The Conformal Type Problem for Riemann Surfaces Through Discrete Analysis of Extended Speiser Graphs

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January 11, 2025

Abstract

The conformal type (parabolic or hyperbolic) of a covering surface of the Riemann sphere with n singular values can be determined by the type (recurrent or transient respectively) of the corresponding extended Speiser graph. In this paper, we look at the extended Speiser graphs of some covering surfaces whose conformal type is known via analytic methods. For a covering surface known to be parabolic, we use discrete techniques of shorting and discrete extremal length to verify that the type of its extended Speiser graph is recurrent. For a covering surface known to be hyperbolic we provide ideas of possible approaches for transience of the extended Speiser graph.

1 Introduction

The origins of the conformal type problem date back to the establishment of the Riemann Mapping Theorem and its generalization, the Uniformization Theorem. Various forms of the Uniformization theorem were known in the 19th century, with rigorous proofs provided in the early 20th century. The Riemann Mapping Theorem states that every nonempty simply connected proper open subset of the complex plane is conformally equivalent to the unit disk, whereas the Uniformization Theorem states that every simply connected Riemann surface is conformally equivalent to the sphere, unit disk, or complex plane [5]. In particular, Riemann surfaces that are not compact must be conformally equivalent to either the unit disk or the complex plane, and therefore can be categorized into two types: hyperbolic and parabolic, respectively.

The particular type of non-compact Riemann surface we analyze in this paper are infinite-sheeted covering surfaces of an n-punctured Riemann sphere. Each of these covering surfaces has a unique representation as a combinatorial object known as its Speiser graph [7]. This graph describes the covering surface completely up to the location of the punctures, and since the location of the punctures does not affect the type, the graph must completely determine the type of the surface [1].

The problem of determining the conformal type of a covering surface from its graph has attracted a lot of attention since the 1930's. Many results that relate properties of the surface and its corresponding Speiser graph have been found, with most being either sharp but not useful, or useful but not sharp [1]. However, in [1], Doyle found a result that encapsulates both: by modifying the Speiser graph slightly to obtain a so-called "extended Speiser graph", the type of the surface (parabolic or hyperbolic) corresponds exactly with the type of a random walk on its extended Speiser graph (recurrent or transient, respectively).

The goal of this paper is to determine the conformal type of covering surfaces of the Riemann sphere with n punctures by finding the type of a random walk on the corresponding extended Speiser graphs. In Section 2, we provide some preliminaries on conformal type, Speiser graphs, graph types, and determining graph types. Then, in the following sections we examine two covering surfaces whose types are known through surface cutting-pasting methods, but whose extended Speiser graphs have not yet been studied. In Section 3, we prove that for a surface proven to be parabolic in [8], its extended Speiser graph is indeed recurrent. We do so by using the method of extremal length [4, 6]. In Section 4, we describe two possible approaches, namely the method of flows and embedding a transient tree [2], to prove that for a surface proven to be hyperbolic in [3], its extended Speiser graph is indeed transient. While we were not able to prove that the graph is transient, we believe our approach can lead to a proof of transience. If successful, these methods could be applied to determine the types of many similar extended Speiser graphs.

2 Preliminaries

In this section, we explain the conformal type problem, recurrent and transient graph types, how to construct Speiser graphs and extended Speiser graphs, and techniques to determine the type of graphs. We will use these preliminaries throughout the rest of the paper.

2.1 The Conformal Type Problem

The conformal type problem involves finding the conformal type of non-compact Riemann surfaces. In this subsection, we define conformal type.

Theorem 2.1 (Uniformization Theorem [5]). *Every simply connected Riemann surface is conformally equivalent to exactly one of the following three Riemann surfaces:*

1. *the complex plane*
2. *the open unit disk*
3. *the Riemann sphere.*

A consequence of Theorem 2.1, all non-compact Riemann surfaces are conformally equivalent to either the complex plane or the unit disk. Thus, non-compact Riemann surfaces can be categorized into two types.

Definition 2.2. The conformal type of a non-compact Riemann surface S is *parabolic* if there exists a conformal map between S and the complex plane. Likewise, the conformal type of a Riemann surface S is *hyperbolic* if there exists a conformal map between S and the unit disk.

