A RIGIDITY RESULT FOR AXISYMMETRIC TORIC RICCI SOLITONS

SHIQIAO ZHANG

Abstract. We examine a non-axisymmetric perturbation of a family of axisymmetric toric Einstein manifolds and Ricci solitons studied in Firester–Tsiamis (2024). We establish a rigidity result stating that these axisymmetric Ricci solitons do not admit constant-angle non-axisymmetric perturbations except for conformally flat cases. For these new cases, our result leads to an explicit description of the Einstein metrics and a classification of the Ricci solitons under a volume-collapsing ansatz.

1. Introduction

Ricci solitons are the self-similar solutions to the Ricci flow $\partial_t g = -2\operatorname{Ric}(g)$. Characterized by the quasi-Einstein metric equation $\operatorname{Ric}(g) + \frac{1}{2}\mathcal{L}_V g = \lambda g$, Ricci solitons model the formation of singularities in the Ricci flow; see [Cao09] for a survey of the role of Ricci solitons in the singularity study of the Ricci flow. In [FT24], Firester and Tsiamis investigated a family of axisymmetric toric Ricci solitons in dimension four. They classified them based on geometric properties motivated by known families of Ricci solitons and studied the geometric behavior of new solutions. We show that these axisymmetric Ricci solitons are rigid with respect to constant-angle non-axisymmetric perturbations; specifically, no such perturbations exist unless the Ricci soliton is conformally flat. This rigidity comes from an off-diagonal obstruction in the Ricci soliton equation that disappears when the metric is axisymmetric. Based on this rigidity result, we transfer a classification by Firester and Tsiamis to constant-angle non-axisymmetric toric Ricci solitons.

Understanding the behavior of Ricci solitons in dimensions four and above is essential for the singularity study of the Ricci flow in higher dimensions, as a foundational work of Bamler [Bam18] showed that the singular sets of Ricci flows with bounded scalar curvature have codimension at least four. There is no known classification of four-dimensional Ricci solitons except in the Kähler setting, where Bamler, Cifarelli, Conlon, and Deruelle [Bam+24] completed the classification of shrinking Kähler–Ricci solitons in complex dimension two. In [App23], Appleton showed that four-dimensional Ricci solitons can exhibit multiple blow-up behaviors in the same soliton, including orbifold singularities, calling for the research of Ricci solitons with singularities or incompleteness. For example, Bamler, Chow, Deng, Ma, and Zhang [Bam+22] classified the tangent flows at infinity of steady soliton singularity models in dimension four; Alexakis, Chen, and Fournodavlos [ACF15] constructed a new family of spherically symmetric singular Ricci solitons in all dimensions and studied their stability under spherically symmetric perturbations; and Hui [Hui24] showed the existence of singular rotationally symmetric steady and expanding gradient Ricci solitons of the form $g = da^2/h(a^2) + a^2 g_{\mathbb{S}_n}$ in dimension $n+2 \geq 4$.

Many known examples of four-dimensional Ricci solitons are cohomogeneity two metrics, which are metrics that admit an isometric torus action. In [FT24], Firester and Tsiamis studied a family of axisymmetric toric Ricci solitons given locally by

$$g = \frac{1}{q(x,y)^2}\left(\frac{dx^2}{A(x)} + \frac{dy^2}{B(y)} + A(x)\,ds^2 + B(y)\,dt^2\right). \tag{$\star$}$$

The axisymmetric property means that the torus fibers with coordinates (s,t) are orthogonal, so the metric is diagonal. Firester and Tsiamis classified Ricci solitons of the form ($\star$) whose base metric

$$\frac{1}{q(x,y)^2}\left(\frac{dx^2}{A(x)} + \frac{dy^2}{B(y)}\right) \tag{1.1}$$

is conformally cylindrical, which means that the function q is homogeneous of degree 1. This volume-collapsing ansatz ensures that the metric ($\star$) is asymptotically cylindrical and allows reducing the Ricci soliton equation to an ordinary differential equation. See Section 2 for a summary of the results and methods of Firester and Tsiamis that we make use of.

