PRIMES 2024: ENTRANCE PROBLEM SET

Notation. We let $\mathbb{Z}$ and $\mathbb{R}$ denote the set of integers and the set of real numbers, respectively. Also, we let $\mathbb{P}$, $\mathbb{N}$, and $\mathbb{N}_0$ denote the set of primes, positive integers, nonnegative integers, respectively.

General Math Problems

Problem G1 (General). Hogwarts has quite peculiar habits and games.

(a) Gryffindor fans tell the truth when Gryffindor wins and lie when it loses. Fans of Hufflepuff, Ravenclaw, and Slytherin behave similarly. After two matches of quidditch with the participation of these four teams (with no draws), among the wizards who watched the broadcast, 500 answered positively to the question "Do you support Gryffindor?", 600 answered positively to the question "Do you support Hufflepuff?", 300 answered positively to the question "Do you support Ravenclaw?", and 200 answered positively to the question "Do you support Slytherin?". How many wizards support each of the teams? Note: Each wizard is fan of exactly one of the teams.

(b) There is a bucket of N candies leftover from Halloween ($N \geq 2$). Two friends, Hermione Granger and Ron Weasley, take turns to disappear candies from the bucket as follows. The first turn, Hermione must disappear at least one candy and cannot disappear all of the candies. Then taking turns, each of them must disappear at least one candy and at most $9/4$ times the number of candies disappeared by her/his friend in the previous turn. The winner is the one disappearing the last candy. Assume that Hermione and Ron play optimally.

  (i) For which numbers N does Hermione have a winning strategy? Justifying your answer.

  (ii) Answer the previous question replacing $9/4$ by 3.

Comments: Proposed by Leonid Rybnikov (part (a)) and Nitya Mani (part (b)).

Solution.

(a) Let A and B be the winning teams, and let a and b be the numbers of fans of A and B, respectively. Similarly, let C and D be the losing teams, and let c and d be the numbers of fans of C and D, respectively. When a wizard is asked "Do you support A?",

- the wizard will answer positively if he/she is an A fan (because A won, and so he/she will tell the truth),
- the wizard will answer positively if he/she is either a C or a D fan (because C and D lost, and so he/she will lie), and
- the wizard will answer negatively if he/she is a B fan (because B won and so he/she will tell the truth).

Therefore the total number of positive answers to the question "Do you support A?" is $a+c+d$. Similarly, the total number of positive answers to the question "Do you support B?" is $b+c+d$. On the other hand, when a wizard is asked "Do you support C?",

- the wizard will answer negatively if he/she is either an A or a B fan (because A and B won, and so he/she will tell the truth),
- the wizard will answer negatively if he/she is a C fan (because C lost, and so he/she will lie), and

- the wizard will answer positively if he/she is a D fan (because D lost, and so he/she will lie).

Thus, the total number of positive answers to the question "Do you support C?" is d. In a similar manner, we see that the total number of positive answers to the question "Do you support D?" is c. Now observe that both numbers $a+c+d$ and $b+c+d$ are greater than or equal to both numbers c and d. Therefore we conclude that Ravenclaw and Slytherin lost and had, respectively, 200 and 300 fans. This means that there were $500-200-300=0$ Gryffindor fans and there were $600-200-300=100$ Hufflepuff fans. □

(b) (i) Here is the general strategy for 9/4. Compute the Zeckendorf representation of N; that is, write N as a sum of nonconsecutive Fibonacci numbers, which can be done uniquely. The first player wins unless N is a Fibonacci number; the optimal strategy involves computing the Zeckendorf representation of the current number and then removing the smallest part of the Zeckendorf representation (i.e. the smallest Fibonacci number in the sum).

(ii) When we replace 9/4 by 3, the strategy is similar except that we compute the generalized Zeckendorf representation of N with respect to the recurrence $P_n = P_{n-1} + P_{n-4}$ with initial conditions $0, 1, 2, 3, 4, 6$. Player 2 wins if $N = P_n$ for some n, and player 1 has a similar winning strategy.