In this paper, we find the conformal type (parabolic or hyperbolic) of an infinite-sheeted covering surface of an n-punctured Riemann sphere, which is a non-compact Riemann surface. We find the conformal type of the surface through the lens of a different type problem: the type problem for graphs.

2.2 The Type Problem for Graphs

Next, we introduce the notion of the type of a connected graph G, as well as some related theorems.

Definition 2.3. Assume that G is an infinite non-directed connected graph of bounded degree. Given two vertices x, y of G, let xy be the edge connecting them (or set of edges if there are multiple). For each edge (or set of edges) xy of G, we assign a weight $C_{xy} > 0$ called a *conductance*. The graph G together with the conductances $C = (C_{xy})$ is called a *network* and denoted by (G, C).

Definition 2.4. Given a network (G, C) and a vertex v, we define the *random walk* starting at v as a chance process where the walker starts at v and has probability P_{xy} of moving from vertex x to vertex y. This probability is

$$P_{xy} = \frac{C_{xy}}{C_x},$$

where $C_x = \sum_y C_{xy}$.

Under the definition of a random walk, if the graph G in a network (G, C) has multiple edges between two vertices, we can replace the set of edges with a single edge of the same conductance.

Definition 2.5. We define the *simple random walk* starting at a vertex v on a graph G as the random walk on the network (G, C) where all the conductances are equal. A simple random walk moves along each adjacent edge with the same probability.

Definition 2.6. The type of a random walk on network (G, C) (or simple random walk on graph G) starting at vertex v is *recurrent* if the walk returns to v at some point with probability 1. The type of a random walk on network (G, C) (or simple random walk on graph G) starting at vertex v is *transient* if the walk returns to v at some point with probability less than 1.

Theorem 2.7 ([2, pg.102]). *The type of a random walk on a network (G, C) (or simple random walk on graph G) is independent of the starting vertex.*

Since the type of a random walk is independent of the starting vertex, in the following sections, we will use "the type of the network (G, C)" to refer to the type of a random walk on (G, C) and we will use "the type of the graph G" to refer to the type of a simple random walk on G.

We will also use the following theorems on graph type.

Theorem 2.8 ([2, pg.102]). *Let (G, C) be a network. Suppose there exists constants $u, v > 0$ such that for every edge xy of G we have $0 < u < C_{xy} < v < \infty$. Then the type of the network (G, C) is the same as the type of the graph G.*

Definition 2.9. A *subnetwork* of a network (G, C) is defined as a network $(\overline{G}, \overline{C})$ consisting of a subgraph $\overline{G}$ of the graph G and the corresponding conductances $\overline{C}$ of the edges of $\overline{G}$. The edges of $\overline{G}$ in the subnetwork have the same conductances as in the original network.

Theorem 2.10 ([2, pg.74-75]). *All subgraphs of recurrent graphs are recurrent and all graphs containing a transient subgraph are transient.*

2.3 Relating Surfaces and Graphs

Now that we have defined the conformal type of surfaces and the type of a graph, in this subsection, we describe how to relate the two type problems through the extended Speiser graph [1].

Before introducing the extended Speiser graph, we must first introduce the notion of a Speiser graph [7]. An example Speiser graph is shown in Figure 2.

Definition 2.11. The *Speiser graph* is a combinatorial graph associated to an infinitely-sheeted covering surface of the n-punctured Riemann sphere.
To construct the Speiser graph, first draw a simple closed curve C passing through the n punctures of the Riemann sphere. The curve C will separate the sphere into two halves which we call S_a and S_b. Meanwhile, each section of curve between two adjacent punctures will be labeled $C_1, C_2, \ldots, C_n$, as shown in Figure 1. Cutting the covering surface along the curve C yields many pieces: some that cover S_a and some that cover S_b.
The vertices of the Speiser graph represent pieces of the covering surface formed by cutting along curve C, which are labeled a or b according to whether they cover S_a or S_b, respectively. Two vertices are connected by an edge if and only if the two corresponding pieces of the covering surface are connected along a section of curve C_i. The edge between them is labeled i accordingly.

To determine the conformal type of a covering surface, we will use an extended Speiser graph, which was introduced in [1].

Definition 2.12. Given a Speiser graph, the *extended Speiser graph* is constructed by quadrangulating all faces of the original Speiser graph that have an infinite number of edges. That is, we glue in lattices along the vertices of the infinite edged face.