We consider a non-axisymmetric perturbation

$$g = \frac{1}{q(x,y)^2}\left(\frac{dx^2}{A(x)} + \frac{dy^2}{B(y)} + A(x)\,ds^2 + B(y)\,dt^2 + 2\sqrt{A(x)B(y)}\cos\theta\,ds\,dt\right) \tag{$\star\star$}$$

of the metric ($\star$), in which the torus fibers are placed at a constant angle $0 < \theta < \pi/2$; it would be interesting to understand Ricci solitons where θ varies in the base (x, y). The metric ($\star\star$) recovers the axisymmetric metric ($\star$) when θ is identically $\pi/2$. In Theorem 1.1, we show a rigidity result for non-axisymmetric Ricci solitons of the form ($\star\star$), which requires that A and B be constants in addition to the same system of differential equations (2.3) to (2.5) arising from the metric ($\star$). Here, the subscripts on q denote partial derivatives, and function arguments are omitted for brevity.

Theorem 1.1. *A non-axisymmetric metric ($\star\star$) is a Ricci soliton if and only if A and B are constants and there are functions $S^x(x, y)$ and $S^y(x, y)$ satisfying the system of equations*

$$\partial_y S^x + \partial_x S^y = 4q_x q_y, \qquad \partial_x S^x = 2q_x^2, \qquad \partial_y S^y = 2q_y^2, \tag{1.2}$$

$$\lambda = q_{xx}A + q_{yy}B - \frac{q_x^2 A + q_y^2 B}{q} - \frac{S^x q_x A + S^y q_y B}{q^2}. \tag{1.3}$$

In particular, any such metric is conformally flat. It is only in the axisymmetric case $\theta = \pi/2$ that there are many Ricci solitons, which are therefore rigid with respect to constant-angle non-axisymmetric perturbations. Theorem 1.1 leads to an explicit description of Einstein metrics of the form ($\star\star$), which are characterized by the Einstein metric equation $\mathrm{Ric}(g) = \lambda g$.

Corollary 1.2. *A non-axisymmetric metric ($\star\star$) is an Einstein metric if and only if A and B are constants and $q(x, y) = ax + by + c$ for some constants a, b, and c, with $\lambda = -3(a^2A + b^2B)$ in the Einstein metric equation.*

Under the conformally cylindrical ansatz, the additional constraint that A and B are constants narrows down the classification of Ricci solitons of the form ($\star\star$).

Corollary 1.3. *A non-axisymmetric metric ($\star\star$) whose base metric (1.1) is conformally cylindrical is one of*

(i) an Einstein metric as described in Corollary 1.2;
(ii) a product of two Ricci soliton surfaces with the same constant λ defined in the Ricci soliton equation; or
(iii) a product of a flat surface with an Einstein surface, given by

$$g = \frac{1}{f(x)^2}\left(\frac{dx^2}{A} + \frac{dy^2}{B} + A\,ds^2 + B\,dt^2\right)$$

up to translating, rescaling, and swapping x and y, where $f \in \{\cosh, \sinh, \sin\}$, and A and B are constants.

2. The axisymmetric case

In Theorem 1.1 in [FT24], Firester and Tsiamis classified Ricci solitons of the form ($\star$) whose base metric (1.1) is conformally cylindrical. We summarize this classification in Theorem 2.1. Recall that the base metric (1.1) being conformally cylindrical means that the function q is homogeneous of degree 1.

Theorem 2.1 (Firester–Tsiamis). *When the base metric (1.1) is conformally cylindrical, the metric ($\star$) is one of*

(i) an Einstein metric, specifically the Riemannian Schwarzschild metric

$$g = \frac{1}{x^2}\left(\frac{dx^2}{A(x)} + \frac{dy^2}{B(y)} + A(x)\,ds^2 + B(y)\,dt^2\right) \tag{2.1}$$

for some cubic A and quadratic B, possibly degenerate, or the Plebański–Demiański metric

$$g = \frac{1}{(x \pm y)^2}\left(\frac{dx^2}{a_0 - P(x)} + \frac{dy^2}{b_0 + P(y)} + (a_0 - P(x))\,ds^2 + (b_0 + P(y))\,dt^2\right) \tag{2.2}$$

for some constants a_0 and b_0 and possibly degenerate cubic P, up to translating, rescaling, and swapping x and y;