□

Problem G2 (Elementary Geometry). Suppose that each edge of a given convex hexagon has distance 1 to the origin (this means, each edge is contained in a line whose distance to the origin equals 1). What is the minimum possible area enclosed by this hexagon? Justify your answer.

Comments: Proposed by Jingze Zhu.

Solution. The space of hexagon is compact, so the minimizer exists and is non-degenerate (i.e the intersection point between consecutive edges are points within finite distance.)

Suppose that we take the minimizer and label the vertices as $A_1, \ldots, A_6$ in clockwise fashion and each corresponding interior angles are $\alpha_1, \ldots, \alpha_6$, respectively. We can draw a line from O to each segment A_iA_{i+1} with the intersection point to be H_i. Let $|A_iH_i| = L_i$ and $|H_iA_{i+1}| = R_i$. We assume that L_i, R_i are nonnegative, for other cases the proof will be similar. We will rotate the vertex A_iA_{i+1} counterclockwise around H_1 by an angle θ and see how the area changes. By drawing pictures and inspecting the triangles, we find that the area change is

$$\Delta Area = \frac{L_i^2 \sin(\theta)\sin(\alpha_i)}{\sin(\alpha_i + \theta)} - \frac{R_i^2 \sin(\theta)\sin(\alpha_{i+1})}{\sin(\alpha_{i+1} - \theta)}$$

Note that we will take $|\theta|$ small enough. If $L_i > R_i$, then we can find small $\theta > 0$ such that $\Delta Area > 0$. this will contradict the fact that we have taken the minimizer. If $L_i < R_i$, in the same way we can find $\theta < 0$ and $|\theta|$ is small enough such that $\Delta Area > 0$, a contradiction again. So $L_i = R_i$. By Pythagoras theorem, it is not hard to see that

$$L_i^2 + |OH_i|^2 = |OA_i^2|, \quad R_i^2 + |OH_i|^2 = |OA_{i+1}^2|$$

Thus, $L_i = R_{i-1}$. Inductively, we will see that $L_1 = L_2 = \cdots = L_6 = R_1 = \cdots = R_6$. Then all edges are of the same length and all angles α_i are the same. So the minimum area is achieved by the regular hexagon, whose area is $2\sqrt{3}$. □

Problem G3 (Number Theory). For any positive $a, b \in \mathbb{Z}$, we define $\text{pow}(a,b)$ inductively in the following way: $\text{pow}(a,1) = a$ and $\text{pow}(a,b) = a^{\text{pow}(a,b-1)}$ if $b \geq 2$.

(a) Prove that for any positive $k, n \in \mathbb{Z}$ with $\gcd(k, n) = 1$, there exists $c \in \mathbb{Z}$ with $0 \le c < n$ and $M \in \mathbb{N}$ such that $\mathrm{pow}(k, m) \equiv c \pmod n$ for all $m \in \mathbb{Z}$ such that $m \ge M$: we denote c by $f_n(k)$.

(b) Prove that for every positive integer n, the inclusion $(\mathbb{Z}/n\mathbb{Z})^\times \subseteq \mathrm{Im}(f_n)$ holds, where $\mathrm{Im}(f_n)$ is the image of the function $f_n \colon \mathbb{Z} \to \mathbb{Z}$.

Comments: Proposed by Benjamin Li.

Solution.

(a) We proceed by induction on n. The case $n = 1$ is obvious. Now fix $k, n \in \mathbb{Z}$ with $n > 1$, and assume that the statement we wish to prove holds for any pair (n', k') of positive integers such that $n' < n$. Since $\varphi(n) < n$, there exists $c' \in \mathbb{Z}$ with $0 \le c' < \varphi(n)$ such that $\mathrm{pow}(k, m) \equiv c' \pmod{\varphi(n)}$ for all sufficiently large m. Thus, $\mathrm{pow}(k, m') \equiv k^{\mathrm{pow}(k, m'-1)} \equiv k^{c'} \pmod n$ for all sufficiently large m', as desired. Note: Also, we have obtained the following $f_n(k) \equiv k^{f_{\varphi(n)}(k)} \pmod n$. □

