Counting spanning trees with linear algebra

Denis Liabakh, Maksym Skulysh, Maryna Lubimova

June 2024
Definitions

Definition 1 (Graph)
A simple undirected graph $G$ is a pair $(V, E)$, where $V$ is a set and $E$ is a symmetric subset of $V \times V \setminus \{(x, x), x \in V\}$. The elements of $V$ are called the vertices of $G$ and the elements of $E$ are called the edges of $G$.

Definition 2 (Path)
A path is a non-empty subgraph $P = (V_P, E_P)$ of the graph $G$ of the form $V_P = \{x_0, x_1, \ldots, x_k\}$, $E_P = \{x_0x_1, x_1x_2, \ldots, x_{k-1}x_k\}$, where the $x_i$ are all distinct.

Definition 3 (Connected graph)
A non-empty graph $G$ is called connected if any two of its vertices are linked by a path in $G$. 

Denis Liabakh, Maksym Skulys, Maryna Lubimova
Counting spanning trees with linear algebra
2
Definitions

Definition 1 (Graph)
A simple undirected graph $G$ is a pair $(V, E)$, where $V$ is a set and $E$ is a symmetric subset of $V \times V \setminus \{(x, x), x \in V\}$. The elements of $V$ are called the vertices of $G$ and the elements of $E$ are called the edges of $G$.

Definition 2 (Path)
A path is a non-empty subgraph $P = (V_P, E_P)$ of the graph $G$ of the form

$$V_P = \{x_0, x_1, \ldots, x_k\} \quad E_P = \{x_0x_1, x_1x_2, \ldots, x_{k-1}x_k\},$$

where the $x_i$ are all distinct.
Definitions

Definition 1 (Graph)
A simple undirected graph $G$ is a pair $(V, E)$, where $V$ is a set and $E$ is a symmetric subset of $V \times V \setminus \{(x, x), x \in V\}$. The elements of $V$ are called the vertices of $G$ and the elements of $E$ are called the edges of $G$.

Definition 2 (Path)
A path is a non-empty subgraph $P = (V_P, E_P)$ of the graph $G$ of the form

$$V_P = \{x_0, x_1, \ldots, x_k\} \quad E_P = \{x_0x_1, x_1x_2, \ldots, x_{k-1}x_k\},$$

where the $x_i$ are all distinct.

Definition 3 (Connected graph)
A non-empty graph $G$ is called connected if any two of its vertices are linked by a path in $G$. 
**Definition 4 (Tree)**

A simple connected graph $T$ is called *tree* if it is minimally connected, i.e., $T$ is connected but $T - e$ is disconnected for every edge $e \in T$. 

**Definition 5 (Spanning tree)**

If $G$ is a connected graph, we say that $T$ is a *spanning tree* of $G$ if $G$ and $T$ have the same vertex set, and each edge of $T$ is also an edge of $G$. 

Definition 4 (Tree)
A simple connected graph $T$ is called tree if it is minimally connected, i.e. $T$ is connected but $T - e$ is disconnected for every edge $e \in T$.

Definition 5 (Spanning tree)
If $G$ is a connected graph, we say that $T$ is a spanning tree of $G$ if $G$ and $T$ have the same vertex set, and each edge of $T$ is also an edge of $G$. 
The graph on $V = \{1, \cdots, 7\}$ with edge set $E = \{\{1, 2\}, \{2, 5\}, \{3, 4\}, \{4, 5\}, \{5, 7\}\}$
You are given a finite simple connected graph $G$. How to calculate number of spanning trees of $G$?
Theorem 6 (Matrix-Tree theorem)

Let $U$ be a simple undirected graph. Let \( \{v_1, v_2, \ldots, v_n\} \) be the vertices of $U$. Define \((n - 1) \times (n - 1)\) matrix $L_0$ by

\[
\ell_{ij} = \begin{cases} 
\text{the degree of } v_i \text{ if } i = j, \\
-1 \text{ if } i \neq j, \text{ and } v_i \text{ and } v_j \text{ are adjacent, and} \\
0 \text{ otherwise}
\end{cases}
\]

where \(1 \leq i, j \leq n - 1\). Then $U$ has exactly $\det L_0$ spanning trees.
Definition 7 (Matrix)
The matrix size $m \times n$ with real or complex entries is a rectangular array or table filled with real or complex numbers.
Definition 7 (Matrix)

The matrix size $m \times n$ with real or complex entries is a rectangular array or table filled with real or complex numbers.

$$I = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{bmatrix}$$
Definition 7 (Matrix)

The matrix size \( m \times n \) with real or complex entries is a rectangular array or table filled with real or complex numbers.