(ii) *a product of two Ricci soliton surfaces with the same constant λ defined in the Ricci soliton equation;*
(iii) *a product of a flat surface with an Einstein surface;*
(iv) *a metric locally conformal to $\mathbb{S}^2 \times \mathbb{H}^2$ or $\mathbb{R}^4$, given explicitly by*

$$g = \frac{1}{y^2 f(x/y)^2}\left(\frac{dx^2}{a_0 - x^2} + \frac{dy^2}{b_0 + y^2} + (a_0 - x^2)\, ds^2 + (b_0 + y^2)\, dt^2\right)$$

for some constants a_0 and b_0 and solution $f(z)$ to the ordinary differential equation

$$\begin{aligned} 0 &= -3(a_0(f')^2 + b_0(f - zf')^2)^2 - \lambda(a_0(f')^2 + b_0(f - zf')^2) \\ &\quad + (4a_0 - b_0z^2)b_0 f^3 f'' + (a_0 + b_0z^2)^2 f f''((f')^2 - ff'') + \lambda(a_0 + b_0z^2) f f'' \\ &\quad - (a_0 + b_0z^2)(b_0 z f - (a_0 + b_0z^2)f') f^2 f^{(3)}; \end{aligned}$$

or

(v) *a steady singular Ricci soliton given by*

$$g = \frac{1}{x^{2\alpha} y^{2(1-\alpha)}}\left(\frac{dx^2}{F_{\alpha,c,k_1}(x)} + \frac{dy^2}{F_{1-\alpha,-c,k_2}(y)} + F_{\alpha,c,k_1}(x)\, ds^2 + F_{1-\alpha,-c,k_2}(y)\, dt^2\right)$$

where

$$F_{\alpha,c,k}(t) \coloneqq ct^2 + kt^{\frac{2\alpha^2}{2\alpha-1}+1}$$

for some constants c, k_1, k_2, and $\alpha \neq 1/2$.

To establish the classification in Theorem 2.1, Firester and Tsiamis introduced auxiliary functions (2.7) and reduced the Ricci soliton equation into the system of differential equations (2.3) to (2.5). This result is stated in Corollary 2.4 of [FT24]. Based on this result, they specialized to the case where the base metric (1.1) is conformally cylindrical and further reduced the Ricci soliton equation to an ordinary differential equation.

Theorem 2.2 (Firester–Tsiamis)**.** *When the base metric (1.1) is conformally cylindrical, the metric ($\star$) is a Ricci soliton if and only if there are functions $S^x(x, y)$ and $S^y(x, y)$ satisfying the system of differential equations*

$$\partial_y S^x + \partial_x S^y = 4q_x q_y, \qquad \partial_x S^x = 2q_x^2, \qquad \partial_y S^y = 2q_y^2, \tag{2.3}$$

$$w \coloneqq A'' - \frac{S^x}{q^2} A' = B'' - \frac{S^y}{q^2} B', \tag{2.4}$$

$$-\frac{\lambda}{q} = \frac{1}{2} w - \frac{q_x A' + q_{xx} A + q_y B' + q_{yy} B}{q} + \frac{q_x^2 A + q_y^2 B}{q^2} + \frac{S^x q_x A + S^y q_y B}{q^3}. \tag{2.5}$$

Firester and Tsiamis's proof of Theorem 2.2 uses the following result, stated and proved as Claim 2.3 in [FT24].

Lemma 2.3 (Firester–Tsiamis)**.** *If the metric ($\star$) satisfies the Ricci soliton equation $\mathrm{Ric}(g) + \frac{1}{2}\mathcal{L}_V g = \lambda g$, then there is some smooth vector field of the form $\tilde{V} = \tilde{V}^x(x, y)\, \partial_x + \tilde{V}^y(x, y)\, \partial_y$ such that $\mathrm{Ric}(g) + \frac{1}{2}\mathcal{L}_{\tilde{V}} g = \lambda g$.*

We now review Firester and Tsiamis's proof of Theorem 2.2, which forms the basis of the proof of our rigidity result in Section 3.1.

