PRIMES MATH PROBLEM SET
PRIMES 2024
DUE NOVEMBER 30, 2023

Dear PRIMES applicant:

This is the PRIMES 2023 Math Problem Set. Please send us your solutions as part of your PRIMES application by November 30, 2023. For complete rules, see the following link: http: //math.mit.edu/research/highschool/primes/apply.php

- Note that this set contains two parts: "General Math Problems" and "Advanced Math Problems." Please, solve as many problems as you can in both parts.
- You can type the solutions or write them up by hand and then scan them. Please attach your solutions to the application as a PDF file. The name of the attached file must start with your last name, for example, "`etingof-solutions.pdf`" or similar. Include your full name in the heading of the file.
- Please, write not only answers, but also proofs (and partial solutions/results/ideas if you cannot completely solve the problem). Besides the admission process, your solutions will be used to decide which projects would be most suitable for you if you are accepted to PRIMES.
- Submissions in LATEX are preferred, but handwritten submissions are also accepted.
- You are allowed to use any resources to solve these problems, except other people's help. This means that you can use calculators, computers, books, and the Internet. However, if you consult books or Internet sites, please give us a reference.
- **Note that posting these problems on problem-solving websites before the application deadline is strictly forbidden!** Applicants who do so will be disqualified, and their parents and recommenders will be notified. Note that some of these problems are tricky. We recommend that you do not leave them for the last day. Instead, think about them, on and off, over some time, perhaps several days.

Why it makes no sense to cheat

PRIMES expects its participants to adhere to MIT rules and standards for honesty and integrity in academic studies. As a result, **any cases of plagiarism, unauthorized collaboration, cheating, or facilitating academic dishonesty during the application process or during the work at PRIMES may result in immediate disqualification from the program, at the sole discretion of PRIMES.** In addition, PRIMES reserves the right to notify a participant's parents, schools, and/or recommenders in the event it determines that a participant did not adhere to these expectations. For explanation of these expectations, see *What is Academic Integrity?*, integrity.mit.edu.

Moreover, even if someone gets into PRIMES by cheating, it would immediately become apparent that their background is weaker than expected, and they are not ready for research. This would prompt an additional investigation with serious consequences. By trying to get into PRIMES by cheating, students run very serious risks of exposing their weak background and damaging their college admissions prospects.

Note: This entrance problem set is larger than those of previous years, so we expect competitive applicants to solve at least 60% of the problems (unlike previous years, when competitive applicants were expected to solve at least 70% of the problems). However, we encourage you to apply if you can solve at least 40% of the problems.

ENJOY!

PRIMES 2024: ENTRANCE PROBLEM SET

Notation. We let $\mathbb{Z}$ and $\mathbb{R}$ denote the set of integers and the set of real numbers, respectively. Also, we let $\mathbb{P}$, $\mathbb{N}$, and $\mathbb{N}_0$ denote the set of primes, positive integers, and nonnegative integers, respectively.

General Math Problems

Problem G1. Hogwarts has quite peculiar habits and games.

(a) Gryffindor fans tell the truth when Gryffindor wins and lie when it loses. Fans of Hufflepuff, Ravenclaw, and Slytherin behave similarly. After two matches of quidditch with the participation of these four teams (with no draws and each team playing exactly one game), among the wizards who watched the broadcast, 500 answered positively to the question "Do you support Gryffindor?", 600 answered positively to the question "Do you support Hufflepuff?", 300 answered positively to the question "Do you support Ravenclaw?", and 200 answered positively to the question "Do you support Slytherin?". How many wizards support each of the teams? Note: Each wizard is fan of exactly one of the teams.

(b) There is a bucket of N candies leftover from Halloween ($N \geq 2$). Two friends, Hermione Granger and Ron Weasley, take turns to disappear candies from the bucket as follows. The first turn, Hermione must disappear at least one candy and cannot disappear all of the candies. Then taking turns, each of them must disappear at least one candy and at most $9/4$ times the number of candies disappeared by her/his friend in the previous turn. The winner is the one disappearing the last candy. Assume that Hermione and Ron play optimally.

(i) For which numbers N does Hermione have a winning strategy? Justifying your answer.

(ii) Answer the previous question replacing $9/4$ by 3.

Problem G2. Suppose that each edge of a given convex hexagon has distance 1 to the origin (this means, each edge is contained in a line whose distance to the origin equals 1). What is the minimum possible area enclosed by this hexagon? Justify your answer.

Problem G3. For any positive $a, b \in \mathbb{Z}$, we define $\mathrm{pow}(a, b)$ inductively in the following way: $\mathrm{pow}(a, 1) = a$ and $\mathrm{pow}(a, b) = a^{\mathrm{pow}(a,b-1)}$ if $b \geq 2$.