(b) To easy notation, we use $L(n)$ to denote the least common multiple of $n, \varphi(n), \varphi(\varphi(n)), \ldots$, while we use $P(n)$ to denote the largest prime factor of n. Note that the value of $f_n(k)$ only depends on $k \pmod{L(n)}$, and $P(\varphi(n)) \le P(n)$. To solve this problem, we use induction on n to prove a stronger statement: for any b such that $\gcd(n, b) = 1$, there exists k such that $\gcd(k, L(n)) = 1$ and $f_n(k) = b$. For $n = 1$, the corresponding statement is obviously true. Now assume $n > 1$. Set $p = P(n)$, and write $n = n_0 p^t$, where $P(n_0) \le p$. We split the rest of the solution into the following two cases.

Case 1: $t = 1$. Take any b_1 and b_2 such that $\gcd(n_0, b_1) = \gcd(p, b_2) = 1$. We want to find k such that $f_{n_0}(k) = b_1$, $f_p(k) = b_2$, and $\gcd(k, L(n)) = 1$.

Claim: For any positive $n_1, n_2, b \in \mathbb{Z}$ with $\gcd(n_1, b) = 1$, there exists a positive $k \in \mathbb{Z}$ such that $k \equiv b \pmod{n_1}$ and $\gcd(k, n_2) = 1$.

Proof of Claim: Set $d := \gcd(n_1, n_2)$, and write $n_2 = n_2' d$, where all prime factors of d are also factors of n_1, and n_2' shares no common factor with n_1. Then $\gcd(b, d) = 1$, and we want to find ℓ such that $\gcd(\ell n_1 + b, n_2' d) = 1$. First, we must have $\gcd(\ell n_1 + b, d) = 1$, so it suffices to find ℓ such that $\gcd(\ell n_1 + b, n_2') = 1$, which is possible because $\gcd(n_2', n_1) = 1$. The claim is now established.

Let $L(n) = pN$, then $P(N) < p$ and $\varphi(n) \mid N$. From our induction hypothesis, we can find k_0 coprime with $L(n_0)$ such that $f_{n_0}(k_0) = b_1$. Thus, the established claim guarantees the existence of k_1 such that $k_1 \equiv k_0 \pmod{L(n_0)}$ and $\gcd(k_1, NL(n_0)) = 1$. Now let $f_{\varphi(n)}(k_1) = b \in (\mathbb{Z}/\varphi(n)\mathbb{Z})^\times$. Consider the homomorphism $(\mathbb{Z}/p\mathbb{Z})^\times \to (\mathbb{Z}/p\mathbb{Z})^\times$ given by the assignments $x \mapsto x^b$. This map is an isomorphism with inverse map $(\mathbb{Z}/p\mathbb{Z})^\times \to (\mathbb{Z}/p\mathbb{Z})^\times$ defined via the assignments $x \mapsto x^{b^{-1}}$, where b^{-1} is the inverse of b in $(\mathbb{Z}/(p-1)\mathbb{Z})^\times$. Therefore we can pick $x \in (\mathbb{Z}/p\mathbb{Z})^\times$ such that $x^b \equiv b_2 \pmod p$. Now we can find k such that $k \equiv k_1 \pmod{NL(n_0)}$ and $k \equiv x \pmod p$, which is the desired k.

Case 2: $t > 1$. Take any b_1 and b_2 such that $\gcd(n_0, b_1) = \gcd(p, b_2) = 1$. We want to find k such that $f_{n_0}(k) = b_1$, $f_{p^t}(k) = b_2$, and $\gcd(k, L(n)) = 1$. Our induction hypothesis ensures the existence of k_0 such that $\gcd(k_0, L(n_0 p^{t-1})) = 1$, $f_{n_0}(k_0) = b_1$, and $f_{p^{t-1}}(k_0) \equiv b_2 \pmod{p^{t-1}}$. Now let $L(n_0) = N_0$, $L(n_0 p^{t-1}) = p^{t-1} N_1$, and $L(n_0 p^t) = p^t N_2$. Then $p \nmid N_0 N_1 N_2$. Write $N = N_0 N_1 N_2$, and set $b_2' := f_{p^t}(k_0)$. Observe that $b_2' \equiv b_2 \pmod{p^{t-1}}$. Now consider $k := k_1 + \ell p^{t-1} N$. We have

