ON THE DOUBLE OF AN ENDYMION ALGEBRA IN CHARACTERISTIC 2

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Abstract. We consider $\mathfrak{E}_{3,-}(q)$, a Yetter-Drinfeld module with a point and a pale block of dimension 3 first introduced by Andruskiewitsch, Angiono, and Giusti in 2021. Working over an arbitrary field of characteristic 2, we compute its Nichols algebra and show that it contains, as a subalgebra, an embedded copy of the restricted Jordan plane. We then calculate its Drinfeld double, explicitly determining the universal R-matrix of this quantum group and thus finding a solution to the quantum Yang-Baxter equation. We also consider the classical $q = 1$ case, classifying all simple modules over this restricted double. Finally, following a parallel construction, we compute the reflective algebra of $\Bbbk$ with respect to the double and its associated universal K-matrix, thus finding a solution to the quantum reflection equation.

Contents

1. Introduction

Drinfeld first introduced his double construction [9] as a method to systematically recover solutions, called universal R-matrices, to the quantum Yang-Baxter equation, which has applications in quantum integrable models and knot theory. Shortly before the establishment of the double, Drinfeld [8] and Jimbo [12] independently introduced quantized enveloping algebras, which they recovered as quotients of a topological notion of this more general construction. These Drinfeld-Jimbo quantum groups motivate and relate to features of any arbitrary double.

The Drinfeld-Jimbo quantum group can be constructed from its positive part, or a Nichols algebra [15] of diagonal-type. In 2006, Heckenberger classified all finite-dimensional diagonal-type Nichols algebras [10]. This encourages studying the lesser-known Nichols algebras of group-type and the more general doubles that arise from them. In fact, through the Andruskiewitsch-Schneider lifting method [4], every pointed Hopf algebra (in characteristic 0) can be recovered from a Nichols algebra, making the classification of the latter objects particularly fundamental.

In addition to a double's positive part, its representation category is also rich. For a Hopf algebra H, the representation category of the double $D(H)$ is equivalent to both the category of Yetter-Drinfeld modules over H as well as the Drinfeld center of the monoidal category H-mod.

Parallel to the pursuit of these universal R-matrices (and the interesting doubles that arise from them) is the search for universal K-matrices, or solutions to the quantum reflection equation, which have applications in quantum integrable models with boundaries and knot theory in tori. In [6], Balagović and Kolb introduced K-matrices for any right coideal subalgebra of a Drinfeld-Jimbo quantum group. The reflective algebra construction as introduced by Laugwitz, Walton, and Yakimov [13] generalizes this notion in the same way doubles generalize Drinfeld-Jimbo quantum groups: the reflective algebra of a left comodule algebra of *any* double is equipped with a K-matrix.

In this paper, we will explore several of the aforementioned structures in a specific characteristic 2 case. We recall the necessary background in Section 2. In Section 3, we study $\mathfrak{E}_{3,-}(q)$, a Yetter-Drinfeld module first introduced by Andruskiewitsch, Angiono, and Giusti in [1]. While they worked in characteristic 0 to compute a new Nichols algebra, we work in characteristic 2 to determine its Nichols algebra. Adopting their notation, we call this algebra Endymion, in reference to its pale braiding structure. We will then compute the double of this Endymion algebra using the strategy outlined in [3]. In Section 4, we explicitly determine the universal R-matrix, the original motivation for the double construction. Though the double's size makes classifying all simple modules hard, in Section 5, we explicitly compute all irreducible representations in the classical case $q = 1$. Finally, in Section 6 we explore the parallel process, computing the reflective algebra of $\Bbbk$ with respect to the double and its associated K-matrix.

Throughout the paper, when several analogous computations arise, we present one representative example and omit the others.

2. Background and Preliminaries

2.1. **Conventions.** We utilize the Kronecker delta function $\delta_{i,j} = [i = j]$. We set $\mathbb{I}_n^m := \{k \in \mathbb{Z} \mid n \leq k \leq m\}$ and $\mathbb{I}_n := \{k \in \mathbb{Z} \mid 1 \leq k \leq n\}$. We will also adopt Sweedler notation: for a coalgebra C, the coproduct $\Delta : C \to C \otimes C$ is written as $\Delta(c) = c_{(1)} \otimes c_{(2)}$, for all $c \in C$. For a comodule M over C, the coaction $\delta : M \to C \otimes M$ is written as $\delta(m) = m_{(-1)} \otimes m_{(0)}$, for all $m \in M$. Given a Hopf algebra H, the group of group-like elements in H is denoted by $G(H) = \{g \in H : \Delta(g) = g \otimes g\}$. For $g, h \in G(H)$, the linear space of skew-primitive elements is $P_{g,h}(H) = \{x \in H : \Delta(x) = x \otimes g + h \otimes x\}$. Finally, for a universal matrix $X = X^{(1)} \otimes X^{(2)}$, the triple tensor product X_{ij} (for $1 \leq i \neq j \leq 3$) places $X^{(1)}$ in the ith tensor position, $X^{(2)}$ in the jth, and 1 in the remaining position.

We also assume that the objects introduced in this section are finite-dimensional, as that will hold for all the objects we study in this paper.

2.2. **Quasitriangular Hopf Algebras and Left Comodule Algebras.** The main objects of study are quasitriangular Hopf algebras.

Definition 2.1. A *quasitriangular Hopf algebra* is a Hopf algebra H equipped with a *universal R-matrix*, or an invertible element $R = R^{(1)} \otimes R^{(2)} \in H \otimes H$ such that

$$(\Delta \otimes id)R = R_{13}R_{23}, \quad (id \otimes \Delta)R = R_{13}R_{12}, \quad R\Delta(h)R^{-1} = h_{(2)} \otimes h_{(1)} \quad \forall h \in H.$$

The first two properties, along with the coassociativity of Δ, imply that R satisfies the *quantum Yang-Baxter equation*

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}.$$

The Drinfeld double construction systematically produces quasitriangular Hopf algebras and, with them, solutions to the quantum Yang-Baxter equation.

Definition 2.2 (Majid)**.** The *Drinfeld double* of a (finite-dimensional) Hopf algebra H, denoted $D(H)$, is a Hopf algebra whose underlying coalgebra structure is $H \otimes (H^*)^{\mathrm{op}}$. For all $h \in H$ and $f \in H^*$, the element $h \otimes f \in D(H)$ is denoted $h \bowtie f$. Multiplication in $D(H)$ is given by

$$(h \bowtie f)(h' \bowtie f') = \langle f_{(1)}, h'_{(1)} \rangle \langle f_{(3)}, S(h'_{(3)}) \rangle (hh'_{(2)}) \bowtie f' f_{(2)}.$$

The antipode map satisfies

$$S_{D(H)}(h \bowtie f) = (S(h_{(2)}) \bowtie S^{-1}(f_{(2)})) \langle f_{(1)}, S(h_{(1)}) \rangle \langle f_{(3)}, h_{(3)} \rangle.$$

Majid [14] also explicitly finds an associated R-matrix:

Proposition 2.3 ([14, Proposition 8.2])**.** *The R-matrix of $D(H)$ is $\sum_i (1_H \bowtie e^i) \otimes (e_i \bowtie 1_{H^*})$, where $\{e_i, e^i\}_i$ is a dual basis of H.*

We also consider the parallel structures cemented in [13]. As a quasitriangular Hopf algebra is equipped with a solution to the quantum Yang-Baxter equation, a quasitriangular left comodule algebra is equipped with a solution to the quantum reflection equation.

Definition 2.4. Let H be a quasitriangular Hopf algebra. A *quasitriangular* left H-comodule algebra A is a left H-comodule algebra with coaction given by $\delta : A \to H \otimes A$. It is equipped with a *universal K-matrix*, or an invertible element $K = K^{(1)} \otimes K^{(2)} \in H \otimes A$ such that,

$$(\Delta \otimes id)K = K_{23}R_{21}K_{13}R_{21}^{-1}, \quad (id \otimes \delta)K = R_{21}K_{13}R_{12}, \quad K\delta(a)K^{-1} = \delta(a),$$

for all $a \in A$.

The first property implies that K satisfies the *quantum reflection equation*:

$$R_{23}K_{13}R_{23}^{-1}K_{12} = K_{12}R_{23}K_{13}R_{23}^{-1}.$$

Further, as the double of a Hopf algebra is a quasitriangular Hopf algebra, the reflective algebra of a left comodule algebra is a quasitriangular left comodule algebra. We focus on the trivial left H-comodule algebra $\Bbbk$, as that is what we will work with in this paper.

According to [14, Example 5.30, Lemma 5.26, Corollary 6.9], we have the following definition.

Definition 2.5. Let H be a quasitriangular Hopf algebra. The reflective algebra of $\Bbbk$ with respect to H, denoted $R_H(\Bbbk)$, has the underlying vector space H^* and a H-comodule algebra structure given by, for $\zeta, \xi \in H^*$,

$$\mu_{R_H(\Bbbk)}(\zeta \otimes \xi) = \sum_{i,j} \langle \xi_{(3)}, t_i \rangle \langle \xi_{(1)}, S(t_j) \rangle \langle \zeta_{(2)}, s_i \rangle \langle \zeta_{(3)}, s_j \rangle \xi_{(2)} \zeta_{(1)},$$
$$\delta_{R_H(\Bbbk)}(\xi) = \sum_{i,j,l} \langle \xi_{(1)}, t_j \rangle \langle \xi_{(2)}, h_l \rangle \langle \xi_{(3)}, s_i \rangle s_j t_i \otimes \xi_l,$$

where $\{h_l, \xi_l\}_l$ is a dual basis of H. The quantum K-matrix is given by $\sum_l h_l \otimes \xi_l$.

2.3. Nichols Algebras and Bosonization.

In this paper, we study the double of the bosonization of a Nichols algebra in characteristic 2 and its associated structures. We call the result a *quantum group in characteristic* 2.

Definition 2.6. A *braided vector space* is a vector space V equipped with an isomorphism $c : V \otimes V \to V \otimes V$ that satisfies the *Yang-Baxter equation*:

$$(c \otimes id) \circ (id \otimes c) \circ (c \otimes id) = (id \otimes c) \circ (c \otimes id) \circ (id \otimes c).$$

Braided vector spaces with a compatible module and comodule structure can be realized over Hopf algebras.

Definition 2.7. A *Yetter-Drinfeld module* V over a Hopf algebra H is a vector space with a module structure $\lambda : H \otimes V \to V$ and comodule structure $\delta : V \to H \otimes V$ satisfying,

$$\delta(h \cdot v) = h_{(1)} v_{(-1)} S(h_{(3)}) \otimes h_{(2)} \cdot v_{(0)} \quad \forall h \in H, v \in V.$$

We denote the category of Yetter-Drinfeld modules over H by ${}^H_H\mathcal{YD}$.

The Yetter-Drinfeld structure induces a braiding $c_{V,V} : V \otimes V \to V \otimes V$ given by $c_{V,V}(v \otimes v') = v_{(-1)} \cdot v' \otimes v_{(0)}$, for all $v, v' \in V$. If $c_{V,V} = c$, we say the braided vector space (V, c) can be realized as a Yetter-Drinfeld module over H. From (V, c), and independent of any realization, we can construct a Nichols algebra.

Definition 2.8. Realize $V \in {}^H_H\mathcal{YD}$ for some Hopf algebra H. Let $I(V)$ be the largest coideal of the tensor algebra $T(V)$ contained in $\bigoplus_{n \geq 2} T^n(V)$. The *Nichols algebra* of V is $\mathfrak{B}(V) = T(V)/I(V)$.

We can compute $I(V)$ with skew-derivations. Let $\{v_i : i \in \mathbb{I}_n\}$ be a basis of V and say the coaction on H is defined by $\delta(v_i) = g_i \otimes v_i$. Define the *right skew-derivation* $\partial_i : T(V) \to T(V)$ recursively:

$$\partial_i(1) = 0, \qquad \partial_j(v_i) = \delta_{i,j}, \qquad \partial_i(uv) = \partial_i(u)(g_i \cdot v) + u\partial_i(v),$$

for all $i, j \in \mathbb{I}_n$.

