Cumulants for Asymmetric Additive Convolution

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Yulia's Dream Program

1 Background on probability and cumulants

2 Asymmetric additive convolution of polynomials

3 Main theorems

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Random variable X:

$$P(X = x_1) = p_1;$$

$$P(X = x_n) = p_n:$$

(Atomic) probability measure :

$$= p_1 \; {}_{x_1} + \quad + p_n \; {}_{x_n}:$$

Moments of :

$$m_k(\;) := \sum_{i=1}^{n} p_i x_i^k; \quad k = 1; 2; 3;$$

Uniform random variable X:

$$P(X = x_1) = p_1 = \frac{1}{n};$$

$$P(X = x_n) = p_n = \frac{1}{n}.$$

Uniform probability measure:

$$= \frac{1}{n} \sum_{i=1}^{n} \quad x_i = \frac{1}{n} \quad x_1 + \qquad + \frac{1}{n} \quad x_n.$$

Moments of a uniform probability measure:

$$m_k(\ ) := \frac{\sum_{i=1}^{n} x_i^k}{n}; \quad k = 1; 2; 3;$$

X, Y random variables are **independent** if

$$P(X = a, Y = b) = P(X = a)P(Y = b).$$

For a random variable X define the **cumulants** $\kappa_1(X), \kappa_2(X), \ldots$ as

$$\kappa_n(X) = n! \cdot [z^n] \ln\left(1 + \sum_{n=1}^{\infty} \frac{z^n m_n(X)}{n!}\right), \qquad (1)$$

where $m_n(X)$ $(n = \overline{1, \infty})$ are moments $m_n(\mu)$ of the probability measure μ of X.

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THEOREM 1 (LINEARITY OF CUMULANTS)

For independent variables X, Y and all positive integers n:

$$\kappa_n(X + Y) = \kappa_n(X) + \kappa_n(Y)$$

Sketch of proof:

$$X, Y \text{ are independent} \implies \mathrm{E}[e^{z(X+Y)}] = \mathrm{E}[e^{zX}] \cdot \mathrm{E}[e^{zY}].$$

$$\text{By Taylor expansion: } \mathrm{E}[e^{zX}] = \mathrm{E}\left(1 + \sum_{n=1}^{\infty} \frac{z^n X^n}{n!}\right) = 1 + \sum_{n=1}^{\infty} \frac{z^n m_n(X)}{n!}.$$

$$\exp\left(\sum_{n=1}^{\infty} \frac{\kappa_n(X+Y) z^n}{n!}\right) = \exp\left(\sum_{n=1}^{\infty} \frac{\kappa_n(X) z^n}{n!}\right) \cdot \exp\left(\sum_{n=1}^{\infty} \frac{\kappa_n(Y) z^n}{n!}\right). \quad \square$$

Consider a monic polynomial

$$P(x) = (x \quad r_1)(x \quad r_2) \quad (x \quad r_d);$$

of degree d, where each root r_i is real. Its **empirical root distribution** is:

$$_P := \frac{1}{d} \sum_{i=1}^{d} \ _{r_i} = \frac{1}{d} \ _{r_1} + \quad + \frac{1}{d} \ _{r_d}:$$

Consider a monic polynomial

$$P(x) = (x - r_1)(x - r_2) \cdots (x - r_d),$$

of degree d, where each root r_i is real. Its **empirical root distribution** is:

$$\mu_P := \frac{1}{d} \sum_{i=1}^{d} \delta_{r_i} = \frac{1}{d}\delta_{r_1} + \cdots + \frac{1}{d}\delta_{r_d}.$$

An **even polynomial** is a polynomial on x^2 with **nonnegative** real roots, so any even polynomial $Q(x)$ can be expressed as

$$Q(x) = (x^2 - r_1^2) \cdots (x^2 - r_d^2) = (x - r_1)(x + r_1) \cdots (x - r_d)(x + r_d),$$

where $r_i \geq 0$. The empirical root distribution of Q is:

$$\mu_Q = \frac{1}{2d} \sum_{i=1}^{d} (\delta_{r_i} + \delta_{-r_i}),$$

and has moments:

$$m_k(\mu_Q) := \begin{cases} 0, & \text{for odd } k, \\ \frac{1}{d}\sum_{i=1}^{d} r_i^k, & \text{for even } k. \end{cases}$$

Asymmetric additive convolution

For monic polynomials $p(x)$ and $q(x)$ of degree d,

$$p(x) = \sum_{i=0}^{d} x^{d-i}(-1)^i a_i^p; \qquad q(x) = \sum_{i=0}^{d} x^{d-i}(-1)^i a_i^q;$$

where $a_0^p = a_0^q = 1$, [Marcus-Spielman-Srivastava '22] defined the asymmetric additive convolution $(p \boxplus_d^0 q)(x)$, as

$$(p \boxplus_d^0 q)(x) := \sum_{k=0}^{d} x^{d-k}(-1)^k \sum_{i+j=k} \left(\frac{(d-i)!(d-j)!}{d!(d-k)!}\right)^2 a_i^p a_j^q;$$

i.e.

$$(p \boxplus_d^0 q)(x) := x^d - \left(a_1^p + a_1^q\right)x^{d-1} + \left(a_2^p + \left(\frac{d-1}{d}\right)^2 a_1^p a_1^q + a_2^q\right)x^{d-2} - \cdots$$

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[redacted]

If $p(x)$ and $q(x)$ both have nonnegative real roots, then $(p \boxplus_d^0 q)(x)$ also has nonnegative real roots.