$$f_{p^t}(k) \equiv (k + p^{t-1} N \ell)^{b_2} \equiv k_1^{b_2} + b_2 N p^{t-1} \ell \equiv b_2' + p^{t-1}(b_2 N \ell) \pmod{p^t}.$$

As ℓ runs through $\mathbb{Z}/p\mathbb{Z}$, the right-hand side of the displayed expression will also touch b_2 because $b_2' \equiv b_2 \pmod{p^{p-1}}$ (note that $p \nmid b_2 N$), so we have found the desired k. $\square$

Problem G4 (Algorithm with flavor of number theory).

(a) Describe an algorithm, with proof, to compute all possible ways to write a given positive integer n as the sum of squares of consecutive positive integers. For example, for $n = 25$, we can write $25 = 5^2$ and $25 = 3^2 + 4^2$. Include your code as part of your solution (feel free to use your favorite programming language).

(b) What is the time complexity of your algorithm?

(c) What is the first number that is NOT a perfect square which can be written as the sum of consecutive squares in three different ways? Hint: it is less than 150000.

Comments: Proposed by Arun Kannan.

Solution.

(a) Suppose that n can be written as the sum of l consecutive squares starting at k. Let us see what we can reason about l and k. We can write

$$\begin{aligned} n &= k^2 + (k+1)^2 + \cdots + (k+l-1)^2 \\ &= k^2 + (k^2 + 2k + 1^2) + \cdots + (k^2 + 2k(l-1) + (l-1)^2) \\ &= lk^2 + 2k(1 + 2 + \cdots + (l-1)) + (1^2 + 2^2 + \cdots + (l-1)^2) \\ &= lk^2 + 2k\frac{l(l-1)}{2} + \frac{(l-1)l(2l-1)}{6}. \end{aligned}$$

We can rewrite this as a quadratic equation in k and then solve for k.

$$\begin{aligned} 0 &= k^2 + (l-1)k + \left(\frac{(l-1)(2l-1)}{6} - \frac{n}{l}\right) \\ \implies k &= \frac{-(l-1) \pm \sqrt{(l-1)^2 - 4\left(\frac{(l-1)(2l-1)}{6} - \frac{n}{l}\right)}}{2}. \end{aligned}$$

The discriminant must be nonnegative because k is real, so we need the following inequality to hold:

$$\begin{aligned} (l-1)^2 - 4\left(\frac{(l-1)(2l-1)}{6} - \frac{n}{l}\right) &\geq 0 \\ \implies n &\geq \frac{(l-1)l(l+1)}{12}. \end{aligned}$$

Moreover, for k to be positive, we need the following inequality to hold:

$$\begin{aligned} \frac{-(l-1) + \sqrt{(l-1)^2 - 4\left(\frac{(l-1)(2l-1)}{6} - \frac{n}{l}\right)}}{2} &> 0 \\ \implies n &> \frac{(l-1)l(2l-1)}{6}. \end{aligned}$$

It is easy to see that the second inequality is stronger than the first inequality as $\frac{2l-1}{6} \geq \frac{l+1}{12}$ for all $l \geq 1$. We deduce that for each value of l, there is at most one value of k that works and moreover it suffices to consider l such that $l \geq 1$ and $\frac{(l-1)l(2l-1)}{6} < n$. We can then iterate through all possible values of l and check if the corresponding value of k is an integer. If so, then we can list out the l consecutive squares that sum up to n starting at k^2.