Proposition 2.9 ([11, Proposition 7.3.4])**.** *The Nichols algebra is defined by*

$$\mathfrak{B}(V) = T(V)/\big\langle \bigcup_{m \geq 2} \{x \in T^m(V) : \partial_i(x) = 0 \text{ for all } i \in \mathbb{I}_n\}\big\rangle.$$

The Nichols algebra $\mathfrak{B}(V) \in {}^H_H\mathcal{YD}$ is then a braided Hopf algebra. However, $\mathfrak{B}(V)$ itself is independent of this realization (i.e the choice of H). From $\mathfrak{B}(V)$, we can use bosonization to recover a Hopf structure.

Definition 2.10. Given a braided Hopf algebra $B \in {}^{H}_{H}\mathcal{YD}$, the *bosonization* $B\#H$ is a Hopf algebra with the underlying vector space $B \otimes H$. For all $b \in B$ and $h \in H$, the element $b \otimes h \in B\#H$ is denoted $b\#h$. (Co)multiplication is given by,

$$\begin{aligned}(b\#h)(c\#k) &= b(h_{(1)} \cdot c)\#h_{(2)}k \\ \Delta(b\#h) &= b^{(1)}\#(b^{(2)})_{(-1)}h_{(1)} \otimes (b^{(2)})_{(0)}h_{(2)},\end{aligned}$$

for all $b, c \in B$ and $h, k \in H$. Here, we denote $\Delta_B(b) = b^{(1)} \otimes b^{(2)}$ and $\Delta_H(h) = h_{(1)} \otimes h_{(2)}$ to differentiate between the two comultiplications.

2.4. **The Jordan Plane.** To illustrate the structures described in the previous subsection, we consider the Jordan plane in characteristic 2 (see [5] and [7]). We will show that, as algebras, the Jordan plane is embedded within our Nichols algebra, making this example particularly applicable.

Let $(\widetilde{V}, \tilde{c})$ be a braided vector space with basis $\{v_i : i \in \mathbb{I}_2\}$ and braiding defined on the basis by:

$$\tilde{c}(v_i \otimes v_j)_{i,j\in\mathbb{I}_2} = \begin{bmatrix} v_1 \otimes v_1 & (v_1 + v_2) \otimes v_1 \\ v_1 \otimes v_2 & v_2 \otimes v_2 \end{bmatrix}.$$

We can realize $\widetilde{V} \in {}^{\Bbbk C_2}_{\Bbbk C_2}\mathcal{YD}$ where $C_2 = \langle \sigma \rangle$ is the cyclic group of order 2. The corresponding action and coaction is given by

$$\sigma \cdot v_1 = v_1, \qquad \sigma \cdot v_2 = v_1 + v_2, \qquad \delta(v_i) = \sigma \otimes v_i \quad \forall i \in \mathbb{I}_2.$$

The Nichols algebra $\mathfrak{B}(\widetilde{V})$ in characteristic 2 was computed in [7]; it is presented by generators v_1, v_2 with relations

$$v_1^2 = 0, \quad v_2^4 = 0, \quad v_2^2 v_1 + v_1 v_2^2 + v_1 v_2 v_1 = 0, \quad v_2 v_1 v_2 v_1 + v_1 v_2 v_1 v_2 = 0. \tag{2.1}$$

The set of monomials $\{v_1^{e_1} v_{12}^{e_2} v_2^{e_3} : e_1, e_2 \in \mathbb{I}_0^1, e_3 \in \mathbb{I}_0^3\}$ form a PBW basis of $\mathfrak{B}(\widetilde{V})$ where $v_{12} = v_1 v_2 + v_2 v_1$. By [7, Corollary 3.4], the bosonization $\widetilde{H} := \mathfrak{B}(\tilde{V})\#\Bbbk C_2$ is generated by v_1, v_2, and σ with relations (2.1) and

$$\sigma v_1 = v_1 \sigma, \qquad \sigma v_2 = (v_2 + v_1)\sigma, \qquad \sigma^2 = 1.$$

The coalgebra structure is given by

$$G(H) = \langle \sigma \rangle \quad \text{and} \quad P_{1,\rho}(H) = \operatorname{span}\{v_i : i \in \mathbb{I}_2\}.$$

3. Computing the Double of the Endymion Algebra

From now on, we assume that $\Bbbk$ is a field of characteristic 2. We compute the double of the bosonization of the Nichols algebra of the braided vector space $V := \mathfrak{E}_{3,-}(q), c$ as defined in [1]. Let V be the braided vector space with basis $\{x_i : i \in \mathbb{I}_4\}$ and braiding given by

$$c(x_i \otimes x_j)_{i,j\in\mathbb{I}_4} = \begin{bmatrix} x_1 \otimes x_1 & x_2 \otimes x_1 & x_3 \otimes x_1 & q_{12} x_4 \otimes x_1 \\ x_1 \otimes x_2 & x_2 \otimes x_2 & x_2 \otimes x_3 & q_{12} x_4 \otimes x_2 \\ x_1 \otimes x_3 & x_2 \otimes x_3 & x_3 \otimes x_3 & q_{12} x_4 \otimes x_3 \\ q_{21} x_1 \otimes x_4 & q_{21}(x_1 + x_2) \otimes x_4 & q_{21}(x_2 + x_3) \otimes x_4 & x_4 \otimes x_4 \end{bmatrix},$$

where $q \in \Bbbk^{\times}$ and $q_{12} = q = q_{21}^{-1}$. Here, $\{x_i : i \in \mathbb{I}_3\}$ forms a pale block of dimension 3 and $\{x_4\}$ forms a point. Suppose that q is a primitive m-th root of unity for m

odd. Then, we can realize $V \in {}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$, where $\Bbbk\Gamma := C_m \otimes C_{4m} = \langle g \rangle \otimes \langle h \rangle$. The corresponding action and coaction are given by

$$\begin{aligned}
&g \cdot x_i = x_i, \quad \forall i \in \mathbb{I}_3, && g \cdot x_4 = q_{12} x_4,\\
&h \cdot x_1 = q_{21} x_1, && h \cdot x_2 = q_{21}(x_1 + x_2),\\
&h \cdot x_3 = q_{21}(x_2 + x_3), && h \cdot x_4 = x_4,\\
&\delta(x_i) = g \otimes x_i, \quad \forall i \in \mathbb{I}_3, && \delta(x_4) = h \otimes x_4.
\end{aligned}$$

We first compute $\mathfrak{B}(V)$.

Theorem 3.1. *Let* $z_i = x_4 x_i + q_{21}(x_i + x_{i-1}) x_4$ *for* $i \in \mathbb{I}_2^3$, $w = z_2 x_3 + q_{21}(x_3 + x_2) z_2$, *and* $y = z_2 z_3 + z_3 z_2$. *Then,* $\mathfrak{B}(V)$ *is generated by* x_i, $i \in \mathbb{I}_4$, *with relations*

$$x_i^2 = 0, \qquad x_i x_j = x_j x_i, \quad \forall i \neq j \in \mathbb{I}_3, \tag{3.1}$$

$$x_4^2 = 0, \qquad x_1 x_4 = q_{12} x_4 x_1, \tag{3.2}$$

$$z_2^2 = 0, \qquad z_3^4 = 0, \tag{3.3}$$

$$z_3^2 z_2 + z_2 z_3^2 + z_2 z_3 z_2 = 0, \qquad z_3 z_2 z_3 z_2 + z_2 z_3 z_2 z_3 = 0, \tag{3.4}$$

$$z_3 w + q_{21} w z_3 = 0. \tag{3.5}$$

The set of monomials

$$\{x_1^{m_1} x_2^{m_2} x_3^{m_3} w^d z_2^{n_2} y^e z_3^{n_1} x_4^{m_4} \;:\; m_i, d, e, n_2 \in \mathbb{I}_0^1,\ n_1 \in \mathbb{I}_0^3\}$$

is a basis of $\mathfrak{B}(V)$.

Proof. We adopt a strategy analogous to that used in the characteristic 0 case, as developed in [1, Theorem 4.4].

The relations (3.1) and (3.2) follow from [1]. Note that $\delta(z_2) = \delta(z_3) = hg$, $(hg) \cdot z_2 = z_2$, and $(hg) \cdot z_3 = z_2 + z_3$, so z_2 and z_3 generate the Jordan plane (as an algebra) with skew derivatives defined by $\partial_{z_2} = \partial_4 \partial_1$ and $\partial_{z_3} = \partial_4 \partial_2$. Thus, $\partial_4 \partial_1$ and $\partial_4 \partial_2$ annihilate the relations (3.3) and (3.4). Further, $\partial_i \partial_1$ and $\partial_i \partial_2$ kill z_2 and z_3 for all $i \in \mathbb{I}_3$, so they also annihilate these relations. In particular, since all the skew derivatives of ∂_1 and ∂_2 compute to 0, the relations (3.3) and (3.4) both compute to 0 under ∂_1 and ∂_2. Finally, ∂_3 and ∂_4 kill z_2 and z_3, so they annihilate (3.3) and (3.4) as well. So, the relations (3.3) and (3.4) also hold. Lastly, it is easy to verify that ∂_1, ∂_3, and ∂_4 kill (3.5). Relations (3.1)-(3.3) imply that $x_4 z_2 = q_{21} z_2 x_4$ and $z_2 x_2 = q_{21}(x_2 + x_1) z_2$. Ultimately,

$$\begin{aligned}
x_4 w &= x_4 \left(z_2 x_3 + q_{21}(x_3 + x_2) z_2 \right)\\
&= q_{21} z_2 x_4 x_3 + q_{21} x_4 (x_3 + x_2) z_2\\
&= q_{21} z_2 \left(z_3 + q_{21}(x_3 + x_2) x_4 \right) + q_{21} \left(z_3 + z_2 + q_{21}(x_3 + x_1) x_4 \right) z_2\\
&= q_{21}(z_2 z_3 + z_3 z_2) + q_{21}^2 \left(z_2 x_3 + q_{21}(x_3 + x_2) z_2 \right) x_4 + q_{21}^2 (z_2 x_2 + q_{21}(x_2 + x_1) z_2) x_4\\
&= q_{21} y + q_{21}^2 w x_4.
\end{aligned}$$

The verification that ∂_2 annihilates (3.5) follows from this. So we have verified that all the claimed relations hold.

Now, we prove that the given basis spans the Nichols algebra. Observe that x_1 q-commutes with z_2, z_3, and w, and the following commutations also hold:

$$\begin{aligned}
&w^2 = 0, && z_2 x_2 = q_{21}(x_2 + x_1) z_2,\\
&z_2 x_3 = w + q_{21}(x_3 + x_2) z_2, && x_4 z_2 = q_{21} z_2 x_4,
\end{aligned}$$

$$\begin{aligned}
&z_3x_2 = w + q_{21}(x_2+x_1)z_3, && z_3x_3 = q_{21}(x_3+x_1)z_3,\\
&x_4z_3 = q_{21}(z_3+z_2)x_4, && wx_2 = q_{21}(x_2+x_1)w,\\
&wx_3 = q_{21}(x_3+x_2)w, && x_4w = q_{21}y + q_{21}^2wx_4,\\
&z_2w = q_{21}wz_2, && z_2z_3 = y + z_3z_2,\\
&yx_2 = q_{21}^2x_2y, && yx_3 = q_{21}^2(x_3+x_2)y + q_{21}^2x_1z_2z_3 + q_{21}wz_2,\\
&x_4y = q_{21}^2yx_4, && yz_2 = z_2y,\\
&z_3y = y(z_2+z_3), && yw = q_{21}^2wy.
\end{aligned}$$

In particular, given any element in $T(V)$, we may repeatedly swap any two adjacent variables until it follows the form of the monomials in our basis. Thus, the monomials span the algebra.