Proof of Theorem 2.2. The Ricci soliton equation can be rewritten as

$$\Lambda \coloneqq g^{-1}(\mathrm{Ric}(g) + \tfrac{1}{2}\mathcal{L}_V g) = \lambda I, \tag{2.6}$$

where I denotes the identity tensor. The Ricci tensor of the metric ($\star$) is given by

$$\begin{aligned} \mathrm{Ric}_{xx} &= -\frac{1}{A}\left(\frac{A''}{2} - \frac{2q_x A' + 3q_{xx} A + q_y B' + q_{yy} B}{q} + \frac{3(q_x^2 A + q_y^2 B)}{q^2}\right), \\ \mathrm{Ric}_{xy} &= \frac{2q_{xy}}{q}, \\ \mathrm{Ric}_{ss} &= -A\left(\frac{A''}{2} - \frac{2q_x A' + 3q_{xx} A + q_y B' + q_{yy} B}{q} + \frac{3(q_x^2 A + q_y^2 B)}{q^2}\right), \end{aligned}$$

and the symmetry $(x, A) \leftrightarrow (y, B)$, where all off-diagonal entries of Ric are zero except Ric_{xy} and Ric_{yx}. For the metric ($\star$), Lemma 2.3 allows us to assume without loss of generality that $V = V^x(x, y)\,\partial_x + V^y(x, y)\,\partial_y$ for some functions V^x and V^y. Then, the Lie derivative $\mathcal{L}_V g$ is given by

$$\begin{aligned}(\mathcal{L}_V g)_{xx} &= \frac{2(\partial_x V^x)A - V^x A'}{q^2A^2} - \frac{2(V^x q_x + V^y q_y)}{q^3 A},\\ (\mathcal{L}_V g)_{xy} &= \frac{(\partial_x V^y)A + (\partial_y V^x)B}{q^2AB},\\ (\mathcal{L}_V g)_{ss} &= \frac{V^x A'}{q^2} - \frac{2A(V^x q_x + V^y q_y)}{q^3},\end{aligned}$$

and the same symmetry $(x, A) \leftrightarrow (y, B)$, where all off-diagonal entries of $\mathcal{L}_V g$ are zero except $(\mathcal{L}_V g)_{xy}$ and $(\mathcal{L}_V g)_{yx}$. The Ricci soliton equation (2.6) is equivalent to the system of equations $\Lambda^x{}_y = \Lambda^y{}_x = 0$ and $\Lambda^x{}_x = \Lambda^y{}_y = \Lambda^s{}_s = \Lambda^t{}_t = \lambda$. To reduce these equations, we introduce the auxiliary functions

$$S^x := 2qq_x + \frac{V^x}{A} \qquad \text{and} \qquad S^y := 2qq_y + \frac{V^y}{B}. \tag{2.7}$$

Calculation shows that all off-diagonal entries of Λ are identically zero except $\Lambda^x{}_y$ and $\Lambda^y{}_x$. Both $\Lambda^x{}_y = 0$ and $\Lambda^y{}_x = 0$ reduce to $\partial_y S^x + \partial_x S^y = 4q_x q_y$. Equating pairs of diagonal entries of Λ, the equations $\Lambda^x{}_x = \Lambda^s{}_s$ and $\Lambda^y{}_y = \Lambda^t{}_t$ simplify to $\partial_x S^x = 2q_x^2$ and $\partial_y S^y = 2q_y^2$ respectively, while the equation $\Lambda^x{}_x = \Lambda^y{}_y$ reduces to

$$w(x, y) := A'' - \frac{S^x}{q^2}A' = B'' - \frac{S^y}{q^2}B'.$$

Finally, the equation $\Lambda^x{}_x = \lambda$ simplifies to

$$-\frac{\lambda}{q} = \frac{1}{2}w - \frac{q_x A' + q_{xx}A + q_y B' + q_{yy}B}{q} + \frac{q_x^2 A + q_y^2 B}{q^2} + \frac{S^x q_x A + S^y q_y B}{q^3},$$

thereby reducing the Ricci soliton equation (2.6) to the system of differential equations (2.3) to (2.5). ☐

3. The non-axisymmetric case

3.1. **Rigidity result.** Theorem 1.1 shows that Ricci solitons of the form ($\star\star$) are constrained by an obstacle arising from the $\Lambda^s{}_t = 0$ component of the Ricci soliton equation (3.1). This obstacle disappears when $\theta = \pi/2$, allowing a much larger family of Ricci solitons. To prove Theorem 1.1, we require an analogue of Lemma 2.3 for the metric ($\star\star$). The same proof that Firester and Tsiamis provided for Claim 2.3 in [FT24] also applies to this analogue; we provide a different constructive proof.