(a) Prove that for any positive $k, n \in \mathbb{Z}$ with $\gcd(k, n) = 1$, there exists $c \in \mathbb{Z}$ with $0 \leq c < n$ and $M \in \mathbb{N}$ such that $\mathrm{pow}(k, m) \equiv c \pmod{n}$ for all $m \in \mathbb{Z}$ such that $m \geq M$: we denote the integer c by $f_n(k)$.

(b) Prove that for every positive integer n, the inclusion $(\mathbb{Z}/n\mathbb{Z})^\times \subseteq \mathrm{Im}(f_n)$ holds, where $\mathrm{Im}(f_n)$ is the image of the function $f_n \colon \mathbb{Z} \to \mathbb{Z}$.

Problem G4.

(a) Describe an algorithm, with proof, to compute all possible ways to write a given $n \in \mathbb{N}$ as the sum of squares of consecutive positive integers. For example, for $n = 25$, we can write $25 = 5^2$ and $25 = 3^2 + 4^2$. Include your code as part of your solution (feel free to use your favorite programming language).

(b) What is the time complexity of your algorithm?

(c) What is the first number that is NOT a perfect square which can be written as the sum of squares of consecutive positive integers in three different ways? Hint: it is less than 150000.

Problem G5. A nonempty set S consisting of positive real numbers is called an *additive set* if $x + y \in S$ when $x, y \in S$. Let S be an additive set. An element of S is called *indecomposable* if it is not the sum of two (not necessarily distinct) elements of S, and S is called *decomposable* if every element of S can be written as a finite sum of indecomposable elements (allowing repetitions and sums consisting of only one summand). Prove that if S is an additive set and there exists a strictly decreasing sequence $(x_n)_{n\geq 1}$ such that $\{x_n, x_n - x_{n+1} : n \in \mathbb{N}\} \subseteq S$, then there exists an additive set contained in S that is not decomposable.

Advanced Math Problems

Problem M1. Suppose that a function $g\colon [0,1] \to \mathbb{R}$ is continuous on $[0,1]$ and differentiable on $(0,1)$. Prove that if there exists $c \in \mathbb{R}$ with $0 < c < 1$ such that $\int_0^c g(t)\,dt = 0$, then

$$\frac{2}{1-c}\int_0^1 g(t)\,dt \le \sup\big\{|g'(t)| : t \in (0,1)\big\}.$$

Problem M2. Let b and n be nonnegative integers with $b \ge 2$. If $n = \sum_{i=0}^k a_i b^i$, where $a_k \neq 0$ and $0 \le a_0, \ldots, a_k \le b-1$, then we write $n = [a_k, \ldots, a_0]_b$, and we say that n is *b-ascending* provided that $a_k < a_{k-1} < \cdots < a_0$. Assume that 0 is b-ascending for every b. For example, $158 = [1,5,8]_{10}$ is 10-ascending and also 11-ascending because $158 = [1,3,4]_{11}$, but it is not 12-ascending because $158 = [1,1,2]_{12}$.

(a) For each b, argue that there are only finitely many b-ascending numbers for any b, and find how many b-ascending numbers are there.

(b) Let $P(d,b)$ be the probability that a positive random d-digits number in base b is b-ascending. Assume that $P(0,b) = 1$ for every b. Give a formula for $P(d,b)$ and compute

$$\lim_{b\to\infty} P(d,b) \qquad \text{and} \qquad \lim_{b\to\infty} \sum_{d=0}^{\infty} P(d,b).$$

(c) Given $n \in \mathbb{N}$, justify why there must be only finitely many b such that n is not b-ascending. Set $N(n) := |\{b \in \mathbb{N} : \sqrt{n} < b \le n\}$ and n is not b-ascending$\}|$. Find a formula for $N(n)$.

Problem M3. Let $(A,+)$ be a finite abelian group. For $k \in \mathbb{N}$, a sequence $(a_1, a_2, \ldots, a_k)$ is called a *nuller* of A if $a_1 + a_2 + \cdots + a_k = 0$, and a nuller $(a_1, a_2, \ldots, a_k)$ is called *minimal* if $\sum_{i\in I} a_i \neq 0$ for any nonempty proper subset of $\{1, 2, \ldots, k\}$. Define

$$\rho(A) := \max\left\{\frac{1}{o(a_1)} + \frac{1}{o(a_2)} + \cdots + \frac{1}{o(a_k)} : (a_1, a_2, \ldots, a_k) \text{ is a minimal nuller of } A\right\},$$

where $o(a)$ denotes the order of $a \in A$. Prove that $\rho(A) = 1$ if and only if A is cyclic with $|A| = p^n$ for some $p \in \mathbb{P}$ and $n \in \mathbb{N}_0$.