We now show that the claimed basis is linearly independent. Assume for the sake of contradiction that there exists a nontrivial linear combination of minimal degree:

$$S = \sum k_i x_1^{m_1}x_2^{m_2}x_3^{m_3}w^pz_2^{n_1}y^{n_2}z_3^{n_3}x_4^{m_4} = 0,$$

where $k_i \in \Bbbk$. Then,

$$0 = \partial_4\big(\sum k_i x_1^{m_1}x_2^{m_2}x_3^{m_3}w^dz_2^{n_1}y^ez_3^{n_3}x_4^{m_4}\big) = \sum \delta_{m_4,1}k_i x_1^{m_1}x_2^{m_2}x_3^{m_3}w^dz_2^{n_1}y^ez_3^{n_3}.$$

By the minimality of S's degree, every element in S must have $m_4 = 0$. Further,

$$\begin{aligned}
0 &= \partial_4\partial_1\big(\sum k_i x_1^{m_1}x_2^{m_2}x_3^{m_3}w^dz_2^{n_1}y^ez_3^{n_3}\big) = \sum \Big[x_1^{m_1}x_2^{m_2}x_3^{m_3}w^d \sum k_i\partial_4\partial_1(z_2^{n_1}y^ez_3^{n_3})\Big],\\
0 &= \partial_4\partial_2\big(\sum k_i x_1^{m_1}x_2^{m_2}x_3^{m_3}w^dz_2^{n_1}y^ez_3^{n_3}\big) = \sum \Big[x_1^{m_1}x_2^{m_2}x_3^{m_3}w^d \sum k_i\partial_4\partial_2(z_2^{n_1}y^ez_3^{n_3})\Big],
\end{aligned}$$

where we are grouping the terms by their m_1, m_2, m_3, and d exponents. Since z_2 and z_3 generate the Jordan plane, the monomials $z_2^{n_1}y^ez_3^{n_3}$ are linearly independent. In particular, no nontrivial combination of them is a relation, and therefore no nontrivial combination of them compute to 0 under both $\partial_4\partial_1$ and $\partial_4\partial_2$. By the minimality of S's degree, we must therefore have $n_1 = e = n_3 = 0$. Finally,

$$\begin{aligned}
0 &= \partial_4\partial_1\partial_2\big(\sum k_i x_1^{m_1}x_2^{m_2}x_3^{m_3}w^d\big) = \sum \delta_{d,1}k_i x_1^{m_1}x_2^{m_2}x_3^{m_3},\\
0 &= \partial_3\big(\sum k_i x_1^{m_1}x_2^{m_2}x_3^{m_3}\big) = \sum \delta_{m_3,1}k_i x_1^{m_1}x_2^{m_2},\\
0 &= \partial_2\big(\sum k_i x_1^{m_1}x_2^{m_2}\big) = \sum \delta_{m_2,1}k_i x_1^{m_1},\\
0 &= \partial_1\big(\sum k_i x_1^{m_1}\big) = \sum \delta_{m_1,1}k_i,
\end{aligned}$$

so by the same argument as before, $d = m_1 = m_2 = m_3 = 0$, contradiction as S is nontrivial. Thus, the stated monomials do indeed form a basis.

Finally, assume for the sake of contradiction that there exists some other relation $S' = 0$ not stated. Then, S' can be expressed as a linear combination of the basis monomials; however, this is a clear contradiction as no nontrivial linear combination of the basis monomials can sum to 0. Therefore, this is the final Nichols algebra. $\square$

Let (V^*, c^*) be the dual of (V, c). We now compute the dual $\mathfrak{B}(V)^* \cong \mathfrak{B}(V^*)$ as algebras. Define the functions w_i such that $w_i(x_j) = \delta_{i,j}$, for $i, j \in \mathbb{I}_4$.

Lemma 3.2. *Let* $a_i = w_4 w_i + q_{21}(w_i + w_{i+1}) w_4$ *for* $i \in \mathbb{I}_2$, $b = a_2 w_1 + q_{21}(w_1 + w_2) a_2$, *and* $c = a_1 a_2 + a_2 a_1$. *Then,* $\mathfrak{B}(V^*)$ *is generated by* $w_i, i \in \mathbb{I}_4$, *with relations*

$$w_i^2 = 0, \qquad w_i w_j = w_j w_i, \quad \forall i \neq j \in \mathbb{I}_3, \tag{3.6}$$

$$w_4^2 = 0, \qquad w_3 w_4 = q_{12} w_4 w_3, \tag{3.7}$$

$$a_2^2 = 0, \qquad a_1^4 = 0, \tag{3.8}$$

$$a_1^2 a_2 + a_2 a_1^2 + a_2 a_1 a_2 = 0, \qquad a_1 a_2 a_1 a_2 + a_2 a_1 a_2 a_1 = 0, \tag{3.9}$$

$$a_1 b + q_{21} b a_1 = 0. \tag{3.10}$$

The set of monomials

$$\{w_3^{m_3'} w_2^{m_2'} w_1^{m_1'} b^{d'} a_2^{n_2'} c^{e'} a_1^{n_1'} w_4^{m_4'} \ : \ m_i', d', e', n_2' \in \mathbb{I}_0^1, \ n_1' \in \mathbb{I}_0^3\}$$

is a basis of $\mathfrak{B}(V^*)$.

Proof. Using [2, 1.10, 1.11], we compute the action and coaction of $V^* \in {}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$:

$$\begin{aligned} g \cdot w_i &= w_i, \quad \forall i \in \mathbb{I}_3, & g \cdot w_4 &= q_{21} w_4, \\ h \cdot w_1 &= q_{12}(w_1 + w_2 + w_3), & h \cdot w_2 &= q_{12}(w_2 + w_3), \\ h \cdot w_3 &= q_{12} w_3, & h \cdot w_4 &= w_4, \\ \delta(w_i) &= g^{-1} \otimes w_i, \quad \forall i \in \mathbb{I}_3, & \delta(w_4) &= h^{-1} \otimes w_4. \end{aligned}$$

For example, for $i \in \mathbb{I}_4$, using [2, 1.10],

$$\begin{aligned} \langle h \cdot w_1, x_i \rangle &= \langle w_1, h^{-1} \cdot x_i \rangle \\ &= \langle w_1, q_{12}\left(x_1 + \delta_{i,2} x_2 + \delta_{i,3}(x_2, + x_3)\right) \rangle \\ &= (1 + \delta_{i,4}) q_{12} \end{aligned}$$

and, using [2, 1.11],

$$\begin{aligned} \delta(w_1)_{(-1)} \langle \delta(w_1)_{(0)}, x_i \rangle &= S(x_i) \langle w_1, x_i \rangle \\ &= \delta_{i,1} g x_1. \end{aligned}$$

Mapping $x_1 \mapsto w_3, x_3 \mapsto w_1$, and $x_2 \mapsto w_2$ shows that $(V^*, c^*) \cong (V, c^{-1})$ and so $\mathfrak{B}(V) \cong \mathfrak{B}(V^*)$ by [2, Lemma 1.11]. The stated relations and basis then follow. $\square$

Observe that $\Gamma \cong C_m \otimes C_m \otimes C_4$ by mapping g to g and h to $h^{4i} \otimes h^{mj}$, where i and j are the unique integers (mod m and 4, respectively) satisfying $4i + mj = 1$. Let $p = h^4$ and $r = h^m$; then, g, p, r generate Γ. So we realize $V \in {}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ by the action:

$$\begin{aligned} g \cdot x_i &= x_i, \quad \forall i \in \mathbb{I}_3, & g \cdot x_4 &= q_{12} x_4, \\ p \cdot x_i &= q_{21}^4 x_i, \quad \forall i \in \mathbb{I}_3, & p \cdot x_4 &= x_4, \\ r \cdot x_1 &= x_1, & r \cdot x_2 &= x_1 + x_2, \\ r \cdot x_3 &= (1 + \alpha) x_1 + x_2 + x_3, & r \cdot x_4 &= x_4. \end{aligned}$$

where α is 1 if $m \equiv 1 \pmod 4$ and 0 if $m \equiv 3 \pmod 4$.

Lemma 3.3. *The Hopf algebra $H = \mathfrak{B}(V)\#\Bbbk\Gamma$ is generated by x_i, $i \in \mathbb{I}_4$, and g, p, r, with relations* (3.1) – (3.5) *and*

$$gx_4 = q_{12}x_4g, \qquad x_ig = gx_i, \quad \forall i \in \mathbb{I}_3 \tag{3.11}$$

$$px_4 = x_4p, \qquad px_i = q_{21}^4x_ip, \quad \forall i \in \mathbb{I}_3, \tag{3.12}$$

$$rx_1 = x_1r, \qquad rx_2 = (x_1 + x_2)r \tag{3.13}$$

$$rx_3 = ((1+\alpha)x_1 + x_2 + x_3)\, r \qquad rx_4 = x_4r, \tag{3.14}$$

$$g^m = p^m = 1 \quad r^4 = 1, \qquad gp = pg, \quad gr = rg, \quad pr = rp. \tag{3.15}$$

The coalgebra structure is given by

$$G(H) = \langle g, p, r\rangle, \qquad P_{1,g}(H) = \operatorname{span}\{x_i : i \in \mathbb{I}_3\}, \qquad P_{1,h}(H) = \operatorname{span}\{x_4\}.$$

The set of monomials

$$\{x_1^{m_1}x_2^{m_2}x_3^{m_3}w^dz_2^{n_2}y^ez_3^{n_1}x_4^{m_4}g^ip^jr^k \\ : \ m_i, d, e, n_2 \in \mathbb{I}_0^1, \ \ n_1, k \in \mathbb{I}_0^3, \ \ i, j \in \mathbb{I}_0^{m-1}\}$$

is a basis of H.

Proof. Since we are working with a group algebra, bosonization simplifies to $\gamma v = (\gamma \cdot v)\gamma$ for $v \in \mathfrak{B}(V)$ and $\gamma \in \Bbbk\Gamma$. Plugging in the actions gives us the stated relations. The stated comultiplication and basis can also easily be verified. ☐

By [3, Proposition 2.2], $H^* \cong \mathfrak{B}(V^*)\#\Bbbk^\Gamma$. Before we can compute this, we need a presentation of $\Bbbk^\Gamma$. Note that we can identify $\Bbbk^\Gamma$ with $\Bbbk^{C_m} \otimes \Bbbk^{C_m} \otimes \Bbbk^{C_4}$ using the map $\lambda_{V,W} : V^* \otimes W^* \to (W \otimes V)^*$ such that $\lambda_{V,W}(f \otimes f')(w \otimes v) = f(v)f'(w)$ for all $v \in V, w \in W, f \in V^*$, and $f' \in W$. Consider the natural dual bases of these dual groups, i.e. the sets of functions f_{g^i}, f_{p^i}, and f_{r^i} such that $f_{g^i}(g^j) = \delta_{i,j}$, $f_{p^i}(p^j) = \delta_{i,j}$, and $f_{r^i}(r_j) = \delta_{i,j}$. Then, we have

$$\Bbbk^\Gamma \cong \Bbbk^{C_m} \otimes \Bbbk^{C_m} \otimes \Bbbk^{C_4} \cong \Bbbk\langle u\rangle \otimes \Bbbk\langle v\rangle \otimes \Bbbk\langle s, t\rangle/(s^2 + s, t^2 + t, st + ts),$$

where $u = \sum_{i=0}^{m-1} q_{12}^i f_{g^i}$, $v = \sum_{i=0}^{m-1} q_{21}^{4i} f_{p^i}$, $s = f_r + f_{r^3}$, and $t = f_r + f_{r^2}$.