Below is a Python program that does just that:

```
from math import sqrt
def consecutive_squares(n):
l = 1
csum = (l-1)*(2*l-1)/6
while n > l*csum:
d = (l-1)**2 - 4*(csum - n/l)

k = (-(l-1) + sqrt(d))/2
if k == int(k):
k = int(k)
squares = "".join(
[str(int(i)) + "^2+" for i in range(k, k+l)]
)
output = str(n) + " = " + squares[:-1]
print(output)
l=l+1
csum = (l-1)*(2*l-1)/6
```

□

(b) The time complexity is $O(n^{1/3})$ because the number of values of l that we need to check is bounded above by $n^{1/3}$ because $\frac{(l-1)l(2l-1)}{6} < n$ and $l \geq 1$. □

(c) The first number that is not a perfect square which can be written as the sum of consecutive squares in three different ways is 147441.

$$\begin{aligned} 147441 &= 85^2 + 86^2 + \cdots + 101^2 \\ &= 29^2 + 30^2 + \cdots + 77^2 \\ &= 18^2 + 19^2 + \cdots + 76^2. \end{aligned}$$

This can be found in a matter of seconds by checking each number between 1 and 150000 using the program above. □

Remark: There can be different solutions to this problem. For instance, there is a dynamic programming approach that can compute the answer in $O(n^{4/3})$ time complexity by computing the answer for all $k \leq n$.

Problem G5 (Elementary Algebra and Sequences). A nonempty set S consisting of positive real numbers is called an *additive set* if $x + y \in S$ when $x, y \in S$. Let S be an additive set. An element of S is called *indecomposable* if it is not the sum of two (not necessarily distinct) elements of S, and S is called *decomposable* if every element of S can be written as a finite sum of indecomposable elements (allowing repetitions and sums consisting of only one summand). Prove that if S is an additive set and there exists a strictly decreasing sequence $(x_n)_{n\geq 1}$ such that $\{x_n, x_n - x_{n+1} : n \in \mathbb{N}\} \subseteq S$, then there exists an additive set contained in S that is not decomposable.

Comments: Proposed by Felix Gotti.

Solution. Let us introduce some useful terminology. For $m,n \in \mathbb{Z}$ with $m \le n$, we set $[\![m,n]\!] := \{i \in \mathbb{Z} : m \le i \le n\}$. Let S be an additive set. We let $\mathscr{A}(S)$ denote the set consisting of all indecomposable elements of S. For $s,t \in S$ such that $t-s=0$ or $t-s \in S$, we write $s \mid_S t$ and say that s *divides* t in S. A sequence $(s_n)_{n\ge 1}$ of elements of S is called *ascending* if $s_{n+1} \mid_S s_n$ for every $n \in \mathbb{N}$, while $(s_n)_{n\ge1}$ is said to *stabilize* if there exists $N \in \mathbb{N}$ such that $s_n = s_N$ for every $n \ge N$. Finally, if R is a set consisting of positive real numbers, we let $\langle R\rangle$ denote the set consisting of all finite sums of elements in R (allowing repetitions), which is precisely the smallest additive set containing the set R.

Let S be an additive set containing a sequence described in the statement of the problem. If S is not decomposable we are done. So we assume that S is decomposable. Let us first verify that there is a sequence $(q_n)_{n\ge0}$ of elements of S such that $q_n - q_{n+1} \in \mathscr{A}(S)$ for every $n \in \mathbb{N}$. To do so, let $(\ell_n)_{n\ge0}$ be a sequence with terms in S as described in the statement of the problem. Take $q_0 = \ell_0$. Since $q_0 - \ell_1 \in S$ and S is decomposable, we can take $a_1 \in \mathscr{A}(S)$ such that $a_1 \mid_M q_0 - \ell_1$. After setting $q_1 := q_0 - a_1$, we see that $\ell_1 \mid_S q_1$. Now suppose that we have found $q_0, \dots, q_n \in S$ such that $q_{j-1} - q_j \in \mathscr{A}(S)$ and $\ell_j \mid_M q_j$ for every $j \in [\![1,n]\!]$. As $\ell_{n+1} \mid_S \ell_n$ and $\ell_n \mid_S q_n$, we can choose $a_{n+1} \in \mathscr{A}(S)$ so that $a_{n+1} \mid_S q_n - \ell_{n+1}$. Then we can set $q_{n+1} := q_n - a_{n+1}$. From $a_{n+1} \mid_S q_n - \ell_{n+1}$ and $q_{n+1} := q_n - a_{n+1}$, we can readily deduce that $\ell_{n+1} \mid_S q_{n+1}$. So we can assume the existence of a sequence $(q_n)_{n\ge0}$ with terms in S and a sequence $(a_n)_{n\ge1}$ with terms in $\mathscr{A}(S)$ such that $q_n := q_{n+1} + a_{n+1}$ for every $n \in \mathbb{N}_0$. Note that the series $\sum_{n\in\mathbb{N}} a_n$ converges as its sequence of partial sums is bounded by q_0; indeed, $q_0 = q_n + \sum_{i=1}^n a_i$ for every $n \in \mathbb{N}$.

