Color plane groups

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Section 1

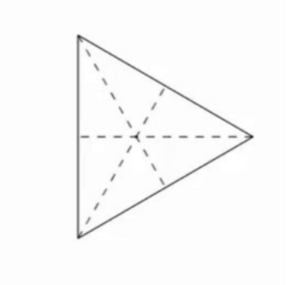
Introduction to wallpaper groups

Dihedral group

Definition

Dihedral group, denoted D_n, is the group of symmetries of a regular n-gon, which includes rotations and reflections.

Consider regular triangle and its symmetries:



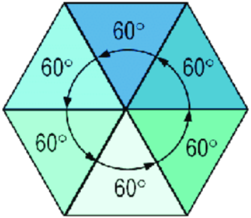
- rotational symmetries of order 3
- reflections about dotted lines

Thus, it is D_3.
Dihedral group D_n is a group of order $2n$.

Cyclic group

Definition

Cyclic group is the group of symmetries of a regular n-gon without reflections, denoted C_n.

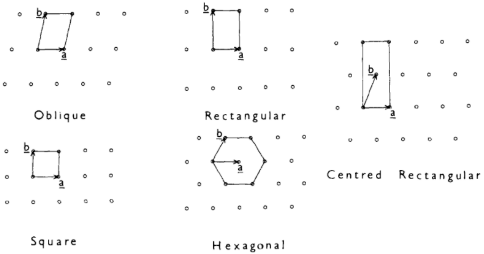


Lattice

Definition

Lattice spanned by vectors $\vec{a}$ and $\vec{b}$ is the set of linear combinations $x\vec{a} + y\vec{b}$, where x, y $\in \mathbb{Z}$.

There are 5 types of lattices in 2-dimensional plane:

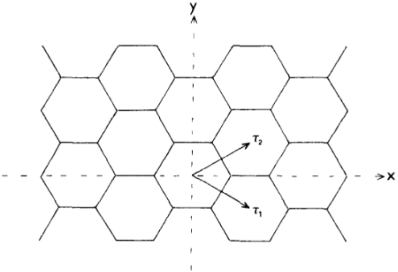


Wallpaper pattern

Definition

A **wallpaper pattern** is a graphical pattern covering the whole Euclidean plane and generated by translation of a pattern by two independent translations.

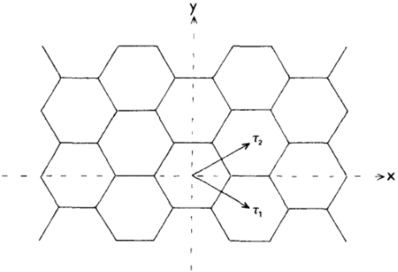
Wallpaper has certain symmetries, which are keeping it unchanged. Let us consider following example:



Wallpaper pattern

Each such pattern has certain symmetries. For example:

- reflection about x and y axes
- rotational symmetry of order 3 around the center of each hexagon
- translations by vectors τ_1 and τ_2



Wallpaper group

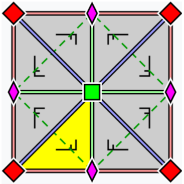
Definition

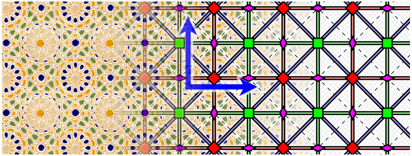
Wallpaper group is a group of symmetries of wallpaper pattern.

- Wallpaper group is a subgroup of E_2 (group of plane isometries)
- Its translation subgroup (T, a lattice) is generated by two independent translations

Examples of wallpaper groups

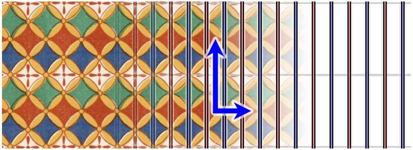
- two rotation centres of order four (red squares)
- four distinct reflections (green and blue lines)
- glide reflections (green dotted lines)
- rotations of order 2 in magenta rhombuses
- D_4 point group, square lattice, $p4m$ wallpaper group





Examples of wallpaper groups

- no rotations
- all parallel reflection axes
- D_1 point group, rectangular lattice, *pm* wallpaper group



Wallpaper groups

There are 17 wallpaper groups:

p1	cm	cmm	p3
p2	pmm	p4	p3m1
pm	pmg	p4m	p31m
pg	pgg	p4g	p6
			p6m

Now a word about notation.