Lemma 3.4. *The Hopf algebra $H^* \cong \mathfrak{B}(V^*)\#\Bbbk^\Gamma$ is generated by w_i, $i \in \mathbb{I}_4$, u, v, s, t with relations* (3.6) – (3.10) *and*

$$w_is = sw_i, \quad \forall i \in \mathbb{I}_3, \qquad w_4s = sw_4 + w_4, \tag{3.16}$$

$$w_it = tw_i, \quad \forall i \in \mathbb{I}_3, \qquad w_4t = tw_4 + (1+\alpha)w_4 + sw_4, \tag{3.17}$$

$$w_iu = q_{12}uw_i, \quad \forall i \in \mathbb{I}_3, \qquad w_4u = uw_4, \tag{3.18}$$

$$w_iv = vw_i, \quad \forall i \in \mathbb{I}_3, \qquad w_4v = q_{21}vw_4, \tag{3.19}$$

$$u^m = v^m = 1, \qquad s^2 = s, \quad t^2 = t, \tag{3.20}$$

$$uv = vu, \quad us = su, \quad ut = tu, \qquad vs = sv, \quad vt = tv, \quad st = ts. \tag{3.21}$$

The coalgebra structure is given by

$$\begin{aligned}
&G(H^*) = \langle u, v\rangle, \quad P_{1,1}(H^*) = \operatorname{span}\{s\},\\
&P_{1,v}(H^*) = \operatorname{span}\{w_3\}, \quad P_{1,u}(H^*) = \operatorname{span}\{w_4\},\\
&\Delta(t) = t \otimes 1 + 1 \otimes t + s \otimes s,\\
&\Delta(w_1) = w_1 \otimes 1 + v \otimes w_1 + sv \otimes w_2 + (\alpha s + t)v \otimes w_3,\\
&\Delta(w_2) = w_2 \otimes 1 + v \otimes w_2 + sv \otimes w_3.
\end{aligned}$$

The set of monomials

$$\begin{aligned}\{u^{i'} s^{k'_1} t^{k'_2} v^{j'} w_3^{m'_3} w_2^{m'_2} w_1^{m'_1} b^{d'} a_2^{n'_2} c^{e'} a_1^{n'_1} w_4^{m'_4}& \\ : m'_i, d', e', n'_2, k'_i \in \mathbb{I}_0^1,\ \ n'_1 \in \mathbb{I}_0^3,\ \ i', j' \in \mathbb{I}_0^{m-1}\}&\end{aligned}$$

is a basis of H^*.

Proof. The proof of [3, Proposition 2.2] offers a realization of $V^* \in {}^{\Bbbk^\Gamma}_{\Bbbk^\Gamma}\mathcal{YD}$. For $f \in \Bbbk^\Gamma$, $\{v_i, v_i^*\}_i$ a dual basis of V, and $w \in V^*$,

$$f \cdot w = \sum_i \langle f, v_{i(-1)}\rangle \langle w, v_{i(0)}\rangle w_i.$$

Thus, we can compute the actions:

$$\begin{aligned} &s \cdot w_i = 0, \quad \forall i \in \mathbb{I}_3, && s \cdot w_4 = w_4,\\ &t \cdot w_i = 0, \quad \forall i \in \mathbb{I}_3, && t \cdot w_4 = \alpha w_4,\\ &u \cdot w_i = q_{21} w_i, \quad \forall i \in \mathbb{I}_3, && u \cdot w_4 = w_4,\\ &v \cdot w_i = w_i, \quad \forall i \in \mathbb{I}_3 && v \cdot w_4 = q_{12} w_4.\end{aligned}$$

This is sufficient to compute the relations stated. As an example of this computation, we will explicitly show $tw_4 = w_4 t + \alpha w_4 + w_4 s$:

$$\begin{aligned} tw_4 &= (t_{(1)} \cdot w_4) t_{(2)}\\ &= (1 \cdot w_4)t + (t \cdot w_4)1 + (s \cdot w_4)s\\ &= \alpha w_4 + w_4 t + w_4 s.\end{aligned}$$

To compute the coproduct, we also need the action of $V^* \in {}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$. We use our work in Lemma 3.2, replacing h with the generators p and r:

$$\begin{aligned} &g \cdot w_i = w_i, \quad \forall i \in \mathbb{I}_3, && g \cdot w_4 = q_{21} w_4,\\ &p \cdot w_i = q_{12}^4 w_i, \quad \forall i \in \mathbb{I}_4, && r \cdot w_1 = w_1 + w_2 + \alpha w_3,\\ &r \cdot w_2 = w_2 + w_3, && r \cdot w_3 = w_3,\\ &r \cdot w_4 = w_4. && \end{aligned}$$

For example, we compute $\Delta(w_1)$:

$$\begin{aligned} \Delta(w_1) &= w_1^{(1)} (w_1^{(2)})_{(-1)} \otimes (w_1^{(2)})_{(0)}\\ &= w_1 \otimes 1 + 1 (w_1)_{(-1)} \otimes (w_1)_{(0)}\\ &= w_1 \otimes 1 + \sum_{i,j,k} f_{g^{-i}} f_{p^{-j}} S(f_{r^k}) \otimes (g^i p^j r^k) \cdot w_1\\ &= w_1 \otimes 1 + \sum_k v f_{r^{-k}} \otimes r^k \cdot w_1\\ &= w_1 \otimes 1 + v f_1 \otimes w_1 + f_{r^3} \otimes (w_1 + w_2 + \alpha w_3)\\ &\quad + v f_{r^2} \otimes (w_1 + w_3) + v f_r \otimes (w_1 + w_2 + (1+\alpha) w_3)\\ &= w_1 \otimes 1 + v(f_1 + f_r + f_{r^2} + f_{r^3}) \otimes w_1 + v(f_r + f_{r^3}) \otimes w_2\\ &\quad + (v\alpha(f_r + f_{r^3}) + v(f_r + f_{r^2})) \otimes w_3\\ &= w_1 \otimes 1 + v \otimes w_1 + vs \otimes w_2 + (\alpha s + t) v \otimes w_3.\end{aligned}$$

The other computations collapse similarly. □

Finally, our last step before determining $D(H)$ is computing $D(\Bbbk\Gamma)$. Using Definition 2.2, we see that $D(\Bbbk\Gamma)$ is generated by g, p, r, u, v, s, t with relations (3.15), (3.20), (3.21) and

$$gu = ug, \quad pu = up, \quad ru = ur, \quad gv = vg, \quad pv = vp, \quad rv = vr, \tag{3.22}$$

$$gs = sg, \quad ps = sp, \quad rs = sr, \quad gt = tg, \quad pt = tp, \quad rt = tr. \tag{3.23}$$

We explicitly show that $sg = gs$:

$$\begin{aligned} sg &= 1(g)1(g^{-1})gs + 1(g)s(g^{-1})s + s(g)1(g^{-1})s \\ &= gs + (\varepsilon \otimes \varepsilon \otimes s)(g^{-1} \otimes 1 \otimes 1)s + (\varepsilon \otimes \varepsilon \otimes s)(g \otimes 1 \otimes 1)s \\ &= gs \end{aligned}$$

since $s(1) = (f_r + f_{r^3})(1) = 0$.

Theorem 3.5. *$D(H)$ is generated by x_i, w_i, $i \in \mathbb{I}_4$, g, p, r, u, s, t, v with relations* (3.1) – (3.23) *and*

$$\begin{aligned}
&ux_i = q_{12}x_iu, \quad \forall i \in \mathbb{I}_3, && ux_4 = x_4u, \\
&vx_i = x_iv, \quad \forall i \in \mathbb{I}_3, && vx_4 = q_{21}x_4v, \\
&sx_i = x_is, \quad \forall i \in \mathbb{I}_3, && sx_4 = x_4s + x_4, \\
&tx_i = x_it, \quad \forall i \in \mathbb{I}_3, && tx_4 = x_4t + (1+\alpha)x_4 + x_4s, \\
&w_ig = gw_i, \quad \forall i \in \mathbb{I}_3, && w_4g = q_{12}gw_4, \\
&w_ip = q_{21}^4pw_i, \quad \forall i \in \mathbb{I}_3, && w_4p = pw_4, \\
&w_1r = rw_1 + rw_2 + (1+\alpha)rw_3, && w_2r = rw_2 + rw_3, \\
&w_ir = rw_i, \quad \forall i \in \mathbb{I}_3^4 && w_ix_j = x_jw_i, \quad \forall j < i \in \mathbb{I}_3, \\
&w_ix_i = 1 + x_iw_i + gv, \quad \forall i \in \mathbb{I}_3, && w_ix_{i+1} = x_{i+1}w_i + gsv, \quad \forall i \in \mathbb{I}_2, \\
&w_1x_3 = x_3w_1 + \alpha gsv + gvt, && w_4x_i = q_{12}x_iw_4, \quad \forall i \in \mathbb{I}_3, \\
&w_4x_4 = 1 + x_4w_4 + hu, && w_ix_4 = q_{21}x_4(w_i + w_{i+1}), \quad \forall i \in \mathbb{I}_2 \\
&w_3x_4 = q_{21}x_4w_3.
\end{aligned}$$

The coalgebra structure is as in Lemma 3.3 and Lemma 3.4. The set of monomials

$$\{x_1^{m_1}x_2^{m_2}x_3^{m_3}w^dz_2^{n_2}y^ez_3^{n_1}x_4^{m_4}g^ip^jr^ku^{i'}s^{k_1'}t^{k_2'}v^{j'}w_3^{m_3'}w_2^{m_2'}w_1^{m_1'}b^{d'}a_2^{n_2'}c^{e'}a_1^{n_1'}w_4^{m_4'} : \\ m_i, m_i', d, d', e, e', n_2, n_2', k_i' \in \mathbb{I}_0^1, \quad n_1, n_1', k \in \mathbb{I}_0^3, \quad i, j, i', j' \in \mathbb{I}_0^{m-1}\}.$$

is a basis of $D(H)$.

Proof. This follows from [3, Proposition 2.4]. An example calculation using each of the three different formulas is shown:

$$\begin{aligned} sx_4 &= 1(h)x_4s + s(h)x_41 \\ &= x_4s + [(1+\alpha)(\varepsilon \otimes \varepsilon \otimes s)(1 \otimes h^{m+1} \otimes h^{3m}) + \alpha(\varepsilon \otimes \varepsilon \otimes s)(1 \otimes h^{3m+1} \otimes h^m)]x_4 \\ &= x_4s + x_4, \end{aligned}$$

$$\begin{aligned} w_1r &= \sum_l w_1(r \cdot x_l)rw_l \\ &= w_1(x_1)rw_1 + w_1(x_1 + x_2)rw_2 + w_1\left((1+\alpha)x_1 + x_2 + x_3\right)rw_3 + w_1(x_4)rw_4 \end{aligned}$$

$$= r\left(w_1 + w_2 + (1+\alpha)w_3\right),$$

$$\begin{aligned} w_1x_3 &= w_1(x_3) + w_1(g\cdot x_1)x_3w_1 \\ &\quad + \sum_{i,j,k} w_1(g\cdot x_1)w_1(g^ip^jr^k\cdot x_3)\langle \varepsilon\otimes w_3, S(x_3)\rangle g f_{g^i}f_{p^j}f_{r^k} \\ &= x_3w_1 + \sum_{i,j,k} w_1(g^ip^jr^k\cdot x_3) g f_{g^i}f_{p^j}f_{r^k} \\ &= x_3w_1 + v\langle w_1, (1+\alpha)x_1 + x_2 + x_3\rangle g f_r + v\langle w_1, x_1 + x_3\rangle g f_{r^2} \\ &\quad + v\langle w_1, \alpha(1+\alpha)x_1 + x_2 + x_3\rangle g f_{r^3} + v\langle w_1, x_3\rangle g f_1 \\ &= x_3w_1 + \alpha v g(f_r + f_{r^3}) + vg(f_r + f_{r^2}) \\ &= x_3w_1 + \alpha v g s + v g t. \end{aligned}$$

The others follow analogously. Since $D(H) \cong H\otimes (H^*)^{\mathrm{op}}$ as coalgebras, the coproduct of the double is adopted from that of H and H^*. The claimed basis is also clear. □

Remark. Using the counit and antipode axioms, we can compute S evaluated on the generators of $D(H)$:

$$\begin{aligned} &S(x_i) = g^{-1}x_i \quad \forall i\in\mathbb{I}_3, && S(x_4) = h^{-1}x_4, \\ &S(w_1) = v^{-1}w_1 + sv^{-1}w_2 + (\alpha s + s + t)v^{-1}w_3, && S(w_2) = v^{-1}w_2 + sv^{-1}w_3, \\ &S(w_3) = v^{-1}w_3, && S(w_4) = u^{-1}w_4, \\ &S(g) = g^{-1}, \quad S(p) = p^{-1}, \quad S(r) = r^{-1}, && S(u) = u^{-1}, \quad S(v) = v^{-1} \\ &S(s) = s, && S(t) = s + t. \end{aligned}$$

We show the computation for $S(w_1)$. It is easy to verify $\varepsilon(s) = \varepsilon(t) = 0$ and $\varepsilon(v) = 1$. Then,

$$w_1 = \varepsilon(w_1)1 + \varepsilon(v)w_1 + \varepsilon(sv)w_2 + \varepsilon(\alpha s + t)w_3 \implies \varepsilon(w_1) = 0.$$

Since $S(v) = v^{-1}$, $S(s) = s$, and $S(t) = s + t$,

$$\begin{aligned} \varepsilon(w_1)1 &= S(w_1)1 + S(v)w_1 + S(sv)w_2 + S(\alpha s + t)w_3 \\ \implies S(w_1) &= v^{-1}w_1 + sv^{-1}w_2 + (\alpha s + s + t)v^{-1}w_3, \end{aligned}$$

as desired. Observe that $S^2 = \mathrm{id}$ on all generators; thus, $D(H)$ is involutive with pivot 1.

