. Fix $s \in S$. The maximum term of $s - d$ must not be less than N' since otherwise the maximum term of $(s - d) + d$ is N'. Similarly, the maximum term of $s - d$ must not be greater than N' as otherwise the maximum term of $(s - d) + d$ is greater than N' as well. Thus, $s - d$ has maximum term N', so $\frac{1}{p_{N'} p_{N'+2}} \mid_{M_P} s - d$ for each $s \in S$, implying that d is not an MCD of

S in M_P. Thus, any MCD d of S can be written as

$$d = c_0(d) + \frac{n_1(d)}{p_1} + \frac{n_2(d)}{p_2} + \sum_{j=1}^{N} c_j(d) \frac{1}{p_j p_{j+2}}.$$

However, $d \le \min S$ is required as well, so $n_1(d), n_2(d)$, and each $c_j(d)$ is bounded for every index $j \in [\![0, N]\!]$. Hence only finitely many such d exist. We can therefore conclude that M_P is an MCD-finite monoid. □

4.2. **Ascent of the MCD-Finite Property.** We proceed to prove that the MCD-finite property ascends to both power monoids and restricted power monoids.

Theorem 4.3. *Let M be an MCD-finite monoid. Then the following statements hold.*

(1) *$\mathscr{P}_{\text{fin}}(M)$ is an MCD-finite monoid.*

(2) *$\mathscr{P}_{\text{fin},\mathscr{U}}(M)$ is an MCD-finite monoid.*

Proof. We assume that M is an additive monoid.

(1) Suppose, towards a contradiction, that the power monoid $\mathscr{P}_{\text{fin}}(M)$ is not MCD-finite. Then we can pick $S_1, \dots, S_n \in \mathscr{P}_{\text{fin}}(M)$ such that the subset $\mathscr{S} := \{S_1, \dots, S_n\}$ of $\mathscr{P}_{\text{fin}}(M)$ has infinitely many pairwise non-associate MCDs in $\mathscr{P}_{\text{fin}}(M)$, that is, $\text{mcd}_{\mathscr{P}_{\text{fin}}(M)}(\mathscr{S})$ has infinite cardinality. As $\mathscr{S}$ consists of finite sets, we can pick $\ell_0 \in \mathbb{N}$ large enough so that

$$\ell_0 \ge n \max\{|S_i| \colon i \in [\![1, n]\!]\}.$$

Fix $U \in \text{mcd}_{\mathscr{P}_{\text{fin}}(M)}(\mathscr{S})$, and notice that for each $i \in [\![1, n]\!]$, the fact that U divides S_i in $\mathscr{P}_{\text{fin}}(M)$ ensures the existence of an element $P_i \in \mathscr{P}_{\text{fin}}(M)$ such that

$$U + P_i = S_i. \tag{4.5}$$

Now for each $i \in [\![1, n]\!]$, one can consider the following family of sets:

$$\mathscr{U}_i := \{U + \{p\} : p \in P_i\}.$$

Observe also that $|\mathscr{U}_i| \le |P_i|$ for any choice of $i \in [\![1, n]\!]$. Now fix $u \in U$ and notice that for any index $i \in [\![1, n]\!]$, the inclusion $\{u\} + P_i \subseteq U + P_i = S_i$ implies that $|P_i| \le |S_i|$. After combining the inequalities in the last two sentences, we obtain that $|\mathscr{U}_i| \le |S_i| \le \ell_0$. Thus, for each $i \in [\![1, n]\!]$, an arbitrary element of $\mathscr{U}_i$ has the form $U + \{p\}$ for some $p \in P_i$ and so $U + \{p\} \subseteq U + P_i = S_i$, which in turn implies that $|U + \{p\}| \le |S_i| \le \ell_0$. Therefore $\mathscr{U}_i$ is a family of sets with a bounded number of elements, there are finitely many possible $\mathscr{U}_i$. Therefore there are finitely many n_0-tuples of $\mathscr{U}_i$.