"*p*" and "*c*" refer to primitive and centered lattice resp.

Symbol for reflection is *m* (mirror) and *g* denotes glide reflection.

1, 2, 3, 4 and 6 are rotation orders.

Point group

Definition

Let G be a wallpaper group, T - its translation subgroup. Then **point group** (denoted as H) will be defined as:
$H = G/T$.

Wallpaper groups are classified according to its point group. The order of rotation in a wallpaper group is 2, 3, 4 or 6. So, a point group of a wallpaper pattern can only be
$C_1, C_2, C_3, C_4, C_6, D_1, D_2, D_3, D_4, D_6$.
Their orders are:
1, 2, 3, 4, 6, 2, 4, 6, 8, 12.

Wallpaper groups: recap

- Wallpaper pattern is a pattern covering the whole plane and generated by two independent translations.
- Wallpaper group is the group of symmetries of a wallpaper pattern.
- There are 17 wallpaper groups.
- Each wallpaper group has a translation subgroup T (one of 5 lattices) and a point group H (one of 10 of the form C_n, D_n)

Section 2

Color groups

Definition of a color groups

A **color group** consists of a wallpaper group G and its subgroup G_1 of a finite index N.
Two such pairs $G_1 \subset G$ and $G_1' \subset G'$ are equivalent if there is an isomorphism $G \to G'$ which maps G_1 to G_1'.

Our goal is to enumerate the color groups of given index.

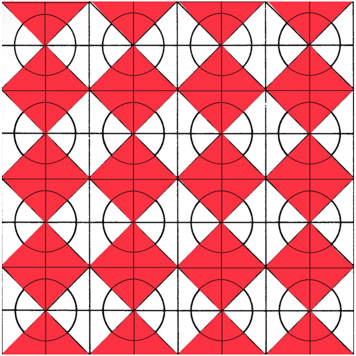


Figure: Coloring in 2 colors

Class and lattice equal color groups

- A color group $G_1 \subset G$ is called **class-equal** if $H_1 = H$, where H_1, H are point groups of G_1, G respectively.
- A color group $G_1 \subset G$ is called **lattice-equal** if $T_1 = T$, where T_1, T are translation subgroups of G_1, G respectively.

Generally, $[G : G_1] = [H : H_1] \cdot [T : T_1]$.

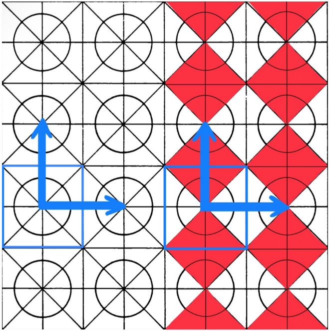


Figure: Lattice-equal color group

"Two-level theorem"

Theorem

Any color group $G_1 \subset G$ can be expressed uniquely as the composition of a lattice-equal color group $G' \subset G$ and a class-equal color group $G_1 \subset G'$.

$$
\begin{tikzcd}
T \arrow[r, hook] & G \arrow[r, two heads] \arrow[d, phantom, "\cup"] & H \arrow[d, phantom, "\cup"] \\
T \arrow[r, hook] \arrow[d, phantom, "\cup"] & G' \arrow[r, two heads] \arrow[d, phantom, "\cup"] & H_1 \\
T_1 \arrow[r, hook] & G_1 \arrow[r, two heads] & H_1
\end{tikzcd}
$$

Figure: "Two-level theorem"

Visualization of color groups

Following Wieting, wallpaper group G can be visualized as a tiling, on which the group G acts simply transitively.
Color group $G_1 \subset G$ is visualized as a colored subset of tiles on which G_1 acts transitively.