4. The R-Matrix of the Double

Drinfeld doubles are guaranteed to have a R-matrix. From Proposition 2.3, it suffices to find a dual basis of $D(H)$. We start by computing the dual basis of $\mathfrak{B}(V)$. Alternately, write the basis of $\mathfrak{B}(V)$ from Theorem 3.1 as the set of monomials

$$S := \left\{x_1^{m_1}x_2^{m_2}x_3^{m_3}z_3^{n_1}(z_3^2)^{n_2}w^dy^ez_2^{n_3}x_4^{m_4} \;:\; m_i, d, e, n_i \in \mathbb{I}_0^1\right\}.$$

Lemma 4.1. *Define*

$$\begin{aligned} z_2^* &= w_4w_1 + q_{12}w_1w_4, \\ z_3^* &= w_4w_2 + q_{12}w_2w_4, \\ (z_3^2)^* &= z_2^*z_3^* + z_3^*z_2^* = q_{21}w_4w^* + q_{12}w^*w_4, \end{aligned}$$

$$
\begin{aligned}
w^* &= z_2^* w_2 + q_{12} w_2 z_2^* = z_3^* w_1 + q_{12} w_1 z_3^*,\\
y^* &= (z_2^*)^2.
\end{aligned}
$$

The set of monomials

$$S^* := \left\{ w_1^{m_1} w_2^{m_2} w_3^{m_3} {z_3^*}^{n_1} (z_3^2)^{*n_2} {w^*}^{d} {y^*}^{e} {z_2^*}^{n_3} {w_4^*}^{m_4} \ : \ m_i, d, e, n_i \in \mathbb{I}_0^1 \right\}$$

is the dual basis of S.

Proof. We will use the notation $\Delta(\alpha) = \sum \alpha^{(1)} \otimes \alpha^{(2)}$ through this proof to help differentiate between indices and comultiplication markings. We have the following computations:

$$
\begin{aligned}
\Delta(x_i) &= x_i \otimes 1 + 1 \otimes x_i \quad \forall i \in \mathbb{I}_4,\\
\Delta(z_2) &= z_2 \otimes 1 + 1 \otimes z_2 + x_4 \otimes x_1,\\
\Delta(z_3) &= z_3 \otimes 1 + 1 \otimes z_3 + x_4 \otimes x_2,\\
\Delta(z_3^2) &= z_3^2 \otimes 1 + 1 \otimes z_3^2 + z_2 \otimes z_3 + x_4 \otimes x_1 z_3 + q_{12} x_4 \otimes w,\\
\Delta(w) &= w \otimes 1 + 1 \otimes w + z_2 \otimes x_2 + x_4 \otimes x_1 x_2 + z_3 \otimes x_1,\\
\Delta(y) &= y \otimes 1 + 1 \otimes y + z_2 \otimes z_2 + z_2 x_4 \otimes x_1 + x_4 \otimes x_1 x_2.
\end{aligned}
$$

For example, for z_2,

$$
\begin{aligned}
\Delta(z_2) &= \Delta\left(x_4 x_2 + q_{21}(x_2 + x_1) x_4\right)\\
&= \Big(id \otimes c \otimes id\Big)\Big((x_4 \otimes 1 + 1 \otimes x_4)(x_2 \otimes 1 + 1 \otimes x_2)\\
&\qquad + q_{21}\Big(((x_2 + x_1) \otimes 1 + 1 \otimes (x_2 + x_1))(x_4 \otimes 1 + 1 \otimes x_4)\Big)\Big)\\
&= z_2 \otimes 1 + 1 \otimes z_2 + \Big(x_4 \otimes x_2 + q_{21}(x_2 + x_1) \otimes x_4\\
&\qquad + q_{21}\Big((x_2 \otimes x_1) \otimes x_4 + q_{12}(x_2 + x_1) \otimes x_4\Big)\Big)\\
&= z_2 \otimes 1 + 1 \otimes z_2 + x_1 \otimes x_4.
\end{aligned}
$$

The other computations follow analogously.

A key observation will be the following: x_1, x_2, and x_3 have coaction g, g acts by a constant on every generator, and h acts by a constant on z_3^2, w, y, z_2, and x_4. Particularly, for arbitrary $\gamma \in S$ and some $k_l \in \Bbbk$,

$$
\begin{aligned}
\Delta(\gamma) &= \Delta(x_1^{m_1} x_2^{m_2} x_3^{m_3} w^d z_3^{n_1} (z_3^2)^{n_2} y^e z_2^{n_3} x_4^{m_4})\\
&= (id \otimes c \otimes id)\left(\sum x_1^{(1)} \otimes x_1^{(2)}\right)^{m_1} \cdots \left(\sum x_4^{(1)} \otimes x_4^{(2)}\right)^{m_4}\\
&= \sum_l k_l \left(\left(x_1^{(1)}\right)^{m_1} \cdots \left(x_4^{(1)}\right)^{m_4}\right) \otimes \left(\left(x_1^{(2)}\right)^{m_1} \cdots \left(x_4^{(2)}\right)^{m_4}\right).
\end{aligned}
$$

We proceed with strong induction on the sum of the exponents appearing in

$$\rho := x_1^{m_1'} x_2^{m_2'} x_3^{m_3'} w^{d'} z_3^{n_1'} (z_3^2)^{n_2'} y^{e'} z_2^{n_3'} x_4^{m_4'} \in S$$

to show that

$$\rho^* := w_1^{m_1'} w_2^{m_2'} w_3^{m_3'} {w^*}^{d'} {z_3^*}^{n_1'} (z_3^2)^{*n_2'} {y^*}^{e'} {z_2^*}^{n_3'} w_4^{m_4'} \in S^*$$

is indeed the dual of ρ. We prove $\langle \rho^*, \gamma \rangle = 1$ if and only if $\gamma = \rho$.

We can manually check that this holds when the sum of the exponents of ρ is 1. As an example, we show that z_2^* is indeed the dual of z_2. It suffices to check z_2^*'s pairing with basis monomials in S of degree 2. We show that $\langle z_2^*, w_1w_4\rangle = 0$; the other verifications are clear.

$$\begin{aligned}\langle z_2^*, x_1x_4\rangle &= (w_4\otimes w_1 + q_{12}w_1\otimes w_4)(\Delta x_1x_4)\\ &= (w_4\otimes w_1 + q_{12}w_1\otimes w_4)(x_1x_4\otimes 1 + x_1\otimes x_4 + q_{12}x_4\otimes x_1 + x_4x_1\otimes 1)\\ &= q_{12}(w_4\otimes w_1)(x_4\otimes x_1) + q_{12}(w_1\otimes w_4)(x_1\otimes x_4)\\ &= 0,\end{aligned}$$

as desired. Now, we prove the statement for an arbitrary ρ with multiple generators.

We say a pure tensor whose components are basis monomials in S is *ordered* if its components concatenate to ρ (i.e. $\gamma_1\otimes\gamma_2$ for $\gamma_1,\gamma_2\in S$ is *ordered* if $\overline{\gamma_1\gamma_2}=\gamma$). We say the tensor is *unordered* otherwise.

Consider some pure tensor $\nu_1\otimes\nu_2$ in the summation expansion of $\Delta(x_i)^{m_i}$, $i\in\mathbb{I}_4$, $\Delta(z_2)^{n_2}, \Delta(z_3)^{n_3}, \Delta(z_3^2)^{n_2}, \Delta(w)^d$ or $\Delta(y)^e$. We say $\nu_1\otimes\nu_2$ has *zero contribution* if

$$\sum_l k_l\left(\left(x_1^{(1)}\right)^{m_1}\cdots\nu_1\cdots\left(x_4^{(1)}\right)^{m_4}\right)\otimes\left(\left(x_1^{(2)}\right)^{m_1}\cdots\nu_2\cdots\left(x_4^{(2)}\right)^{m_4}\right)$$

can be written as a linear combination of unordered tensors. For brevity, we denote

$$L \coloneqq \left(\left(x_1^{(1)}\right)^{m_1}\cdots\nu_1\cdots\left(x_4^{(1)}\right)^{m_4}\right) \text{ and } R \coloneqq \left(\left(x_1^{(2)}\right)^{m_1}\cdots\nu_2\cdots\left(x_4^{(2)}\right)^{m_4}\right).$$

We classify the only possible contributing tensors through the following sequence of claims.

Claim 1: *Any tensor $\nu_1\otimes\nu_2$ such that $\nu_1\neq 1$ and ν_2 contains an x_1 term has zero contribution.*

Proof. Since x_1 commutes with everything (up to a constant), R can be written as a linear combination of basis monomials that have a nonzero power of x_1. Since L has degree greater than one, $L\otimes R$ is a sum of unordered tensors, as desired. $\diamond$

Claim 2: *The tensor $x_4\otimes x_2$ in $\Delta(z_3)$ has zero contribution.*

Proof. Observe that, for $\nu_1\otimes\nu_2 = x_4\otimes x_2$, we can write L as a linear combination of basis monomials that have a nonzero power of x_4, z_2, z_3, z_3^2, w, or y. Further, we can write R as a linear combination of basis monomials that have a nonzero power of x_2. So, $L\otimes R$ is a sum of unordered tensors. $\diamond$

Claim 3: *The tensors $z_2\otimes z_3$ in $\Delta(z_3^2)$, $z_2\otimes x_2$ in $\Delta(w)$, and $z_2\otimes z_2\in\Delta(y)$ have zero contribution.*

Proof. If $(w^{(2)})^d, (y^{(2)})^e, (z_2^{(2)})^{n_3}$, or $(x_4^{(2)})^{m_4}$ contain x_1, we finish by Claim 1. Otherwise, $(w^{(1)})^d$, $(y^{(1)})^e$, $(z_2^{(1)})^{n_3}$, and $(x_4^{(1)})^{m_4}$ contain only x_4, w, z_2 and y. Thus, since z_2 commutes (up to a constant) with w and y, L necessarily contains z_2. Similarly, R necessarily contains one of x_2, z_2, z_3, z_3^2, w, or y. Thus, $L\otimes R$ is a sum of unordered tensors. $\diamond$

Claim 4: *The tensor $x_4\otimes w$ in $\Delta(z_3)^2$ has zero contribution.*

Proof. Claim 1, Claim 2, and Claim 3 cover everything except when:

$$\begin{aligned} &x_i^{(1)}, x_i^{(2)} \in \{1, x_i\}, \quad \forall i \in \mathbb{I}_4, && z_j^{(1)}, z_j^{(2)} = \{1, z_j\}, \quad \forall j \in \mathbb{I}_2^3, \\ &(z_3^2)^{(1)}(z_3^2)^{(2)} \in \{1, z_3^2\}, && y^{(1)}, y^{(2)} \in \{1, y\}. \end{aligned}$$

Then, since x_4 commutes (up to a constant factor) with z_2 and y and "creates" a y term when commuting with w, L contains a x_4 or y. Further, R contains a w. So, $L \otimes R$ is a sum of unordered tensors. ◇

Therefore, the only possible contributing tensors have $\nu_1 = 1$ or $\nu_2 = 1$. So,

$$\begin{aligned} \Delta(\gamma) = {} & x_1^{m_1} x_2^{m_2} x_3^{m_3} w^d z_3^{n_1} (z_3^2)^{n_2} y^e z_2^{n_3} x_4^{m_4} \otimes 1 \\ & + \delta_{m_4,1} x_1^{m_1} x_2^{m_2} x_3^{m_3} w^d z_3^{n_1} (z_3^2)^{n_2} y^e z_2^{n_3} \otimes x_4^{m_4} \\ & + \delta_{n_3,1} x_1^{m_1} x_2^{m_2} x_3^{m_3} w^d z_3^{n_1} (z_3^2)^{n_2} y^e \otimes z_2^{n_3} x_4^{m_4} \\ & \qquad \vdots \\ & + 1 \otimes x_1^{m_1} x_2^{m_2} x_3^{m_3} w^d z_3^{n_1} (z_3^2)^{n_2} y^e z_2^{n_3} x_4^{m_4} \\ & + \sum \text{unordered tensors.} \end{aligned}$$

Importantly, the only ordered tensors of $\Delta(\gamma)$ have the concatenation of their components equal to γ.