As $\text{mcd}_{\mathscr{P}_{\text{fin}}(M)}(\mathscr{S})$ is an infinite set, it follows from the Pigeonhole Principle, that there exists a set $\mathscr{U}$ with infinitely many non-associate MCDs of $\mathscr{S}$ in $\mathscr{P}_{\text{fin}}(M)$ corresponding to the same n_0-tuple of $\mathscr{U}_i$.

Fix $U \in \mathscr{U}$. Since U is an MCD of $\mathscr{S}$ in $\mathscr{P}_{\text{fin}}(M)$, if $d \in M$ divided all of the elements of $P := \bigcup_{i=1}^{n} P_i$, then $U + \{d\}$ would be a common divisor of $\mathscr{S}$, and the maximality of U would imply that $d \in M^\times$. For a fixed index $i \in [\![1, n]\!]$, we now split up the P into the new elements $p_1, \dots, p_m$ of M, and also split up the n_0-tuple of $\mathscr{U}_i$ into the sets $A_1, \dots, A_m$. From Equation (4.5), we obtain the following equation:

$$A_j = U + \{p_j\}. \tag{4.6}$$

Then, since $\{p_1, \dots, p_m\}$ has no common divisors in M, the set U must be an MCD of the the elements $A_1, \dots, A_m$. Without loss of generality, we assume that the A_j and p_j are pairwise distinct and non-associate.

Let n be the cardinality of U and take $u_1, \dots, u_n \in M$ such that $U = \{u_1, \dots, u_n\}$. Note that $|A_1|= |U|= n$, so n is constant across all such MCDs U. Also define $a_{jk} := u_k + p_j$ for all $k \in [\![1, n]\!]$ and $j \in [\![1, m]\!]$. From Equation (4.6), we have $A_j = \{a_{j1}, a_{j2}, \dots, a_{jn}\}$. Now, we "swap rows and columns" and iterate over k instead of over j and define

$$B_k := \{a_{1k}, a_{2k}, \dots, a_{mk}\} = u_k + Q, \tag{4.7}$$

where $Q := \{p_j : j \in [\![1, m]\!]\}$. From our previous argument, we can deduce that there is no $d \in M \setminus M^\times$ such that $d \mid_M p_j$ for each $j \in [\![1, m]\!]$. Also, since there are infinitely many such non-associate U, there are infinitely many corresponding non-associate Q since $Q = B_k - u_k$ and, for every pair of distinct U_1, U_2, there must exist a k such that the corresponding u_k differ.

Now, we will construct a set of elements in M that have infinitely many MCDs. Let U_r be an infinite sequence of non-associate U satisfying the condition. We denote the corresponding u_k as u_{kr}, the corresponding Q as Q_r, and the p_j as p_{jr}. Fix U_0. Since $u_k \in M$ and B_k is independent of U, by Equation (4.7), we have $B_k = u_k + Q_0 = u_{kr} + Q_r$, so setting $g_r := u_k - u_{kr} \in \mathscr{G}(M)$ implies $Q_r = g_r + Q_0$. Fix $a := a_{11}$. We claim that the set $a + Q_0$ has infinitely many MCDs of the form

$$m_r := a + p_{10} - p_{1r}.$$

First, note that $m_r = a - g_r$ and so $(a+Q_0)-m_r = (a+Q_0)-(a-g_r) = Q_0+g_r = Q_r \subseteq M$. Also, from the previous paragraph, we have that the elements of Q_r have no nonunit common divisor, so it remains to show that m_r are elements of M and non-associate.

For the first part, note that $m_r = (a-p_{1r})+p_{10} = u_{1r}+p_{10} \in M$. To prove non-associate, note that $m_r \sim m_s$ if and only if $p_{1r} \sim p_{1s}$. However, this would imply that $g_r \sim g_s$ so $Q_r \sim Q_s$, contradiction as we assumed the Q_i are non-associate. Thus, these are infinitely many MCDs in M, contradiction.