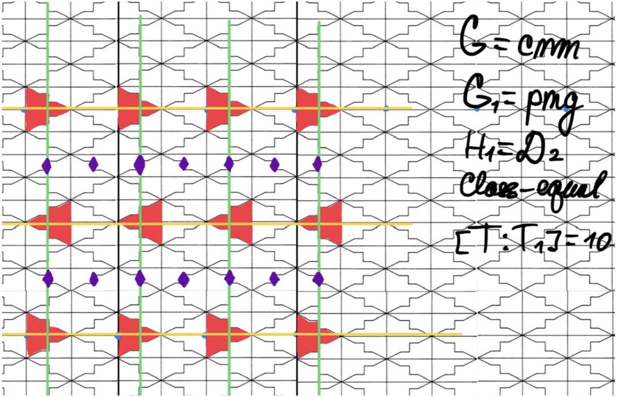


Figure: $G \cong \mathrm{cmm}, G_1 \cong \mathrm{pmg}, [G : G_1] = 10$

Visualization of color groups

Following Wieting, wallpaper group G can be visualized as a tiling, on which the group G acts simply transitively.
Color group $G_1 \subset G$ is visualized as a colored subset of tiles on which G_1 acts transitively.

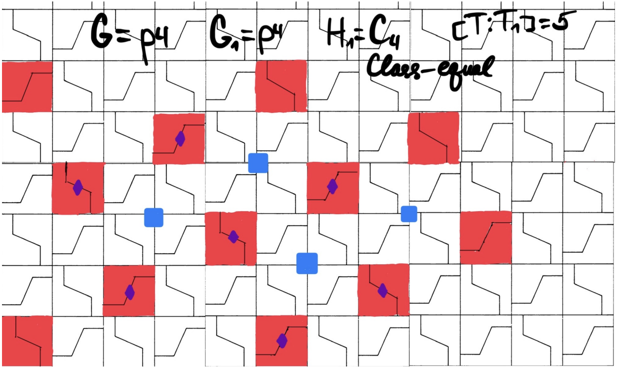


Figure: $G \cong \mathrm{p4}, G_1 \cong \mathrm{p4}, [G : G_1] = 5$

Section 3

Enumeration results

Procedure

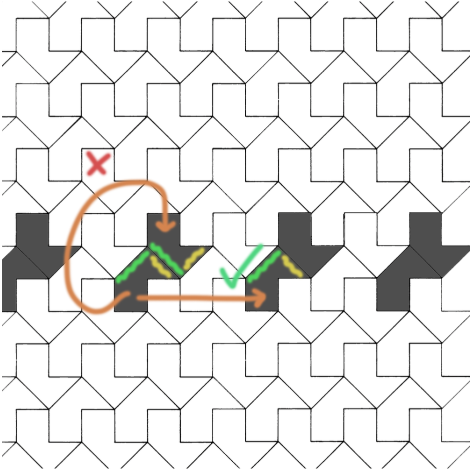
What did we do?

- We counted the number of inequivalent colorings (i.e. color groups) in N colors for all 17 wallpaper groups.

How did we do it?

- By looking at all possible valid colorings and identifying which are equivalent and which are not.

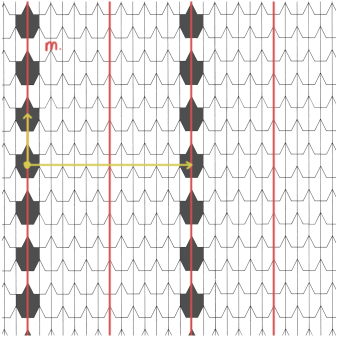
Invalid coloring

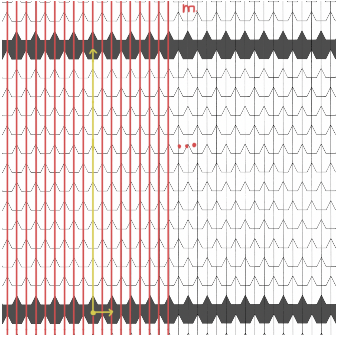


Symmetry group G_1 must act **transitively** on the set of colored tiles.