Now we will construct a strictly increasing sequence $(k_n)_{n\ge1}$ of positive numbers with $a_1 > a_{k_1} + a_{k_2}$ and satisfying that, for every $n \in \mathbb{N}$, the following conditions hold:

(1) $q_0 \notin \langle a_{k_{2i-1}} + a_{k_{2i}} \mid i \in [\![1,n]\!]\rangle$, and

(2) the set of indecomposable elements of $\langle a_{k_{2i-1}} + a_{k_{2i}} \mid i \in [\![1,n]\!]\rangle$ is $\{a_{k_{2i-1}} + a_{k_{2i}} \mid i \in [\![1,n]\!]\}$.

We proceed inductively. Take $k_1 = 2$. The set $\mathbb{Q}_{>0} \cap \{q_0/n - a_2 \mid n \in \mathbb{N}\}$ is clearly finite. Since $(a_n)_{n\ge1}$ converges to zero, there is an $i \in \mathbb{N}$ with $i \ge 3$ such that $a_1 > a_2 + a_i$ and $a_i \notin \{q_0/n - a_2 \mid n \in \mathbb{N}\}$. After taking $k_2 = i > k_1$, we see that $a_1 > a_{k_1} + a_{k_2}$ and $q_0 \notin \langle a_{k_1} + a_{k_2}\rangle$. In addition, it is clear that $a_{k_1} + a_{k_2}$ is the only indecomposable of $\langle a_{k_1} + a_{k_2}\rangle$. Now suppose that we have already found for some $n \in \mathbb{N}$, positive integers $k_1, \dots, k_{2n}$ with $k_1 < \dots < k_{2n}$ satisfying conditions (1) and (2) above. Take $k_{2n+1} = k_{2n} + 1$, and consider the set

$$Q := q_0 - \langle a_{k_{2i-1}} + a_{k_{2i}} \mid i \in [\![1,n]\!]\rangle.$$

It is clear that $Q_{>0}$ is finite, and so after defining

$$Q' := \Big\{ \frac{q}{n} - a_{k_{2n+1}} \ \Big|\ q \in Q \text{ and } n \in \mathbb{N} \Big\},$$

we obtain that $Q'_{>0}$ is also a finite set. So there exists $j_1 \in \mathbb{N}$ such that $a_j \notin Q'$ for any $j \in \mathbb{N}$ with $j \ge j_1$. Now consider the set

$$T := \bigcup_{i=1}^{n} \Big(a_{k_{2i-1}} + a_{k_{2i}} - \langle a_{k_{2j-1}} + a_{k_{2j}} \mid j \in [\![i+1,n]\!]\rangle \Big).$$