Finally, since the sum of ρ's exponents is greater than 1, we can write ρ as the concatenation of ρ_1 and ρ_2 where $\rho_1, \rho_2 > 1$. By our inductive hypothesis, $\rho^* = \overline{\rho_1^* \rho_2^*}$. Then,

$$\begin{aligned} \langle \rho^*, \gamma \rangle &= (\rho_1^*, \rho_2^*) \otimes \Delta(\gamma) \\ &= \sum_l \langle \rho_1^*, \gamma_l^{(1)} \rangle \langle \rho_2^*, \gamma_l^{(2)} \rangle. \end{aligned}$$

Again by our inductive hypothesis, this expression equals 1 if and only if $\gamma_l^{(1)} = \rho_1$ and $\gamma_l^{(2)} = \rho_2$ for some l. Particularly, $\gamma_l^{(1)} \otimes \gamma_l^{(2)}$ must be ordered. So,

$$\gamma = \gamma_l^{(1)} \gamma_l^{(2)} = \rho_1 \rho_2 = \rho,$$

and we have completed our induction and shown that S^* is indeed dual to S. It suffices now to show that S^* is in fact a basis of $\mathfrak{B}(V^*)$. We have the following relations:

$$\begin{aligned} a_1 &= z_2^* + (q_{12} + q_{21}) w_1 w_4 + q_{21} w_2 w_4 \\ a_2 &= z_3^* + (q_{12} + q_{21}) w_2 w_4 + q_{21} w_3 w_4 \\ b &= w^* + (q_{12} + q_{21})(w_1 z_3^* + w_2 z_2^* + q_{12} w_1 w_2 w_4) + q_{12} w_3 z_3^* + w_2 w_3 w_4 \\ c &= (z_3^2)^* + (q_{12} + q_{21})^2 (w_1 z_3^* w_4 + w_2 z_2^* w_4) + (1 + q_{21}^2) w_3 z_2^* w_4 + q_{21}^2 w_3 z_3^* w_4 \end{aligned}$$

In particular, each monomial of the basis from Lemma 3.2 can be written in terms of S^*; thus, S^* spans $\mathfrak{B}(V^*)$. Moreover, since we have 256 monomials of S^* and the dimension of $\mathfrak{B}(V^*)$ is precisely 256, we have that S^* does indeed form a basis. □

Now, let $T := \{g^i p^j r^k : i, k \in \mathbb{I}_m, r \in \mathbb{I}_4\}$ be the basis of $\Bbbk\Gamma$. We compute the dual basis to T.

Lemma 4.2. *Define the following recursively:*

$$f_{r^0} = st + s + t + 1, \qquad f_r = st, \qquad f_{r^2} = st + t,$$

$$f_{r^3} = st + s, \qquad f_{g^i} = \sum_{j=0}^{m-1} q_{21}^{ij} u^j, \qquad f_{p^i} = \sum_{j=0}^{m-1} q_{12}^{4ij} v^j,$$

$$f_{g^i p^j r^k} = f_{g^i} f_{p^j} f_{r^k}.$$

The set of monomials

$$T^* := \{f_{g^i p^j r^k} : i, j \in \mathbb{I}_m, r \in \mathbb{I}_4\}$$

is the dual basis of T.

Proof. See the description of $\Bbbk^\Gamma$ given before Lemma 3.4. The computations for $f_{g^i p^j r^k}$, the dual of $g^i p^j r^k$, follow. □

We are ready to present the R-matrix.

Theorem 4.3. *The* R*-matrix of* $D(H)$ *is* $\sum_l e^l \otimes e_l$ *where*

$$\{e^l\} := \{w_1^{m_1} w_2^{m_2} w_3^{m_3} z_3^{*n_1} (z_3^2)^{*n_2} w^{*d} y^{*e} z_2^{*n_3} w_4^{*m_4} f_{g^i p^j r^k} \\ : m_i, d, e, n_i \in \mathbb{I}_0^1,\ i, j \in \mathbb{I}_m,\ k \in \mathbb{I}_4\}$$

is the dual basis of

$$\{e_l\} := \{x_1^{m_1} x_2^{m_2} x_3^{m_3} z_3^{n_1} (z_3^2)^{n_2} w^d y^e z_2^{n_3} x_4^{m_4} g^i p^j r^k \\ : m_i, d, e, n_i \in \mathbb{I}_0^1,\ i, j \in \mathbb{I}_m,\ k \in \mathbb{I}_4\}.$$

Proof. This is immediate from Lemma 4.1 and Lemma 4.2 and Proposition 2.3. □

Remark. Since $D(H)$ has pivot 1, the quantum trace devolves to the classical trace. The resulting knot invariant from our R-matrix is thus not very interesting.

5. IRREDUCIBLE REPRESENTATIONS AT $q = 1$

The equivalence $D(H)$-mod $\simeq {}^H_H\mathcal{YD}$ makes the classification of all simple modules over $D(H)$ particularly interesting. However, the complexity and size of $D(H)$ make it difficult to explicitly compute all such irreducibles. We instead consider the simpler, classical case where $q = 1$, so $\alpha = g = p = v = u = 1$ and $h = r$. We let D be the double at $q = 1$; it admits a triangular decomposition

$$D = D^{>0} \otimes D^0 \otimes D^{<0} := \mathfrak{B}(V) \otimes (\Bbbk\Gamma \otimes \Bbbk^\Gamma) \otimes \mathfrak{B}(V^*).$$

We present all simple modules over D in this section. For completeness, we initially show how D is presented using Theorem 3.5.

Theorem 5.1. D *is generated by* x_i, w_i, $i \in \mathbb{I}_4$, h, s, t *with relations*

$$\begin{aligned}
&x_i^2 = 0, \quad \forall i \in \mathbb{I}_3, && x_i x_j = x_j x_i, \quad \forall i \neq j \in \mathbb{I}_3, \\
&x_4^2 = 0, && x_1 x_4 = x_4 x_1, \\
&z_2^2 = 0, && z_3^4 = 0, \\
&z_3^2 z_2 + z_2 z_3^2 + z_2 z_3 z_2 = 0, && z_3 z_2 z_3 z_2 + z_2 z_3 z_2 z_3 = 0, \\
&z_3 w = w z_3, && h x_1 = x_1 h, \\
&h x_2 = (x_1 + x_2) h, && h x_3 = (x_2 + x_3) h,
\end{aligned}$$

$$\begin{aligned}
&hx_4 = x_4h, && w_i^2 = 0, \quad \forall i \in \mathbb{I}_3,\\
&w_iw_j = w_jw_i, \quad \forall i \neq j \in \mathbb{I}_3, && w_4^2 = 0,\\
&w_3w_4 = w_4w_3, && a_2^2 = 0,\\
&a_1^4 = 0, && a_1^2a_2 + a_2a_1^2 + a_2a_1a_2 = 0,\\
&a_1a_2a_1a_2 + a_2a_1a_2a_1 = 0, && a_1b + ba_1 = 0,\\
&w_is = sw_i, \quad \forall i \in \mathbb{I}_3, && w_4s = sw_4 + w_4,\\
&w_it = tw_i, \quad \forall i \in \mathbb{I}_3, && w_4t = (s+t)w_4,\\
&sx_i = x_is, \quad \forall i \in \mathbb{I}_3, && sx_4 = x_4s + x_4,\\
&tx_i = x_it, \quad \forall i \in \mathbb{I}_3, && tx_4 = x_4(s+t),\\
&w_1h = h(w_1 + w_2), && w_2h = h(w_2 + w_3),\\
&w_ih = hw_i, \quad \forall i \in \mathbb{I}_3^4, && w_ix_j = x_jw_i, \quad \forall j \leq i \in \mathbb{I}_3,\\
&w_ix_{i+1} = x_{i+1}w_i + s, \quad \forall i \in \mathbb{I}_2, && w_1x_3 = x_3w_1 + s + t,\\
&w_4x_i = x_iw_4, \quad \forall i \in \mathbb{I}_3, && w_4x_4 = 1 + x_4w_4 + h,\\
&w_1x_4 = x_4(w_1 + w_2), && w_2x_4 = x_4(w_2 + w_3),\\
&w_3x_4 = x_4w_3, && h^4 = 1,\\
&s^2 = s, && t^2 = t,\\
&st = ts.
\end{aligned}$$

The coalgebra structure is given by

$$\begin{aligned}
&G(D) = \langle h \rangle, \quad P_{1,1}(D) = \operatorname{span}\{x_1, x_2, x_3, w_1, w_2, s\}, \quad P_{1,h} = \operatorname{span}\{x_4\},\\
&\Delta(w_1) = w_1 \otimes 1 + 1 \otimes w_1 + s \otimes w_2 + (s+t) \otimes w_3,\\
&\Delta(w_2) = w_2 \otimes 1 + 1 \otimes w_2 + s \otimes w_3,\\
&\Delta(t) = t \otimes 1 + 1 \otimes t + s \otimes s.
\end{aligned}$$

□

We now define four modules U_0, U_1, W_0, W_1 such that irrep $D = \{U_0, U_1, W_0, W_1\}$. We help describe each simple module through their weight decompositions with respect to D^0. The weights are then $\{\lambda_{\delta_s,\delta_t} : \delta_s, \delta_t \in \mathbb{I}_0^1\}$ where $\lambda_{\delta_s,\delta_t}$ is the 1-dimensional module over D^0 with action given by:

$$s \cdot 1 = \delta_s, \qquad t \cdot 1 = \delta_t, \qquad h \cdot 1 = 1, \qquad x_i \cdot 1 = 0, \quad \forall i \in \mathbb{I}_4.$$

(1) U_0 is the trivial 1-dimensional module over D where elements of D act by their counit. It admits the weight decomposition $U_0 = (U_0)_{\lambda_{0,0}}$.

(2) $U_1 := \Bbbk\{1, w_1\}$ is the 2-dimensional module where h and t act by id and the other generators of D act by 0. It admits the weight decomposition $U_1 = (U_1)_{\lambda_{0,1}}$.