(2) Once again, assume that M is an MCD-finite monoid, and set $\mathscr{P} := \mathscr{P}_{\text{fin},\mathscr{U}}(M)$. To argue that $\mathscr{P}$ is an MCD-finite monoid, let S be a nonempty finite subset of $\mathscr{P}$. If there does not exist an MCD of S, it is trivially MCD-finite. Otherwise, let A be an element of S, and P be an MCD of S. Note that $P \mid_{\mathscr{P}} A$, so there exists a $C \in \mathscr{P}$ such that $A = P + C$. Since $C \in \mathscr{P}$, there exists a unit $u \in M$ such that $u \in C$, so $u + P \subseteq C + P = A$. Then $P \subseteq -u + A$. Let $\mathscr{F}$ denote the family of subsets of A. Since $A \in \mathscr{P}$, there are finitely many elements in A, so there are finitely many elements of $\mathscr{F}$. Note that P is associate to an element of $\mathscr{F}$, and denote this element by $f(P)$. If $f(P_1) = f(P_2) = T$, then $P_1 \sim T \sim P_2$ so $P_1 \sim P_2$, so f is injective up to associativity. Since the image of f is a subset of $\mathscr{F}$ and $\mathscr{F}$ is finite, there are finitely many MCDs of S up to associates. Hence $\mathscr{P}$ is an MCD-finite monoid, which concludes our proof. □

In the proof of Theorem 4.3 we have indeed proved a statement that is stronger than the ascent of the MCD-finite property to finitary power monoids. With the notation as in the mentioned proof, we say that D in $\mathscr{P}_{\text{fin}}(M)$ is a *p-MCD* of a finite nonempty subset $\mathscr{S}$ of $\mathscr{P}_{\text{fin}}(M)$ provided that the elements of $\bigcup_{T \in S}(T - D)$ have no nonunit common divisors or, equivalently, that there does not exist any element $m \in M \setminus M^\times$ such that the singleton $\{m\}$ divides $\bigcup_{T \in S}(T - D)$ in $\mathscr{P}_{\text{fin}}(M)$. Observe that the p-MCD property is a generalization of the MCD property, as $D + \{m\}$ is a common divisor if such a singleton $\{m\}$ exists.

Corollary 4.4. *If M is an MCD-finite monoid, then $\mathscr{P}_{\text{fin}}(M)$ is p-MCD-finite.*

Proof. If a finite nonempty subset $\mathscr{S}$ of $\mathscr{P}_{\text{fin}}(M)$ has finitely many p-MCDs, then $\mathscr{S}$ has only finitely many MCDs. Assume that M is an MCD-finite monoid. If $\mathscr{P}_{\text{fin}}(M)$ is an MCD monoid, then $\mathscr{P}_{\text{fin}}(M)$ is a p-MCD monoid, whence, or p-MCD-finite implies MCD-finite. □

4.3. **The IDF Property Ascends Over the Class of MCD-finite Monoids.** In this section, we will establish two main results. First, we argue the ascent of the IDF property to power monoids provided that the ground monoid is MCD-finite. This result mimics the direct implication of Theorem 5.1. Then we establish a result parallel to the reverse implication of Theorem 5.1 for power monoids over the class of linearly orderable monoids.

In parallel to this result, we prove in this section that the IDF property ascends to power monoids over the class of linearly orderable MCD-finite monoids.

Theorem 4.5. *Let M be a linearly orderable monoid that is also an MCD-finite monoid. If M is an IDF monoid, then $\mathscr{P}_{\text{fin}}(M)$ is also an IDF monoid.*

Proof. Let M be a linearly orderable MCD-finite monoid, and further assume that M has the IDF property. As M is a linearly orderable monoid, we obtain that the subset $\mathscr{S}_M$ consisting of all the singleton subsets of M is a divisor-closed submonoid of $\mathscr{P}_{\text{fin}}(M)$. Thus, every unit of $\mathscr{P}_{\text{fin}}(M)$ must belong to $\mathscr{S}_M$ and so the group of units of both $\mathscr{P}_{\text{fin}}(M)$ and $\mathscr{S}_M$ is

$$\mathscr{U} := \{\{u\} : u \in \mathscr{U}(M)\},$$

while the set of atoms of $\mathscr{S}_M$ is that consisting of all the atoms of $\mathscr{P}_{\text{fin}}(M)$ that belongs to $\mathscr{S}_M$. It suffices to fix an element S of the power monoid $\mathscr{P}_{\text{fin}}(M)$ and then argue that the set $\mathscr{A}(S)$ consisting of all the atoms of $\mathscr{P}_{\text{fin}}(M)$ that divides S is finite up to associates. It suffices to argue that the set $\mathscr{A}_1(S)$ (resp., $\mathscr{A}_{\geq 2}(S)$) consisting of all the singletons (resp., non-singletons) in $\mathscr{A}(S)$ is finite up to associates.