Positive real-rootedness

If $p(x) = x^d + a_1 x^{d \quad 1} + \quad + a_d$ has nonnegative real roots $r_1^2; \ldots; r_d^2$, for some $r_1; \ldots; r_d \quad 0$, then

$$p(x^2) = x^{2d} + a_1 x^{2d \quad 2} + \quad + a_d$$

has (2d) real roots $\quad r_1; \ldots; \quad r_d$.

The empirical root distribution

$$^{p} := \frac{1}{2d} \sum_{i=1}^{d} (\, _{r_i} + \quad _{r_i}):$$

If $p(x) = x^d + a_1 x^{d-1} + \cdots + a_d$ has nonnegative real roots $r_1^2, \ldots, r_d^2$, for some $r_1, \ldots, r_d \geq 0$, then

$$p(x^2) = x^{2d} + a_1 x^{2d-2} + \cdots + a_d$$

has $(2d)$ real roots $\pm r_1, \ldots, \pm r_d$.

The empirical root distribution

$$\mu_p := \frac{1}{2d} \sum_{i=1}^{d} (\delta_{r_i} + \delta_{-r_i}).$$

Probabilistic point of view: The polynomial operation $(p, q) \mapsto p \boxplus_d^0 q$ is "equivalent" to an operation $(\mu_p, \mu_q) \mapsto \mu_p \boxplus_d^0 \mu_q$.

This is similar to X, Y independent and $(X, Y) \mapsto X + Y$.

Our goal: define cumulants that "linearize" the asymmetric additive convolution.

DEFINITION 3

For the monic polynomial $p(x) = \sum_{i=0}^{d} x^{d-i}(-1)^i a_i^p$, with $a_0^p = 1$, define the **asymmetric cumulants** of $p(x)$ by:

$$\kappa_{2\ell}^p = 2\ell\,[y^{2\ell}] \ln\left(1 + \sum_{n=1}^{d} \left(\frac{(d-n)!}{2^n d!}\right)^2 a_n^p y^{2n}\right); \quad \text{for all } \ell = 1, 2, \dots, d. \tag{2}$$

Theorem 4

For any monic polynomials $p(x); q(x)$ of degree d, we have

$${}^{p}_{2k}\ {}^{0}_{d}q = {}^{p}_{2k} + {}^{q}_{2k}; \quad \textit{for all } k = 1; 2; :::; d.$$

Thus, the asymmetric cumulants linearize asymmetric additive convolution.

Examples:

Recall: $\quad (p\ {}^{0}_{d}\ q)(x) := x^d \quad a_1^p + a_1^q \ \ x^{d \quad 1} + \ \ a_2^p + \frac{d \quad 1}{d}^{\ 2} a_1^p a_1^q + a_2^q \ \ x^{d \quad 2} \qquad :$

For $k = 1$:

$${}^{p}_{2} = \frac{a_1^p}{2d^2}; \qquad {}^{q}_{2} = \frac{a_1^q}{2d^2}; \qquad {}^{p}_{2}\ {}^{0}_{d}q = \frac{a_1^p + a_1^q}{2d^2} = {}^{p}_{2} + {}^{q}_{2};$$

For $k = 2$:

$${}^{p}_{4} = \frac{a_2^p}{4d^2(1 \quad d)^2} \quad \frac{(a_1^p)^2}{8d^4}; \qquad {}^{q}_{4} = \frac{a_2^q}{4d^2(1 \quad d)^2} \quad \frac{(a_1^q)^2}{8d^4};$$

$${}^{p}_{4}\ {}^{0}_{d}q = \frac{a_2^p + \frac{d \quad 1}{d}^{\ 2} a_1^p a_1^q + a_2^q}{4d^2(1 \quad d)^2} \quad \frac{(a_1^p + a_1^q)^2}{8d^4} = {}^{p}_{4} + {}^{q}_{4}.$$

We want formulas between asymmetric cumulants and moments:

$$
\begin{aligned}
{}_2 &= \frac{m_2}{2d}; \\
{}_4 &= \frac{m_4}{8d(d \quad 1)^2} + \frac{2d \quad 1}{8d^2(d \quad 1)^2} m_2^2;
\end{aligned}
$$

and

$$
\begin{aligned}
m_2 &= \quad 2d \;\; {}_2; \\
m_4 &= \quad 8d(d \quad 1)^2 \;\; {}_4 \quad 4d(2d \quad 1) \;\; {}_2^2; \\
m_6 &= \quad 16d(d \quad 1)^2(d \quad 2)^2 \;\; {}_6 + 16d(d \quad 1)^2(7d \quad 6) \;\; {}_4 \;\; {}_2 \quad 8d(2d \quad 1)^2 \;\; {}_2^3.
\end{aligned}
$$

Theorem 5

(i) For all $k \in \mathbb{Z}_{\geq 1}$:

$$_{2k} = \frac{1}{(2k-1)!\; 2^{2k}} \sum_{\in P^{even}(2k)} (-2d)^{\#(\,)} \prod_{B \in} (|B|-1)! \; m \sum_{:} \frac{(-1)^{\#(\,)-1}(\#(\,)-1)}{\prod_{D\in} (-d)^{2}_{\frac{|D|}{2}}}.$$

(ii) For all $k \in \mathbb{Z}_{\geq 1}$:

$$m_{2k} = \frac{2^{2k-1}}{d\;(2k-1)!} \sum_{\in P^{even}(2k)} \prod_{B\in} (|B|-1)! \sum_{:} (-1)^{\#(\,)}(\#(\,)-1)! \prod_{D\in} (-d)^{2}_{\frac{|D|}{2}}.$$