(3) For δ_t either 0 or 1, $W_{\delta_t} := \Bbbk\{w_1^m,\ w_1^mw_2,\ w_1^mb,\ w_1^ma_1^n : m \in \mathbb{I}_0^1, n \in \mathbb{I}_2\}$ is the 10-dimensional module where x_1 and w_3 act by 0, w_1 and w_2 act by D's multiplication (i.e. $w_i \cdot \omega = w_i\omega$, for $i \in \mathbb{I}_2$ and $\omega \in W_{\delta_t}$), and the other generators act as follows:

$$\begin{aligned}
&h \cdot w_1^m = (w_1 + w_2)^m, && h \cdot w_1^mw_2 = w_1^mw_2,\\
&s \cdot w_1^m = w_1^m, && s \cdot w_1^mw_2 = w_1^mw_2,
\end{aligned}$$

$$\begin{aligned}
&t\cdot w_1^m=\delta_t w_1^m, && t\cdot w_1^m w_2=\delta_t w_1^m w_2,\\
&x_2\cdot w_1^m=\delta_{m,1}, && x_2\cdot w_1^m w_2=\delta_{m,1}w_2,\\
&x_3\cdot w_1^m=\delta_t\delta_{m,1}, && x_3\cdot w_1^m w_2=(1+\delta_t)\delta_{m,1}w_2,\\
&x_4\cdot w_1^m=0, && x_4\cdot w_1^m w_2=0,\\
&w_4\cdot w_1^m=\delta_{m,1}a_1, && w_4\cdot w_1^m w_2=\delta_{m,1}b,\\
&h\cdot w_1^m b=w_1^m b, && h\cdot w_1^m a_1^n=w_1 a_1^n,\\
&s\cdot w_1^m b=0, && s\cdot w_1^m a_1^n=\delta_{n,2}a_1^n,\\
&t\cdot w_1^m b=\delta_t w_1^m b, && t\cdot w_1^m a_1^n=(\delta_{n,2}+\delta_t)w_1^m a_1^n,\\
&x_2\cdot w_1^m b=0, && x_2\cdot w_1^m a_1^n=\delta_{n,2}\delta_{m,1}a_1^n,\\
&x_3\cdot w_1^m b=\delta_t\delta_{m,1}b+w_1^m a_1, && x_3\cdot w_1^m a_1^n=(\delta_{n,1}+\delta_t)\delta_{m,1}w_1^m a_1^n,\\
&x_4\cdot w_1^m b=0, && x_4\cdot w_1^m a_1^n=\delta_{n,1}w_1^m w_2+\delta_{n,2}w_1^m b,\\
&w_4\cdot w_1^m b=0, && w_4\cdot w_1^m a_1^n=\delta_{n,1}\delta_{m,1}a_1^{n+1}.
\end{aligned}$$

In W_{δ_t}, D^0 acts diagonally on all generators except w_1 and x_3. Let $B\hookrightarrow\mathfrak{B}(V)$ be the subalgebra generated by everything except x_3 and $B'\hookrightarrow\mathfrak{B}(V^*)$ be the subalgebra generated by everything except w_1. Then, viewing W_{δ_t} as a module over $B\otimes D^0\otimes B'\hookrightarrow D$, we have the following weight decompositions:

$$\begin{aligned}
W_0&=(W_0)_{\lambda_{1,0}}\oplus(W_0)_{\lambda_{0,1}}\oplus(W_0)_{\lambda_{1,1}}\\
&=\Bbbk\{w_1^m,\ w_1^m w_2\}\oplus\Bbbk\{w_1^m b,\ w_1^m a_1\}\oplus\Bbbk\{w_1^m a_1^2\},
\end{aligned}$$

$$\begin{aligned}
W_1&=(W_1)_{\lambda_{1,1}}\oplus(W_1)_{\lambda_{0,0}}\oplus(W_1)_{\lambda_{1,0}}\\
&=\Bbbk\{w_1^m,\ w_1^m w_2\}\oplus\Bbbk\{w_1^m b,\ w_1^m a_1\}\oplus\Bbbk\{w_1^m a_1^2\}.
\end{aligned}$$

Lemma 5.2. *U_0, U_1, W_0, and W_1 are simple modules over D.*

Proof. U_0 is obviously simple. Consider some proper, nontrivial, submodule $U_1'\subset U_1$. Say U_1' contains an element $k_1w_1+k_2$ for $k_1,k_2\in\Bbbk$. Then, it must also contain $x_3\cdot(k_1w_1+k_2)=k_1$, contradiction. Thus, U_1 is simple.

Now, consider some proper, nontrivial, submodule $W'\subset W_{\delta_t}$. Consider some nonzero element $z\in W'$ of minimal degree; call this minimal degree d. Let z_d be the (nonzero) degree d component of z. The main observation is the following: $x_i\cdot z\in W'$ for all $i\in\mathbb{I}_4$. So, since z has minimal degree, we must have $x_i\cdot z=x_i\cdot z_d=0$.

- ***d* = 0:** Then, $1\in W'$, contradiction of W' being proper.
- ***d* = 1:** Then, $z_d=k_1w_1+k_2w_2$ for some $k_1,k_2\in\Bbbk$. So $x_2\cdot z_d=k_1=0$ and $x_3\cdot z_d=k_2=0$, contradiction.
- ***d* = 2:** Then, $z_d=k_1w_1w_2+k_2a_1$, so $x_2\cdot z_d=k_1w_2=0$ and $x_3\cdot z_d=k_2w_2=0$. Consequently, $k_1=k_2=0$, contradiction.
- ***d* = 3:** Then, $z_d=k_1b+k_2w_1a_1$, so $x_4\cdot y=k_2w_1w_2=0$ and $x_3\cdot y=k_1b=0$, contradiction.
- ***d* = 4:** Then, $z_d=k_1w_1b+k_2a_1^2$, so $x_4\cdot y=k_2b=0$ and $x_4x_3\cdot y=x_4\cdot(\delta_tb+k_1w_1a_1)=k_1w_1w_2=0$, contradiction.
- ***d* = 5:** Then, $y=k_1w_1a_1^2$, so $x_4\cdot y=k_1w_1b=0$, contradiction.

Thus, there exists no such W', and W_{δ_t} is indeed simple as well. $\square$

In fact, these are the *only* simples over D. To show this, we will use the induction functor to induct from simples over subalgebras of D to simples over D itself. For B a subalgebra of A, the functor $\mathrm{Ind}_B^A : B\text{-mod} \to A\text{-mod}$ takes a left B-module M to the left A-module $A \otimes_B M$ with action given by $a' \cdot (a \otimes_B m) = (a'a) \otimes_B m$ for all $a, a' \in A$ and $m \in M$. Every simple module of A can be recovered as a quotient of $\mathrm{Ind}(M)$ for some left B-module M.

Theorem 5.3. irrep $D = \{U_0, U_1, W_0, W_1\}$.

Proof. We induct with $D^0 \subset D^{\geq 0} := D^{>0} \otimes D^0$ and $D^{\geq 0} \subset D$. We first consider the simple modules D^0. Since $s^2 = s, t^2 = t$, and $h^4 = 1$, s and t must act by 0 or 1 and h must act by a fourth root of unity. In characteristic 2, the only fourth root of unity is 1, so h must act by 1. Thus, the 1-dimensional modules $\{\lambda_{\delta_s,\delta_t} : \delta_s, \delta_t \in \mathbb{I}_0^1\}$ are precisely the simple modules over D^0.

We now consider the simple modules over $D^{\geq 0}$. They are are quotients of $\mathrm{Ind}(\lambda) = D^{\geq 0} \otimes_{D^0} \lambda_{\delta_s,\delta_t}$. As PBW bases, since $\lambda_{\delta_s,\delta_t}$ is one dimensional,

$$\begin{aligned} D^{\geq 0} \otimes_{D^0} \lambda_{\delta_s,\delta_t} &\simeq \lambda_{\delta_s,\delta_t} \otimes_{D^0} D^{\geq 0} \\ &\simeq \lambda_{\delta_s,\delta_t} \otimes_{D^0} D^0 \otimes \mathfrak{B}(V) \\ &\simeq \lambda_{\delta_s,\delta_t} \otimes \mathfrak{B}(V) \\ &:= \mathfrak{B}(V) \cdot \lambda_{\delta_s,\delta_t}. \end{aligned}$$

The largest submodule of $\mathrm{Ind}(\lambda)$ is then $\bigoplus_{n\geq 1} \mathfrak{B}^n(V) \cdot \lambda_{\delta_s,\delta_t}$, so the simple modules of $D^{\geq 0}$ are isomorphic to $\mathfrak{B}(V) \cdot \lambda_{\delta_s,\delta_t} / (\bigoplus_{n\geq 1} \mathfrak{B}^n(V) \cdot \lambda_{\delta_s,\delta_t})$. Since $\bigoplus_{n\geq 1} \mathfrak{B}^n(V)$ is generated by $\{x_i \cdot \lambda : \forall i \in \mathbb{I}_4\}$, we have that $x_i \cdot \lambda = 0 \quad \forall i \in \mathbb{I}_4$ in this quotient. Thus, the simple modules of $D^{\geq 0}$ are the 1-dimensional modules $\Lambda_{\delta_s,\delta_t}$ with action given by

$$s \cdot 1 = \delta_s, \qquad t \cdot 1 = \delta_t, \qquad h \cdot 1 = 1, \qquad x_i \cdot 1 = 0, \quad \forall i \in \mathbb{I}_4.$$

Now, we determine the simple modules over D, which we recover as quotients of $\mathrm{Ind}(\Lambda_{\delta_s,\delta_t}) = D \otimes_{D^{\geq 0}} \Lambda_{\delta_s,\delta_t}$. As before, as PBW bases, since $\Lambda_{\delta_s,\delta_t}$ is 1-dimensional,

$$\begin{aligned} D \otimes_{D^{\geq 0}} \Lambda_{\delta_s,\delta_t} &\simeq \mathfrak{B}(V^*) \otimes D^{\geq 0} \otimes_{D^{\geq 0}} \Lambda_{\delta_s,\delta_t} \\ &\simeq \mathfrak{B}(V^*) \otimes \Lambda_{\delta_s,\delta_t} \\ &:= \mathfrak{B}(V^*) \cdot \Lambda_{\delta_s,\delta_t}. \end{aligned}$$

We have the following commutation relations, using the ones presented in Theorem 3.5:

$$\begin{aligned} &x_1 a_i = a_i x_1, \quad \forall i \in \mathbb{I}_2, && x_2 a_1 = a_1 x_2 + w_4, \\ &x_2 a_2 = a_2 x_2, && x_3 a_i = a_i x_3 + w_4, \quad \forall i \in \mathbb{I}_2 \\ &x_4 a_1 = (a_1 + a_2) x_4 + (w_2 + w_3) h, && x_4 a_2 = a_2 x_4 + w_3 h, \\ &x_1 b = b x_1, && x_2 b = b x_2 + a_2, \\ &x_3 b = b x_3 + a_1 + a_2, && x_4 b = b x_4 + w_2 w_3 h, \\ &x_i c = c x_i, \quad \forall i \in \mathbb{I}_2, && x_3 c = c x_3 + a_2 w_4, \\ &x_4 c = c x_4 + w_3 a_2, && s a_i = a_i (s + 1), \quad \forall i \in \mathbb{I}_2, \\ &t a_i = a_i (s + t + 1), \quad \forall i \in \mathbb{I}_2, && s b = b(s + 1), \\ &t b = b(s + t + 1), && s c = c s, \\ &t c = c(t + 1). \end{aligned}$$

This gives us the actions of $D^{\geq 0}$ on $\mathrm{Ind}(\Lambda_{\delta_s,\delta_t})$; for example

$$x_4 \cdot (a_2 \cdot \Lambda_{\delta_s,\delta_t}) = a_2 \cdot (x_4 \cdot \Lambda_{\delta_s,\delta_t}) + w_3 \cdot (h \cdot \Lambda_{\delta_s,\delta_t}) = w_3 \cdot \Lambda_{\delta_s,\delta_t}.$$

We now proceed with casework on the parameters δ_s and δ_t.

Case 1: $\delta_s = 0, \delta_t = 0$.

Using the relations presented above, we can check that $(\bigoplus_{n\geq 1} \mathfrak{B}(V^*) \cdot \Lambda_{0,0})$ is a submodule of $\mathrm{Ind}(\Lambda_{0,0})$, and U_0 is isomorphic to the quotient. This is trivially the only simple module induced from $(\Lambda_{0,0})$: every proper submodule of $\mathrm{Ind}(\Lambda_{0,0})$ is contained in $(\bigoplus_{n\geq 1} \mathfrak{B}(V^*) \cdot \Lambda_{0,0})$.