To argue that $\mathscr{A}_1(S)$ is finite up to associates, fix $s_0 \in S$ and note that for any $a \in M$ such that the singleton $\{a\}$ divides S in $\mathscr{P}_{\text{fin}}(M)$, it follows that $\{a\}$ divides $\{s_0\}$. This, along with the fact that the singleton that are atoms in $\mathscr{P}_{\text{fin}}(M)$ are also atoms in $\mathscr{S}$, ensures that the set consisting of all singleton subsets of M that are atoms in $\mathscr{P}_{\text{fin}}(M)$ dividing S is

$$\mathscr{A}_1(S) := \{\{a\} : a \in \mathscr{A}(M) \text{ and } a \mid_M s_0\}.$$

As M is an IDF monoid, the set $\{a \in \mathscr{A}(M) : a \mid_M s_0\}$ is finite up to associates and so the fact that M and $\mathscr{S}_M$ are canonically isomorphic ensures that the set $\mathscr{A}_1(S)$ is finite up to associates in $\mathscr{P}_{\text{fin}}(M)$. Consider the function $\mu\colon 2^S/\mathscr{U} \times (M/\mathscr{U}(M)) \to \mathscr{P}_{\text{fin}}(M)/\mathscr{U}$ defined by:

$$\mu(A + \mathscr{U}, \beta + \mathscr{U}(M)) := A - \{\beta\} + \mathscr{U},$$

where β is an MCD of A. To check that μ is a well-defined map, take A_1, A_2 and β_1, β_2 such that $A_2 = A_1 + \{u_1\}$ and $\beta_2 = \beta_1 + u_2$. Then,

$$\begin{aligned}\mu(A_1 + \mathscr{U}, \beta_1 + \mathscr{U}(M)) &= A_1 - \{\beta_1\} + \mathscr{U} \\ &= A_1 - \{\beta_1\} + \{u_1\} - \{u_2\} + \mathscr{U} \\ &= A_2 - \{\beta_2\} + \mathscr{U} \\ &= \mu(A_2 + \mathscr{U}, \beta_2 + \mathscr{U}(M)).\end{aligned}$$

Now, we claim that $\mathscr{A}_{\geq 2}(S)$ is contained inside the range of the function. Take $A \in \mathscr{A}_{\geq 2}(S)$. Then, there exists some finite nonempty subset B of M such that $S = A + B$ in $\mathscr{P}_{\text{fin}}(M)$, and for some $\beta \in B$, we have $A + \beta \subseteq S$. Since A is an atom of $\mathscr{P}_{\text{fin}}(M)$, none of the singletons $\{m\}$ can divide A in $\mathscr{P}_{\text{fin}}(M)$ when $m \in M \setminus \mathscr{U}(M)$, whence β is an MCD of $A + \beta$ in M. So, $A + \mathscr{U} = \mu(A + \beta + \mathscr{U}, \beta + \mathscr{U}(M))$, and is contained inside the range. Since the domain of the function is finite as each subset of S has finitely many MCDs up to associates, the range must also be finite. Since $\mathscr{A}_{\geq 2}(S)$ is a subset of the range, it must be finite. □

It is worth emphasizing that the MCD-finite restriction is not superfluous in the statement of Theorem 4.5: it was proved in [11, Theorem 6.5] that the IDF property does not ascend to power monoids over the class of linearly orderable monoid. We proceeed to exhibit a linearly orderable monoid that is not an MCD-finite monoid.