Operations with matrices

- Addition
- Scalar multiplication
- Multiplication
- Transposing
- Inverting
Definition 8 (Determinant of matrix)

Determinant of a square matrix is an antisymmetric multilinear function of the columns (or of the rows) of a matrix such that $\det I = 1$. 
Properties of determinant

Properties

- \( \det I = 1 \)
- Exchanging two rows (or two columns) reverses the sign of the determinant.
- The determinant is linear in each row (in each column) separately.
- For matrices of equal size \( X \) and \( Y \): \( \det XY = \det X \det Y \)
- For matrix \( X \) of size \( a \times a \) and constant \( c \in \mathbb{C} \): \( \det(cX) = c^a \det X \)
computing determinant: formula with permutations

\[ \text{det } A = \sum_{\pi \in \text{Sym}(n)} \text{sign}(\pi) a_{1\pi(1)} a_{2\pi(2)} \cdots a_{n\pi(n)}, \]

where \( \pi \) ranges over the collection of all permutations of the set \( \{1, 2, \ldots, n\} = [n] \).
Row operations

**Switching rows**

\[
\begin{bmatrix}
  a_{11} & \cdots & a_{1(n-1)} & a_{1n} \\
  \cdots & \cdots & \cdots & \cdots \\
  a_{m1} & \cdots & a_{m(n-1)} & a_{mn}
\end{bmatrix}
\rightarrow
\begin{bmatrix}
  a_{m1} & \cdots & a_{m(n-1)} & a_{mn} \\
  \cdots & \cdots & \cdots & \cdots \\
  a_{11} & \cdots & a_{1(n-1)} & a_{1n}
\end{bmatrix}
\]

**Multiplying row by a non-zero constant**

\[
\begin{bmatrix}
  a_{11} & \cdots & a_{1(n-1)} & a_{1n} \\
  \cdots & \cdots & \cdots & \cdots \\
  a_{m1} & \cdots & a_{m(n-1)} & a_{mn}
\end{bmatrix}
\rightarrow
\begin{bmatrix}
  Ma_{11} & \cdots & Ma_{1(n-1)} & Ma_{1n} \\
  \cdots & \cdots & \cdots & \cdots \\
  Ma_{m1} & \cdots & Ma_{m(n-1)} & Ma_{mn}
\end{bmatrix}
\]

**Adding rows**

\[
\begin{bmatrix}
  a_{11} & \cdots & a_{1(n-1)} & a_{1n} \\
  \cdots & \cdots & \cdots & \cdots \\
  a_{m1} & \cdots & a_{m(n-1)} & a_{mn}
\end{bmatrix}
\rightarrow
\begin{bmatrix}
  a_{11} + a_{m1} & \cdots & a_{1n} + a_{mn} \\
  \cdots & \cdots & \cdots & \cdots \\
  a_{m1} & \cdots & a_{mn}
\end{bmatrix}
\]
Computing determinant: Cofactor formula

Cofactor formula

\[
\text{det } A = \sum_{j=1}^{n} a_{ij} C_{ij}
\]

where \( i \in [n] \) and \( C_{ij} \) equals \((-1)^{i+j} \times \text{determinant of } (n-1) \times (n-1) \) square matrix obtained by removing row \( i \) and column \( j \). \( C_{ij} \) is called a cofactor of \( a_{ij} \).
Example 9

Prove that the number of spanning trees of $K_n$ is $n^{n-2}$ (Cayley’s formula).
Example 9

Prove that the number of spanning trees of $K_n$ is $n^{n-2}$ (Cayley's formula).