Not equivalent

These two colorings are inequivalent

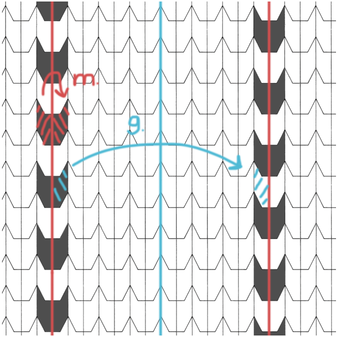


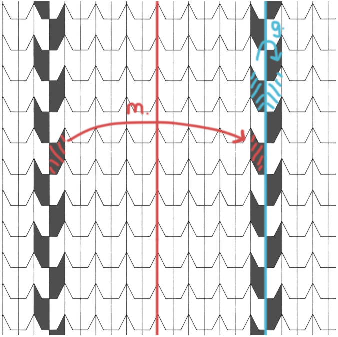


Note: non-colored tiling has symmetry $G \cong$ cm,
colored tiling has symmetry $G_1 \cong$ pm,
index $N = [G : G_1] = 14$.

Equivalent

These two colorings are equivalent





Note: non-colored tiling has symmetry $G \cong$ pm,
colored tiling has symmetry $G_1 \cong$ cm,
index $N = [G : G_1] = 14$

$N = 2$ class-equal (enumeration of class-equal color groups $G_1 \subset G$ of index $N = 2$)

Group G	Class-equal
p1	1
p2	1
p3	0
p4	1
p6	0
cm	2
pm	4
pg	1
cmm	3

Table: for $N = 2$

Group G	Class-equal
pmm	3
pmg	2
pgg	0
p3m1	0
p31m	0
p4m	2
p4g	0
p6m	0

Table: for $N = 2$

$N = 2$ lattice-equal

Group	Lattice-equal
p1	0
p2	1
p3	0
p4	1
p6	1
cm	1
pm	1
pg	1
cmm	2

Table: for $N = 2$

Group	Lattice-equal
pmm	2
pmg	3
pgg	2
p3m1	1
p31m	1
p4m	3
p4g	3
p6m	3

Table: for $N = 2$

$N = p$ class-equal (**no lattice-equal colorings**)

Group	Class-equal
p1	1
p2	1
p3 ($p = 6k + 1$)	1
p4 ($p = 4k + 1$)	1
p6 ($p = 6k + 1$)	1
cm	2
pm	2
pg	2
cmm	1

Table: for $N = p$

Group	Class-equal
pmm	1
pmg	2
pgg	1
p3m1	0
p31m	0
p4m	0
p4g	0
p6m	0

Table: for $N = p$

$N = 2p$ class-equal

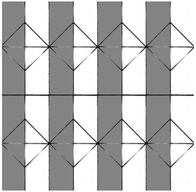
Group	Class-equal
p1	1
p2	1
p3	0
p4 $(p = 4k + 1)$	1
p6	0
cm	4
pm	8
pg	2
cmm	4

Table: for $N = 2p$

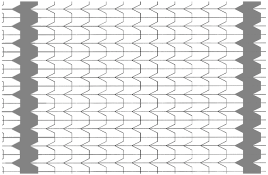
Group	Class-equal
pmm	5
pmg	4
pgg	0
p3m1	0
p31m	0
p4m	0
p4g	0
p6m	0

Table: for $N = 2p$

Two-level theorem example



$$\begin{tikzcd}
& \overset{pmm}{G} \arrow[r, two heads] \arrow[d, phantom, "\cup", "\text{lattice-equal index 2}"'] & \overset{D_2}{H} \arrow[d, phantom, "\cup", "{[H:H_1] = 2}"] \\
T \arrow[r, hook] \arrow[d, phantom, "\cup", "{[T:T_1] = p}"] & \overset{pm}{G'} \arrow[r, two heads] \arrow[d, phantom, "\cup", "\text{class-equal index p}"] & \overset{D_1}{H_1} \\
T_1 \arrow[r, hook] & \overset{pm}{G_1} &
\end{tikzcd}$$