Example 4.6. Consider the additive submonoid $M := \{0\} \cup \mathbb{Q}_{\geq 1}$ of $\mathbb{Q}$. Fix $q, r \in M$ such that $2 < q < r$. Let us argue that the subset $S := \{q, r\}$ of M has infinitely many MCDs (and so it does not have any GCD). Observe that every element in the subset $D_S := [1, q-1] \cap \mathbb{Q}$ of M is a common divisor of S: indeed, for each $d \in D_S$, we see that $q - d \geq q - (q-1) = 1$ and so $r - d > q - d \geq 1$, whence $d \mid_M q$ and $d \mid_M r$. Now fix $\epsilon \in \mathbb{R}$ with $0 < \epsilon < q - 2$. Because $1 < (q-1) - \epsilon < q - 1$, it follows that the infinite subset $M_S := ((q-1) - \epsilon, q-1) \cap \mathbb{Q}$ of M is contained in D_S, and so each $m \in M_S$ is a common divisor of S in M. On the other hand, for each pair $(m, q') \in M_S \times (M \setminus \{0\})$, the inequalities $m + q' \geq m + 1 > q$ hold and so $m + q' \nmid_M q$, whence m is an MCD of S in M. Thus, every element in M_S is an MCD of S, which implies that S has infinitely many non-associate MCDs in M. As a consequence, M is a linearly orderable monoid (under the standard order) that is not an MCD-finite monoid.

The following lemma is well known, but we include its proof here for the sake of completeness.

Lemma 4.7. *Let M be a linearly orderable monoid. Then for any $A, B \in \mathscr{P}_{\text{fin}}(M)$, the following holds:*

$$|A + B| \geq |A| + |B| - 1.$$

Proof. Let $A = \{a_1, a_2, \dots, a_k\}$ where $k \geq 1$ and $a_1 < a_2 < \cdots < a_k$, and let $B = \{b_1, b_2, \dots, b_n\}$, where $n \geq 1$ and $b_1 < b_2 < \cdots < b_n$. Then note that $a_1 + b_j \in A + B$ for all $j \in [\![1, n]\!]$, and $a_i + b_n \in A + B$ for all $i \in [\![1, k]\!]$. Note that, since M is linearly orderable,

$$a_1 + b_1 < a_1 + b_2 < \cdots < a_1 + b_n < a_2 + b_n < \cdots < a_k + a_n.$$

This implies $|A + B| \geq n + k - 1 = |A| + |B| - 1$, as desired. □

Theorem 4.8. *Let M be a linearly orderable IDF monoid. If the finitary power monoid of $\mathscr{P}_{\text{fin}}(M)$ is IDF, then M is MCD-finite.*

Proof. Towards a contradiction, suppose that M is not MCD-finite. Then there exist $n \in \mathbb{N}$ and $a_1, a_2, \dots, a_n$ with infinitely many MCDs, which implies there are infinitely many non-associate $\{b_i : i \in I\}$ (for an infinite set I) such that $a_k - b_i$ for $k \in [\![1, n]\!]$ have no nonunit common divisor. Set

$$S_i := \{a_k - b_i : k \in [\![1, n]\!]\}.$$

Note that $S := \{a_1, \dots, a_n\} = \{b_i\} + S_i$. Since the singleton $\{t\}$ is an atom (resp. unit) of $\mathscr{P}_{\text{fin}}(M)$ if and only if t is an atom (resp. unit) of M, note that each nonunit divisor of S_i has cardinality at least 2. Also, for any nonunits $T_k \in \mathscr{P}_{\text{fin}}(M)$ such that $S_i = \sum_{k=1}^m T_k$, repeatedly applying Lemma 4.7 implies that

$$|S_i| = |T_1| + |T_2| + \cdots + |T_m| - (m-1) \geq 2m - (m-1) = m + 1$$

so $m \leq |S_i| - 1$. Therefore, every factorization of S_i has bounded length, and by taking the longest factorization we obtain that S_i can be factored into atoms of $\mathscr{P}_{\text{fin}}(M)$.