Proof.

$$L_0 = \begin{bmatrix} n - 1 & -1 & \cdots & -1 \\ -1 & n - 1 & \cdots & -1 \\ \cdots \\ -1 & -1 & \cdots & n - 1 \end{bmatrix}$$
Example

Proof.

\[
\begin{pmatrix}
1 & 1 & \ldots & 1 \\
-1 & n-1 & \ldots & -1 \\
\vdots \\
-1 & -1 & \ldots & n-1
\end{pmatrix}
\]
Example

Proof.

\[
\begin{bmatrix}
1 & 1 & \ldots & 1 \\
-1 & n-1 & \ldots & -1 \\
\vdots \\
-1 & -1 & \ldots & n-1
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 1 & \ldots & 1 \\
0 & n & \ldots & 0 \\
\vdots \\
0 & 0 & \ldots & n
\end{bmatrix}
\]
Example

Proof.

\[
\begin{bmatrix}
1 & 1 & \cdots & 1 \\
-1 & n-1 & \cdots & -1 \\
\vdots \\
-1 & -1 & \cdots & n-1
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 1 & \cdots & 1 \\
0 & n & \cdots & 0 \\
\vdots \\
0 & 0 & \cdots & n
\end{bmatrix}
\]

\[
\det L_0 = \begin{vmatrix}
1 & 1 & \cdots & 1 \\
0 & n & \cdots & 0 \\
\vdots \\
0 & 0 & \cdots & n
\end{vmatrix} = n^{n-2}
\]
Definition 9

Directed $G$ graph is defined as follows: $G=(V,E, s, t)$ where $V$ and $E$ are sets and $s$ and $t$ are the functions from $E$ to $V$. For an edge $e$ we think of $s(e)$ as the starting vertex of $e$ and $t(e)$ is the ending vertex of $e$. 

Denis Liabakh, Maksym Skulysh, Maryna Lubimova
Definition 9

Directed $G$ graph is defined as follows: $G=(V,E, s, t)$ where $V$ and $E$ are sets and $s$ and $t$ are the functions from $E$ to $V$. For an edge $e$ we think of $s(e)$ as the starting vertex of $e$ and $t(e)$ is the ending vertex of $e$. 

![Diagram of a directed graph]

- Vertex 1 connected to vertex 2
- Vertex 2 connected to vertex 3
- Vertex 4 connected to vertex 5
- Vertex 5 connected to vertex 6
The case of directed graphs

Definition 10

Let $G$ be a directed graph without loops. Let $\{v_1, v_2, \ldots, v_n\}$ be a vertices of $G$, and let $\{e_1, e_2, \ldots, e_m\}$ denote the edges of $G$. Then the incidence matrix of $G$ is $n \times m$ matrix $A$ defined by

- $a_{ij} = 1$ if $v_i$ is the starting vertex of $e_j$
- $a_{ij} = -1$ if $v_i$ is the ending vertex of $e_j$
- $a_{ij} = 0$ otherwise.

Theorem 11

Let $G$ be a directed graph without loop, and let $A$ be the incidence matrix of $G$. Remove any row of $A$ and let $A_0$ be the remaining matrix. The number of spanning trees of $G$ is $\det A_0 A_0^T$. 
The case of directed graphs

**Definition 10**

Let $G$ be a directed graph without loops. Let $\{v_1, v_2, \ldots, v_n\}$ be a vertices of $G$, and let $\{e_1, e_2, \ldots, e_m\}$ denote the edges of $G$. Then the *incidence matrix* of $G$ is $n \times m$ matrix $A$ defined by

- $a_{ij} = 1$ if $v_i$ is the starting vertex of $e_j$
- $a_{ij} = -1$ if $v_i$ is the ending vertex of $e_j$
- $a_{ij} = 0$ otherwise.

**Theorem 11**

Let $G$ be a directed graph without loop, and let $A$ be the incidence matrix of $G$. Remove any row of $A$ and let $A_0$ be the remaining matrix. The number of spanning trees of $G$ is $\det A_0 A_0^T$. 