$N = 2p$ neither lattice-equal nor class-equal

Group	Neither
p1	0
p2	1
p3	0
p4 ($p = 4k+1$)	$(p+3)/4+1$
p4 ($p = 4k+3$)	$(p+1)/4+1$
p6 ($p = 6k+1$)	1
cm	$(p+1)/2+1$
pm	$(p+1)/2+1$
pg	$(p+1)/2+1$
cmm ($p = 4k+1$)	$(p+3)/4+3$
cmm ($p = 4k+3$)	$(p+1)/4+3$

Table: for $N = 2p$

Group	Neither
pmm ($p = 4k+1$)	$(p+3)/4+3$
pmm ($p = 4k+3$)	$(p+1)/4+3$
pmg	$(p+1)/2+5$
pgg ($p = 4k+1$)	$(p+3)/4+3$
pgg ($p = 4k+3$)	$(p+1)/4+3$
p3m1 ($p = 6k+1$)	1
p31m ($p = 6k+1$)	1
p4m ($p = 4k+1$)	3
p4m ($p = 4k+3$)	2
p4g ($p = 4k+1$)	3
p4g ($p = 4k+3$)	2
p6m ($p = 6k+1$)	1

Table: for $N = 2p$

$N = 2p$ total

Group	Total
p1	1
p2	2
p3	0
p4 $(p = 4k+1)$	$(p+3)/4+2$
p4 $(p = 4k+3)$	$(p+1)/4+1$
p6 $(p = 6k+1)$	1
cm	$(p+1)/2+5$
pm	$(p+1)/2+9$
pg	$(p+1)/2+3$
cmm $(p = 4k+1)$	$(p+3)/4+7$
cmm $(p = 4k+3)$	$(p+1)/4+7$

Table: for $N = 2p$

Group	Total
pmm $(p = 4k+1)$	$(p+3)/4+8$
pmm $(p = 4k+3)$	$(p+1)/4+8$
pmg	$(p+1)/2+9$
pgg $(p = 4k+1)$	$(p+3)/4+3$
pgg $(p = 4k+3)$	$(p+1)/4+3$
p3m1 $(p = 6k+1)$	1
p31m $(p = 6k+1)$	1
p4m $(p = 4k+1)$	3
p4m $(p = 4k+3)$	2
p4g $(p = 4k+1)$	3
p4g $(p = 4k+3)$	2
p6m $(p = 6k+1)$	1

Table: for $N = 2p$

Our results vs Senechal (1979)

G	2	3	4	6	2p
p1	1	1	2	1	1
p2	2	1	3	2	2
p3	0	2	1	1	0
p4	2	0	5	2	$\frac{1}{4}(p+11)$ if $p=4n+1$, $\frac{1}{4}(p+5)$ if $p=4n+3$
p6	1	2	1	5	1 iff $p=6n+1$
pm	5	2	10	11	$\frac{1}{2}(p+1)+9$
pg	2	2	4	5	$\frac{1}{2}(p+1)+3$
cm	3	2	7	7	$\frac{1}{2}(p+1)+5$
pmm	5	1	13	10	$\frac{1}{4}(p+39)$ if $p=4n+1$, $\frac{1}{4}(p+37)$ if $p=4n+3$
pmg	5	2	11	11	$\frac{1}{2}(p+1)+10$
pgg	2	1	4	4	$\frac{1}{4}(p+15)$ if $p=4n+1$, $\frac{1}{4}(p+13)$ if $p=4n+3$
cmm	5	1	11	8	$\frac{1}{4}(p+39)$ if $p=4n+1$, $\frac{1}{4}(p+37)$ if $p=4n+3$
p3m1	1	2	1	4	1 iff $p=6n+1$
p31m	1	2	1	5	1 iff $p=6n+1$
p4m	5	0	13	2	2, +1 if $p=4n+1$
p4g	3	0	7	2	2, +1 if $p=4n+1$
p6m	3	2	2	11	1 iff $p=6n+1$

Thomas Wieting did the computation for the $N \leqslant 60$ and those results match with the generalized formula that we obtained.

References

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