Case 2: $\delta_s = 0, \delta_t = 1$.

Let $\langle w_2, w_3, w_4\rangle$ be the left ideal of $\mathfrak{B}(V^*)$ generated by w_2, w_3, and w_4. Then, $N := (\langle w_2, w_3, w_4\rangle \cdot \Lambda_{0,1})$ is a submodule of $\mathrm{Ind}(\Lambda_{0,1})$, and U_1 is isomorphic to the quotient.

Assume for the sake of contradiction there exists a different proper submodule $N' \subset \mathrm{Ind}(\Lambda_{0,1})$ inducing an irreducible representation of D in the quotient. Since $(N + N')/N \subseteq \mathrm{Ind}(\Lambda_{0,1})/N$ and $(N + N')/N' \subseteq \mathrm{Ind}(\Lambda_{0,1})/N'$, we must have $N + N' = \mathrm{Ind}(\Lambda_{0,1})$ to not contradict irreducibility.

Observe that no element of N has a w_1 in it, and no commutation relation between non-w_1 generators "creates" a w_1. Thus, we must have $w_1 \cdot \Lambda_{0,1} \in N'$. In particular, $x_3w_1 \cdot \Lambda_{0,1} = (\delta_s + \delta_t) \cdot \Lambda_{0,1} = 1 \cdot \Lambda_{0,1} \in N'$, contradiction as N' would then equal $\mathrm{Ind}(\Lambda_{0,1})$. Thus, U_1 is the only simple induced from $\Lambda_{0,1}$.

Case 3: $\delta_s = 1$.

Let $\langle w_3, w_4, a_2, c, a_1^3, w_2a_1, ba_1\rangle$ be a left ideal of $\mathfrak{B}(V^*)$. Then, we can check that $K := (\langle w_3, w_4, a_2, c, a_1^3, w_2a_1, ba_1\rangle \cdot \Lambda_{1,\delta_t})$ is a submodule of $\mathrm{Ind}(\Lambda_{1,\delta_t})$, and W_{δ_t} is isomorphic to the quotient.

Assume for the sake of contradiction there exists a different $K' \subset \mathrm{Ind}(\Lambda_{1,\delta_t})$ inducing an irreducible representation in the quotient (for a fixed δ_t). As in Case 2, $K + K' = \mathrm{Ind}(\Lambda_{1,\delta_t})$, so since K contains no terms with a w_1 in it, $w_1 \cdot \Lambda_{1,\delta_t} \in K'$. But then, $x_2w_1 \cdot \Lambda_{1,\delta_t} = 1 \cdot \Lambda_{1,\delta_t} \in K'$, contradiction. Thus, W_{δ_t} is the only simple induced from Λ_{1,δ_t}.

Therefore, since all simples are quotients of $\mathrm{Ind}(\Lambda_{\delta_s,\delta_t})$, the only simples over D are U_0, U_1, W_0, and W_1. □

6. The Reflective Algebra of $\Bbbk$ with Respect to the Double

The earlier sections studied the double, its R-matrix, and its irreducible representations at $q = 1$. Now, we consider a parallel construction: the reflective algebra of $\Bbbk$ with respect to $D(H)$.

This reflective algebra has the underlying vector space $D(H)^*$. Before presenting it, we thus need to establish the algebra structure of $D(H)^*$. Observe that $D(H)^* \cong H^* \otimes H^{\mathrm{cop}}$ as algebras. $D(H)^*$ then has elements $f \otimes h$ for $f \in H^*$ and $h \in H$ with multiplication structure given by, for $\tilde{h} \bowtie \tilde{f} \in D(H)$,

$$\begin{aligned}\left\langle \mu_{D(H)^*}(f \otimes h), \tilde{h} \bowtie \tilde{f} \right\rangle &= \left\langle f \otimes h, \Delta(\tilde{h} \bowtie \tilde{f} \right\rangle \\ &= \langle f, \tilde{h}_{(1)}\tilde{f}_{(1)}\rangle\langle h, \tilde{h}_{(2)}\tilde{f}_{(2)}\rangle \\ &= \langle f, \tilde{h}_{(1)}\rangle\langle 1, \tilde{f}_{(1)}\rangle\langle 1, \tilde{h}_{(2)}\rangle\langle h, \tilde{f}_{(2)}\rangle.\end{aligned}$$

Inductively, starting from the generators of $D(H)$, the only term in $\Delta(\tilde{h})$ with group-like $\tilde{h}_{(2)}$ has $\tilde{h}_{(1)} = \tilde{h}$ and the only term in $\Delta(\tilde{f})$ with group-like $\tilde{f}_{(1)}$ has $\tilde{f}_{(2)} = \tilde{f}$. Thus, fh is the dual of $f^* \bowtie h^*$. Similarly, we can show that $\left\langle \mu_{D(H)^*}(h \otimes f), \tilde{h} \bowtie \tilde{f} \right\rangle = \delta_{\tilde{h},f^*}\delta_{\tilde{f},h^*}$, so $hf = fh$ in $D(H)^*$.

Theorem 6.1. *The reflective algebra of* $\Bbbk$ *with respect to* $D(H)$ *is generated by* $x_i, w_i,\ i \in \mathbb{I}_4,\ g, p, r, h, u, s, t, v$ *with relations* (3.1) – (3.23) *and*

$$\begin{aligned}
&x_i u = u x_i, \quad \forall i \in \mathbb{I}_4, && x_i v = v x_i, \quad \forall i \in \mathbb{I}_4\\
&x_i s = s x_i, \quad \forall i \in \mathbb{I}_4, && x_i t = t x_i, \quad \forall i \in \mathbb{I}_4,\\
&g w_i = w_i g, \quad \forall i \in \mathbb{I}_4, && p w_i = w_i p, \quad \forall i \in \mathbb{I}_4,\\
&r w_i = w_i r, \quad \forall i \in \mathbb{I}_4, && x_i w_j = w_j x_i, \quad \forall i < j \in \mathbb{I}_4,\\
&x_{i+1} w_i = w_i x_{i+1} + gsv, \quad \forall i \in \mathbb{I}_2, && x_3 w_1 = w_1 x_3 + \alpha gsv + gvt,\\
&x_i w_i = g + w_i x_i + gv, \quad \forall i \in \mathbb{I}_3, && x_4 w_4 = h + w_4 x_4 + hu.
\end{aligned}$$

The comodule structure is given by

$$\begin{aligned}
&\delta(g) = g \otimes g, \quad \delta(p) = p \otimes p, \quad \delta(r) = r \otimes r\\
&\delta(u) = u \otimes u, \quad \delta(v) = v \otimes v,\\
&\delta(s) = 1 \otimes s + s \otimes 1,\\
&\delta(t) = 1 \otimes t + t \otimes 1 + s \otimes s,\\
&\delta(x_i) = 1 \otimes x_i + x_i \otimes g, \quad \forall i \in \mathbb{I}_3,\\
&\delta(x_4) = 1 \otimes x_4 + x_4 \otimes h,\\
&\delta(w_1) = w_1 \otimes 1 + v \otimes w_1 + sv \otimes w_2 + (\alpha s + t)v \otimes w_3,\\
&\delta(w_2) = w_2 \otimes 1 + v \otimes w_2 + sv \otimes w_3,\\
&\delta(w_3) = w_3 \otimes 1 + v \otimes w_3,\\
&\delta(w_4) = w_4 \otimes 1 + u \otimes w_4.
\end{aligned}$$

The quantum K*-matrix is* $\sum_l h_l \otimes \xi_l$*, where*

$$\begin{aligned}
\{\xi_l\} := \{&w_1^{m_1} w_2^{m_2} w_3^{m_3} z_3^{*n_1} (z_3^2)^{*n_2} w^{*d} y^{*e} z_2^{*n_3} w_4^{*m_4}\\
&\cdot f_{g^i p^j r^k} g^{i'} p^{j'} r^{k'} x_1^{m_1'} x_2^{m_2'} x_3^{m_3'} z_3^{n_1'} (z_3^2)^{n_2'} w^{d'} y^{e'} z_2^{n_3'} x_4^{m_4'}\\
&: m_i, d, e, n_i, m_i', d', e', n_i' \in \mathbb{I}_0^1,\ k, k' \in \mathbb{I}_4\}
\end{aligned}$$

is the dual basis to

$$\begin{aligned}
\{h_l\} := \{&x_1^{m_1} x_2^{m_2} x_3^{m_3} z_3^{n_1} (z_3^2)^{n_2} w^d y^e z_2^{n_3} x_4^{m_4} g^i p^j r^k\\
&\cdot f_{g^{i'} p^{j'} r^{k'}} w_1^{m_1'} w_2^{m_2'} w_3^{m_3'} z_3^{*n_1'} (z_3^2)^{*n_2'} w^{*d'} y^{*e'} z_2^{*n_3'} w_4^{*m_4'}\\
&: m_i, d, e, n_i, m_i', d', e', n_i' \in \mathbb{I}_0^1,\ k, k' \in \mathbb{I}_4\}.
\end{aligned}$$

Proof. This follows from Definition 2.5. As an example of the multiplication computation, we show why the relation $w_1 x_1 = 1 + x_1 w_1 + g$ holds in the reflective algebra:

$$\begin{aligned}
\mu_{R_{D(H)}(\Bbbk)}(x_1 \otimes w_1) &= \langle v, 1\rangle\langle 1, 1\rangle\langle 1, f_1\rangle\langle 1, f_1\rangle w_1 x_1 + \langle w_1, x_1\rangle\langle v, 1\rangle\langle x_1, f_1 w_1\rangle\langle 1, f_1\rangle gv\\
&\quad + \langle 1, g\rangle\langle w_1, g^{-1} x_1\rangle\langle g, f_g\rangle\langle x_1, w_1\rangle g\\
&= w_1 x_1 + gv + g.
\end{aligned}$$

Further,

$$\begin{aligned}\mu_{R_{D(H)}(\Bbbk)}(w_1 \otimes x_1) &= \langle 1, 1\rangle\langle g, 1\rangle\langle 1, f_1\rangle\langle 1, f_1\rangle x_1 w_1\\ &= x_1 w_1.\end{aligned}$$

Thus, since $x_1 w_1 = w_1 x_1$ in $D(H)^*$, we have $x_1 w_1 = g + w_1 x_1 + gv$ in the reflective algebra, as desired. The other calculations follow similarly.

Remark that $\{h_l, \xi_l\}_l$ do indeed form a dual basis of $D(H)$ because $\{e_l, e^l\}_l$ forms a dual basis of H. As an example of the comodule computation, we compute the coaction at x_1 :

$$\begin{aligned}\delta_{R_{D(H)}(\Bbbk)}(x_1) = &\sum_{i,j,k,i',j',k'} \langle g, g^i p^j r^k\rangle\langle x_1, g^{i'} p^{j'} r^{k'} f_1 w_1\rangle\langle 1, f_1\rangle f_{g^i p^j r^k} \otimes f_{g^{i'} p^{j'} r^{k'}} x_1\\ &+ \sum_{i,j,k,i',j',k'} \langle g, g^i p^j r^k\rangle\langle g, g^{i'} p^{j'} r^{k'} f_g\rangle\langle x_1, f_1 w_1\rangle f_{g^i p^j r^k} x_1 \otimes f_{g^{i'} p^{j'} r^{k'}} g\\ = &\sum_{i,j,k} f_{g^i p^j r^k} \otimes \sum_{i',j',k'} f_{g^{i'} p^{j'} r^{k'}} x_1 + \sum_{i,j,k} f_{g^i p^j r^k} x_1 \otimes \sum_{i',j',k'} f_{g^{i'} p^{j'} r^{k'}} g\\ = &\, 1 \otimes x_1 + x_1 \otimes g,\end{aligned}$$

where the last line follows as $\sum_{i,j,k} f_{g^i p^j r^k} = \sum_{i',j',k'} f_{g^{i'} p^{j'} r^{k'}} = 1$. The other computations follow analogously. □

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