Since $\mathscr{P}_{\text{fin}}(M)$ is IDF, there are finitely many non-associate atoms dividing S. Let these atoms be $\{d_1, d_2, \dots, d_m\}$, up to associates. Since every atom dividing S_i also divides S, every S_i can be expressed as a finite sum of atoms d_j for $j \in [\![1, m]\!]$, up to associativity. However, for all $i \in I$, there exist $r_j \in \mathbb{N}_0$ such that $S_i \sim_{\mathscr{P}_{\text{fin}}(M)} \sum_{j=1}^m d_j r_j$, and Lemma 4.7 implies $r_j \leq n$ for all $j \in [\![1, m]\!]$. Therefore there are finitely many such sums $\sum_{j=1}^m d_j r_j$ that correspond to at least one S_i, so there exist two S_i that are associate. Since $S = b_i + S_i$, this implies the two b_i are associate, which is a contradiction as we assumed the b_i are not associate. Hence M is MCD-finite, which concludes the proof. □

5. Ascent of the IDF and MCD-finite Properties to Polynomial Extensions

In this final section, we provide two results that should shed some light upon the ascent of both the MCD-finite and the IDF properties to polynomial extensions. First, we establish the ascent of the MCD-finite to polynomial extensions. Then we prove that the PSP property is stronger than the MCD-finite property, and deduce from this fact that the ascent of the IDF property to polynomial extensions over the class of PSP domains [22, Theorem 3.2] is a special case of the ascent of the IDF property to polynomial extensions over the class of MCD-finite domains [12, Theorem 2.1].

5.1. Ascent of MCD-Finite Property – Polynomial Extensions.

The MCD-finite property was introduced by Eftekhari and Khorsandi [12] back in 2018, who were motivated by the study of the ascent of the IDF property to polynomial extensions. The primary result of their paper is the following.

Theorem 5.1. [12, Theorem 2.1] *For an integral domain R, the following conditions are equivalent.*

(a) *R is an MCD-finite IDF domain.*

(b) *$R[x]$ is an IDF domain.*

However, in their paper they did not address the potential ascent of the MCD-finite property to polynomial extension. Although the main algebraic structures of this paper are power monoids, we conclude this section proving the ascent of the MCD-finite property to polynomial extensions.

Theorem 5.2. *Let R be an MCD-finite integral domain. Then the polynomial domain $R[x]$ is also an MCD-finite domain.*

Proof. Let F be the field of fractions of R. Since $F[x]$ is a Euclidean domain, it is a UFD, whence every nonzero polynomial in $F[x]$ has finitely many divisors up to associates. Since the units of $F[x]$ are precisely the scalars in $F^\times$ and the divisibility relation $A(x) \mid_{R[x]} B(x)$ implies $A(x) \mid_{F[x]} B(x)$ for all $A(x), B(x) \in R[x]$, every nonzero polynomial in $R[x]$ has finitely many divisors in $R[x]$ up to multiplication by an nonzero scalar.

Assume, by way of contradiction, that $R[x]$ is not an MCD-finite domain. Then there exists a finite subset S of $R[x]^*$ having infinitely many MCDs in $R[x]$ up to associates. Set $n := |S|$ and let $F_1(x), \dots, F_n(x) \in R[x]$ be the polynomials in S. Let D denote a maximal set of non-associate representatives of the set of divisors of $F_1(x)$ in $F[x]$, and consider the map $\phi\colon S \to S/{\sim_D}$ defined as follows: $\phi\colon m(x) \mapsto [m(x)]_{F[x]}$. As D is finite, it follows from the Pigeonhole Principle that there exist infinitely many MCDs of S that are associate in $F[x]$.

Let $A(x)$ be an MCD of S in $R[x]$, and let $(r_n)_{n\ge 1}$ be a sequence of scalars in F such that $r_nA(x)$ are also MCDs of S in $R[x]$. Now for each index $k \in [\![1,n]\!]$, take a polynomial $B_k(x) \in R[x]$ such that $F_k(x) = A(x)B_k(x)$. Let C denote the subset of R that is the union of the sets of coefficients of the polynomials $B_k(x)$ for every $k \in [\![1,n]\!]$. Observe that because $A(x)$ is an MCD of S in $R[x]$, the set C has no nonunit common divisors in R. Similarly, the set $\frac{1}{r_m}C$ has no nonunit common divisor in R. Let a_1, b_1, and f_1 denote the leading coefficients of the polynomials $A(x)$, $B_1(x)$, and $F_1(x)$, respectively. Thus, $f_1 = a_1b_1$.