Observe that $T_{>0}$ is finite, and so if we set

$$T' := \Big\{ \frac{t}{n} - a_{k_{2n+1}} \ \Big|\ t \in T \text{ and } n \in \mathbb{N} \Big\},$$

we obtain that $T'_{>0}$ is also finite. As a result, there exists $j_2 \in \mathbb{N}$ such that $a_j \notin T'$ for any $j \ge j_2$. So we can take $j \in \mathbb{N}$ large enough so that $j > k_{2n+1}$ and $a_{k_{2n-1}} + a_{k_{2n}} > a_{k_{2n+1}} + a_j$ and $a_j \notin Q' \cup T'$. Now set $k_{2n+2} = j$. Because $a_j \notin Q'$, it follows that Q is disjoint from $\langle a_{k_{2n+1}}, a_{k_{2n+2}}\rangle$, which implies condition (1): $q_0 \notin \big\langle a_{k_{2i-1}} + a_{k_{2i}} \mid i \in [\![1, n+1]\!] \big\rangle$. On the other hand, the fact that $a_j \notin T'$ guarantees that T is disjoint from $\langle a_{k_{2n+1}}, a_{k_{2n+2}} \rangle$. In turn, this implies that

$$a_{k_{2i-1}+a_{k_{2i}}} \notin \big\langle a_{k_{2j-1}} + a_{k_{2j}} \mid j \in [\![i+1, n+1]\!] \big\rangle$$

for any $i \in [\![1, n]\!]$. This, together with the fact that $a_{k_1} + a_{k_2} > \cdots > a_{k_{2n+1}} + a_{k_{2n+2}}$, guarantees condition (2): the set of indecomposable elements of $\big\langle a_{k_{2i-1}} + a_{k_{2i}} \mid i \in [\![1, n+1]\!] \big\rangle$ is $\big\{ a_{k_{2i-1}} + a_{k_{2i}} \mid i \in [\![1, n+1]\!] \big\}$. Hence we have constructed a strictly increasing sequence $(k_n)_{n \in \mathbb{N}}$ of positive integers such that conditions (1) and (2) above hold.

Finally, for each $n \in \mathbb{N}$ set $a'_n := a_{k_{2n-1}} + a_{k_{2n}}$ and $s'_n := \sum_{i=1}^n a'_i$. For every $n \in \mathbb{N}$, it is clear that $s_n \mid_S s_m$ for some $m \in \mathbb{N}$. Since $k_1 = 2$, from $q_0 \geq \sum_{n=1}^\infty a_n$ we obtain that

$$\sum_{n=1}^{\infty} a'_n = \sum_{n=1}^{\infty} a_{k_{2n-1}} + a_{k_{2n}} \leq \sum_{n=2}^{\infty} a_n \leq q_0 - a_1. \tag{0.1}$$

Let $(q'_n)_{n \geq 1}$ be the sequence of rational numbers defined as follows: take $q'_0 = q_0$ and take $q'_n = q_0 - s'_n$ for every $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$, and then take an $m \in \mathbb{N}$ such that $s_m = t + s'_n$ for some $t \in S$. Now we see that $q'_n = q_0 - s'_n = (q_0 - s_m) + (s_m - s'_n) = (q_0 - s_m) + t \in S$. Then the terms of the sequence $(q'_n)_{n \geq 1}$ are indeed in S. Therefore the additive set $N := \langle q'_n, a'_n \mid n \in \mathbb{N} \rangle$ is contained in S. Since $q'_n = q_0 - s'_n = (q_0 - s'_{n+1}) + a'_{n+1} = q'_{n+1} + a'_{n+1}$, it follows that $\mathscr{A}(N) \subseteq A := \{a'_n \mid n \in \mathbb{N}\}$. In addition, for each $m \in \mathbb{N}$, in light of (0.1), we see that

$$q'_m = q_0 - \sum_{i=1}^{n} a'_i \geq q_0 - \sum_{n=1}^{\infty} a'_n \geq q_0 - (q_0 - a_1) = a_1 > a'_n$$

for every $n \in \mathbb{N}$. As a result, none of the elements in the set A is divisible by any q_m in N. Then condition (2) above ensures that every element of A belongs to $\mathscr{A}(N)$. Thus, $\mathscr{A}(N) = \{a'_n \mid n \in \mathbb{N}\}$. Therefore it follows from condition (1) above that $q_0 \notin \langle \mathscr{A}(N) \rangle$, and so N is not decomposable. Thus, we have constructed an additive set contained in S that is not decomposable, which concludes our proof. □