Problem M4. Let $\mathbb{N}_0[x^{\pm 1}]$ denote the set of all Laurent polynomials with coefficients in $\mathbb{N}_0$. For instance, $x^{-2023} + x^{2023}$ belongs to $\mathbb{N}_0[x^{\pm 1}]$ while $x^2 - 2x + 1$ does not. As $\mathbb{N}_0$, the set $\mathbb{N}_0[x^{\pm 1}]$ is closed under both addition and multiplication. We say that a nonzero Laurent polynomial $f = \sum_{i=0}^n c_i x^{k_i} \in \mathbb{N}_0[x^{\pm 1}]$ with $k_0 > k_1 > \cdots > k_n$ has $n+1$ *terms* and is

- *irreducible* if f is not a monic monomial and the equality $f = gh$ for some $g, h \in \mathbb{N}_0[x^{\pm 1}]$ implies that either g or h is a monic monomial of $\mathbb{N}_0[x^{\pm 1}]$,

- *monolithic* if $f = gh$ for some $g, h \in \mathbb{N}_0[x^{\pm 1}]$ implies that either g or h is a monomial of $\mathbb{N}_0[x^{\pm 1}]$, and
- *hyper-monolithic* if f is not a monomial and either $k_0 - k_1 < k_i - k_{i+1}$ for every $i \in \mathbb{Z}$ with $1 \le i \le n-1$ or $k_{n-1} - k_n < k_j - k_{j+1}$ for every $j \in \mathbb{Z}$ with $0 \le j \le n-2$.

In this problem, we will prove that $\mathbb{N}_0[x^{\pm 1}]$ satisfies a condition similar to that of the famous Goldbach's conjecture.

(a) Prove that every non-monomial $f \in \mathbb{N}_0[x^{\pm 1}]$ satisfying that $f(1)$ is an odd number greater than 3 can be written as a sum of at most three irreducibles.

(b) Prove that each hyper-monolithic in $\mathbb{N}_0[x^{\pm 1}]$ is monolithic.

(c) Prove that if $f \in \mathbb{N}_0[x^{\pm 1}]$ has at least three terms, then $f = g + h$ for some $g, h \in \mathbb{N}_0[x^{\pm 1}]$, where g is hyper-monolithic and $h(1) \le g(1)$.

(d) Suppose that $f \in \mathbb{N}_0[x^{\pm 1}]$ has at least three terms and also that $\frac{5f(1)}{6} - 1 \le p \le f(1) - 2$ for some $p \in \mathbb{P}$. Prove that f can be written as the sum of two irreducibles in $\mathbb{N}_0[x^{\pm 1}]$.

(e) Prove that every non-monomial $f \in \mathbb{N}_0[x^{\pm 1}]$ satisfying that $f(1) > 31$ can be written as a sum of at most two irreducibles. NOTE: You can use (without proving it) the Nagura's result that for each $n \in \mathbb{Z}$ with $n \ge 25$, there exists $p \in \mathbb{P}$ such that $n < p < \frac{6n}{5}$.

Problem M5. A (possibly infinite) graph is called a *tree* if it is connected and acyclic. For $k \ge 1$, a tree is called *k-regular* if every vertex has exactly k neighbors.

(a) Prove that, given a vertex v in a $(q+1)$-regular tree and $i \in \mathbb{N}$, there are exactly $(q+1)q^{i-1}$ vertices at distance i from v. In particular, the ball of center v and radius k contains exactly

$$1 + \sum_{i=1}^{k}(q+1)q^{i-1} = 1 + (q+1)\sum_{i=0}^{k-1} q^i = 1 + (1+q)\frac{q^k - 1}{q-1}.$$

A *river* in the tree is an infinite path such that for every vertex in the path, exactly two neighbors are also in the path. Let A and B be two rivers in a $(q+1)$-regular tree that share finitely many vertices. Let c be the number of vertices A and B have in common, and let d be the distance between A and B (i.e., the length of the shortest path connecting a vertex in A and a vertex in B). Also, let α and β be two non-negative integers.

(b) How many vertices are there at distance at most α from A and β from B when $A \cap B$ is empty (in terms of α, β, and d)? Justify your answer.

(c) How many vertices are there at distance at most α from A and β from B when $A \cap B$ is nonempty (in terms of α, β, and c)? Justify your answer.