Finally, we proceed to construct infinitely many MCDs of the finite subset f_1C of R, from which we can obtain our desired contradiction. For every $m \in \mathbb{N}$, the equalities

$$f_1r_m = a_1b_1r_m = (a_1r_m)b_1$$

holds in R because a_1r_m is the leading coefficient of $r_mA(x) \in R[x]$. In addition, the equality $f_1C(f_1r_m)^{-1} = \frac{1}{r_m}C$ holds in R and they have no nonunit common divisors, whence f_1r_m is an MCD of f_1C in R. From the fact that the elements r_m's are not pairwise associates in R, we can now deduce that the elements f_1r_m's are not pairwise associates in R neither, whence they are infinitely many MCDs of the subset f_1C of R, which concludes our proof. □

5.2. **The PSP and the MCD-finite Properties.** In this final section we prove that the PSP property is stronger than the q-GCD property, and deduce that the class consisting of all PSP monoids (resp., domains) is larger than the class consisting of all MCD-finite monoids (resp., domains).

Proposition 5.3. *Every PSP monoid is a q-GCD monoid.*

Proof. Let M be a PSP monoid. Assume, towards a contradiction, that M is not a q-GCD. Then there exists a nonempty finite subset S of M possessing two non-associate MCDs in M, namely, d_1 and d_2. Now set $S_1 := S/d_1$, and then consider the subset

$$I := \bigcup_{s \in S_1} sM.$$

of M, which is clearly an ideal of M. We claim that the only principal ideal of M containing I is M itself. To argue this, suppose that $I \subseteq bM$ for some $b \in M$. Then $s \in bM$ for every $s \in S_1$ and, therefore, b is a common divisor of S_1. Now the fact that d_1 is an MCD of S ensures that $b \in M^\times$. Thus, $bM = M$, and so the only principal ideal of M containing I is M itself. Hence I is primitive ideal.

We proceed to argue that I is not super-primitive. To do so, fix $a \in I$, and then take $s_a \in S_1$ such that $a \in s_a M$ or, equivalently, $\frac{a}{s_a} \in M$. In addition, $\frac{d_1 s_a}{d_2} \in M$ because $s_a \in S_1$ and d_2 is a common divisor of $S = d_1 S_1$. Therefore $\frac{d_1}{d_2} a = \frac{a}{s_a} \frac{d_1 s_a}{d_2} \in M$. As a was take to be an arbitrary element in I, it follows that $\frac{d_1}{d_2} I \subseteq M$. As a consequence,

$$\frac{d_1}{d_2} \in (M : I) = I^{-1}. \tag{5.1}$$

From the fact that d_1 and d_2 are non-associate MCDs of S in M, we deduce that $d_2 \nmid_M d_1$ or, equivalently, $\frac{d_1}{d_2} \notin M$. This, along with (5.1), guarantees that $\{1\} \cdot I^{-1} \not\subseteq M$. This in turn implies that $1 \notin (M : I^{-1}) = I_v$. Therefore the divisorial ideal I_v is a proper ideal of M and, as a consequence, the ideal I is not super-primitive.

Then we have proved that the ideal I is primitive but not super-primitive, which contradicts the fact that M is a PSP monoid. Hence we conclude that M is a q-GCD monoid. □

As an immediately consequence of Proposition 5.3 we obtain that the PSP property is stronger than the MCD-finite property.

Corollary 5.4. *Every PSP monoid (resp., domain) is an MCD-finite monoid (resp., domain).*

Corollary 5.4 reveals that the ascent of the IDF property to polynomial extensions over MCD-finite domains [12, Theorem 2.1] is stronger than the ascent of the IDF property to polynomial extensions over PSP domains [22, Theorem 3.2].

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