Lemma 3.1. *If the metric ($\star\star$) satisfies the Ricci soliton equation* $\mathrm{Ric}(g) + \frac{1}{2}\mathcal{L}_V g = \lambda g$, *then there is some smooth vector field of the form* $\tilde{V} = \tilde{V}^x(x, y)\,\partial_x + \tilde{V}^y(x, y)\,\partial_y$ *such that* $\mathrm{Ric}(g) + \frac{1}{2}\mathcal{L}_{\tilde{V}} g = \lambda g$.

Proof. Suppose that the metric g of the form ($\star\star$) satisfies the Ricci soliton equation with the vector field V. For the Lie derivative $\mathcal{L}_V g$, we may assume $V^s = V^t = 0$ since the coefficients of g depend only on x and y. Calculation shows that the Ricci tensor Ric of the toric metric g is also toric, namely $\mathrm{Ric} = \mathrm{Ric}_{\mathrm{base}} \oplus \mathrm{Ric}_{\mathrm{torus}}$ where the metrics $\mathrm{Ric}_{\mathrm{base}}$ over (x, y) and $\mathrm{Ric}_{\mathrm{torus}}$ over (s, t) both have coefficients depending only on x and y. Hence, $\mathcal{L}_V g = 2(\lambda g - \mathrm{Ric})$ is also toric, meaning that $\mathcal{L}_V g = \mathcal{L}_{\tilde{V}} g$ for the vector field

$$\tilde{V}(x, y) = \frac{1}{|\Omega|}\int_\Omega V(x, y, s, t)\, ds \wedge dt$$

obtained by averaging V over a region Ω in the coordinates (s, t) constituting one period of the toric fibers. ☐

The conditions on q, A, B, S^x, and S^y in Theorem 1.1 are the same as those in Theorem 2.2 with the additional constraint that A and B are constants, so $A' = B' = 0$ and $w(x, y) = 0$. The value of the constant $0 < \theta < \pi/2$ cancels out in the calculations and does not appear in the resulting conditions. When $\theta = \pi/2$ as in Theorem 2.2, however, the equations $\Lambda^s{}_t = 0$ and $\Lambda^t{}_s = 0$ in the proof below hold identically instead, so the constraint that A and B are constants is lifted, resulting in a much wider range of Ricci solitons.

Proof of Theorem 1.1. Recall that the Ricci soliton equation can be rewritten as

$$\Lambda \coloneqq g^{-1}(\mathrm{Ric}(g) + \tfrac{1}{2}\mathcal{L}_V g) = \lambda I \tag{3.1}$$

where I denotes the identity tensor. The Ricci tensor of the metric ($\star\star$) is given by

$$\begin{aligned}
\mathrm{Ric}_{xx} &= -\frac{1}{A}\left(\frac{A''}{2} - \frac{2q_x A' + 3q_{xx}A + q_y B' + q_{yy}B}{q} + \frac{3(q_x^2 A + q_y^2 B)}{q^2} + \frac{(A')^2\cot^2\theta}{8A}\right),\\
\mathrm{Ric}_{xy} &= \frac{2q_{xy}}{q} + \frac{A'B'\cot\theta}{8AB},\\
\mathrm{Ric}_{ss} &= -A\left(\frac{A''}{2} - \frac{2q_x A' + 3q_{xx}A + q_y B' + q_{yy}B}{q}\right.\\
&\qquad \left. + \frac{3(q_x^2 A + q_y^2 B)}{q^2} - \frac{(B(A')^2 + A(B')^2)\cot^2\theta}{8AB}\right),\\
\mathrm{Ric}_{st} &= -\sqrt{AB}\cos\theta\left(\frac{A''+B''}{4} - \frac{3(q_x A' + q_y B') + 2(q_{xx}A + q_{yy}B)}{2q} + \frac{3(q_x^2 A + q_y^2 B)}{q^2}\right)\\
&\qquad + \frac{(B(A')^2 + A(B')^2)\cos\theta}{8\sqrt{AB}\sin^2\theta},
\end{aligned}$$

and the symmetry $(x, A) \leftrightarrow (y, B)$, where all off-diagonal entries of Ric are zero except Ric_{xy}, Ric_{yx}, Ric_{st}, and Ric_{ts}. For the metric ($\star\star$), Lemma 3.1 allows us to assume without loss of generality that $V = V^x(x,y)\,\partial_x + V^y(x,y)\,\partial_y$ for some functions V^x and V^y. Then, the Lie derivative $\mathcal{L}_V g$ is given by

$$\begin{aligned}
(\mathcal{L}_V g)_{xx} &= \frac{2(\partial_x V^x)A - V^x A'}{q^2A^2} - \frac{2(V^x q_x + V^y q_y)}{q^3 A},\\
(\mathcal{L}_V g)_{xy} &= \frac{(\partial_x V^y)A + (\partial_y V^x)B}{q^2AB},\\
(\mathcal{L}_V g)_{ss} &= \frac{V^x A'}{q^2} - \frac{2A(V^x q_x + V^y q_y)}{q^3},\\
(\mathcal{L}_V g)_{st} &= \frac{\cos\theta}{\sqrt{AB}}\left(\frac{V^x A' B + V^y B' A}{2q^2} - \frac{2AB(V^x q_x + V^y q_y)}{q^3}\right),
\end{aligned}$$

and the same symmetry $(x, A) \leftrightarrow (y, B)$, where all off-diagonal entries of $\mathcal{L}_V g$ are zero except $(\mathcal{L}_V g)_{xy}$, $(\mathcal{L}_V g)_{yx}$, $(\mathcal{L}_V g)_{st}$, and $(\mathcal{L}_V g)_{ts}$. The Ricci soliton equation (3.1) is equivalent to the system of equations

$$\Lambda^x{}_y = \Lambda^y{}_x = \Lambda^s{}_t = \Lambda^t{}_s = 0 \qquad \text{and} \qquad \Lambda^x{}_x = \Lambda^y{}_y = \Lambda^s{}_s = \Lambda^t{}_t = \lambda.$$

Given $0 < \theta < \pi/2$, the equations $\Lambda^s{}_t = 0$ and $\Lambda^t{}_s = 0$ simplify to

$$2(A'BV^x - AB'V^y) + 4qAB(A'q_x - B'q_y) + q^2(A'^2B + AB'^2 - 2AB(A'' - B'')) = 0$$

and

$$2(A'BV^x - AB'V^y) + 4qAB(A'q_x - B'q_y) - q^2(A'^2B + AB'^2 + 2AB(A'' - B'')) = 0$$

respectively. Subtracting these equations and dividing through by q^2AB results in

$$\frac{A'^2}{A} + \frac{B'^2}{B} = 0.$$

This forces $A' = B' = 0$ since $A > 0$ and $B > 0$, so A and B are constants. Under this condition, both $\Lambda^x{}_y = 0$ and $\Lambda^y{}_x = 0$ simplify to

$$\frac{\partial_y V^x}{A} + \frac{\partial_x V^y}{B} + 4qq_{xy} = 0,$$

which is equivalent to $\partial_y S^x + \partial_x S^y = 4q_x q_y$ under the introduction of the same auxiliary functions

$$S^x \coloneqq 2qq_x + \frac{V^x}{A} \qquad \text{and} \qquad S^y \coloneqq 2qq_y + \frac{V^y}{B} \tag{3.2}$$

from (2.7). Further calculation shows that the equations $\Lambda^x{}_x = \Lambda^s{}_s$ and $\Lambda^y{}_y = \Lambda^t{}_t$ simplify to $\partial_x S^x = 2q_x^2$ and $\partial_y S^y = 2q_y^2$ respectively, whereas $\Lambda^s{}_s = \Lambda^t{}_t$ holds identically. Finally, the equation $\Lambda^x{}_x = \lambda$ simplifies to

$$\lambda = q_{xx}A + q_{yy}B - \frac{q_x^2 A + q_y^2 B}{q} - \frac{S^x q_x A + S^y q_y B}{q^2}.$$

Hence, we have reduced the Ricci soliton equation (3.1) to the system of equations (1.2) and (1.3). ☐

3.2. **Corollaries.** The description of Einstein metrics of the form ($\star\star$) follows immediately from Theorem 1.1.

Proof of Corollary 1.2. If the metric ($\star\star$) is an Einstein metric, then we have $V = 0$ in Theorem 1.1, so the auxiliary functions (3.2) are $S^x = 2qq_x$ and $S^y = 2qq_y$ respectively. Hence, the system of equations (1.2) reduces to $q_{xx} = q_{xy} = q_{yy} = 0$, so $q = ax + by + c$ for some constants a, b, and c. Finally, (1.3) becomes $\lambda = -3(a^2 A + b^2 B)$. ☐

Finally, we classify Ricci solitons of the form ($\star\star$) whose base metric (1.1) is conformally cylindrical.

Proof of Corollary 1.3. For a Ricci soliton g of the form ($\star\star$), Theorem 1.1 implies that the corresponding diagonal metric ($\star$) is a Ricci soliton classified in Theorem 2.1. There are only three cases where A and B can be constants, namely

(i) when g is an Einstein metric as described in Corollary 1.2;
(ii) when g is a product of two Ricci soliton surfaces with the same constant λ defined in the Ricci soliton equation; and
(iii) when g is a product of a flat surface with an Einstein surface.

We compute the form of g explicitly in the last case. Suppose that $g = g_{\text{Einstein}} \oplus g_{\text{flat}}$ where

$$g_{\text{Einstein}} = \frac{1}{q(x,y)^2}\left(\frac{dx^2}{A} + A\,ds^2\right)$$

is an Einstein metric and

$$g_{\text{flat}} = \frac{1}{q(x,y)^2}\left(\frac{dy^2}{B} + B\,dt^2\right)$$

is a flat metric. The Ricci tensor of g_{flat} is diagonal with entries

$$\mathrm{Ric}_{\text{flat},yy} = \frac{qq_{yy} - q_y^2}{q^2} \qquad \text{and} \qquad \mathrm{Ric}_{\text{flat},tt} = \frac{B^2(qq_{yy} - q_y^2)}{q^2},$$

so the flatness condition reduces to $qq_{yy} - q_y^2 = 0$, whose solution is $q(x,y) = h(x)e^{yf(x)}$ for some functions f and h. Similar calculation shows that the Einstein condition on g_{Einstein} reduces to $qq_{xx} - q_x^2 = \lambda/A$, which becomes

$$e^{2yf}(h(xhf'' + h'') - (h')^2) = \frac{\lambda}{A}. \tag{3.3}$$

Since the left-hand side is constant for all x and y, its partial derivative with respect to y must be zero, which results in

$$f'' = -\frac{2f(hh'' - (h')^2)}{h^2(2yf+1)}. \tag{3.4}$$

Substituting (3.4) into (3.3) results in

$$\frac{e^{2yf}(hh'' - (h')^2)}{2yf+1} = \frac{\lambda}{A}.$$

Setting the partial derivative of the left-hand side with respect to y equal to zero again results in $yf(x) = 0$, which must hold for all x and y. Hence, f is identically zero, and (3.3) reduces to $hh'' - (h')^2 = \lambda/A$, whose solutions, up to translating and rescaling x, are $h(x) = \pm\sqrt{\lambda/A}\cosh x$ for $\lambda \geq 0$ and $h(x) = \pm\sqrt{-\lambda/A}\sinh x$ and $h(x) = \pm\sqrt{-\lambda/A}\sin x$ for $\lambda \leq 0$. We conclude that the metric is given by

$$g = \frac{1}{f(x)^2}\left(\frac{dx^2}{A} + \frac{dy^2}{B} + A\,ds^2 + B\,dt^2\right)$$

up to translating, rescaling, and swapping x and y, where $f \in \{\cosh, \sinh, \sin\}$. ☐

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