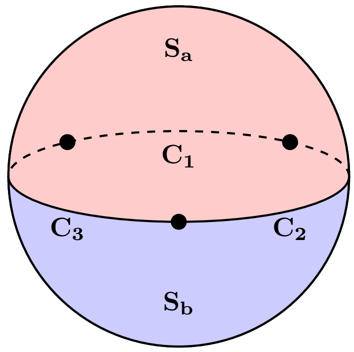

Figure 1: A curve C separating a Riemann sphere with 3 punctures into two halves S_a and S_b

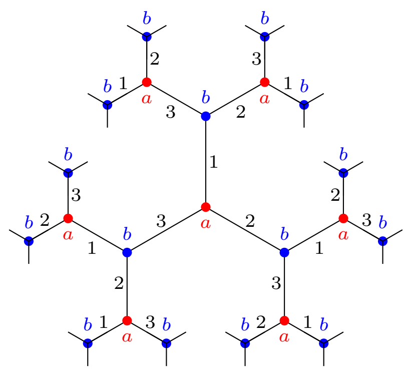

Figure 2: An example of a Speiser graph

Figure 3 shows an example of a Speiser graph with a face of infinite edges, and the extended Speiser graph formed by quadrangulating the face. We call the structure in the quadrangulated region a "half-plane lattice." Figure 4 shows another example, where the half-plane lattice is now crammed into a strip. The latter half-plane lattice shows up often in the extended Speiser graphs we are studying, and we will call such a structure a "folded half-plane lattice."

We will use the type of the extended Speiser graph to find the conformal type of a covering surface by using the theorem below.

Theorem 2.13 (Doyle [1]). *A covering surface of a Riemann Sphere with n punctures is parabolic if and only if the type of the simple random walk on its extended Speiser graph is recurrent. Likewise, a covering surface of a Riemann Sphere with n punctures is hyperbolic if and only if the type of the simple random walk on its extended Speiser graph is transient.*

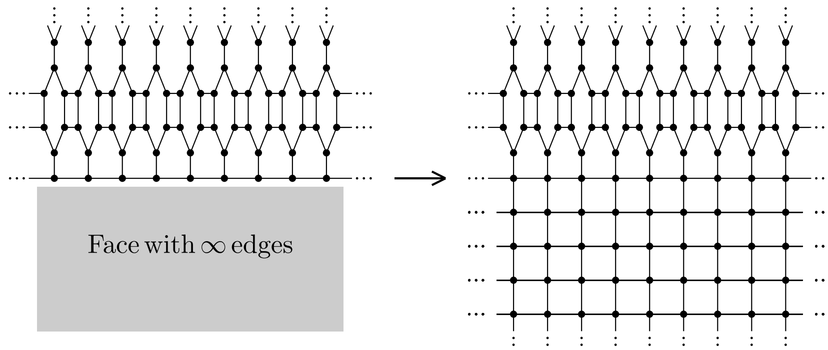

Figure 3: Replacing an infinite edged face with a half-plane lattice

2.4 Techniques to Determine Graphs Type

In this section, we introduce a few techniques to determine the type of a graph.

2.4.1 The Method of Extremal Length

The method of extremal length is used to prove the recurrence of a graph. We give the definition of extremal length and some theorems about its properties.

Definition 2.14. Let G be a locally-finite connected graph. Consider a path t in G and denote by E_t the set of edges of t. Similarly, use E_T to denote the set of edges of a family of paths T in G. Then the *extremal length* of a family of paths T in G, is denoted by $\lambda(T)$ and is defined as

$$\lambda(T)^{-1} = \inf \left\{ \sum_{e \in E_T} \mu(e)^2 \right\},$$

where the infinimum is taken with respect to all density functions μ defined on the edge set E_T, such that for all $t \in T$,

$$\sum_{e \in E_T} \mu(e) \geq 1.$$

Definition 2.15. For a family of paths T in a graph G, the *mass* of T is defined as $\lambda(T)^{-1}$.

The following two theorems describe some properties of extremal length. We use these theorems when applying the method of extremal length.

Theorem 2.16 ([4, pg.24-25])**.** *The extremal length of the family of paths connecting a given vertex to infinity, is infinite for recurrent graphs and finite for transient graphs.*

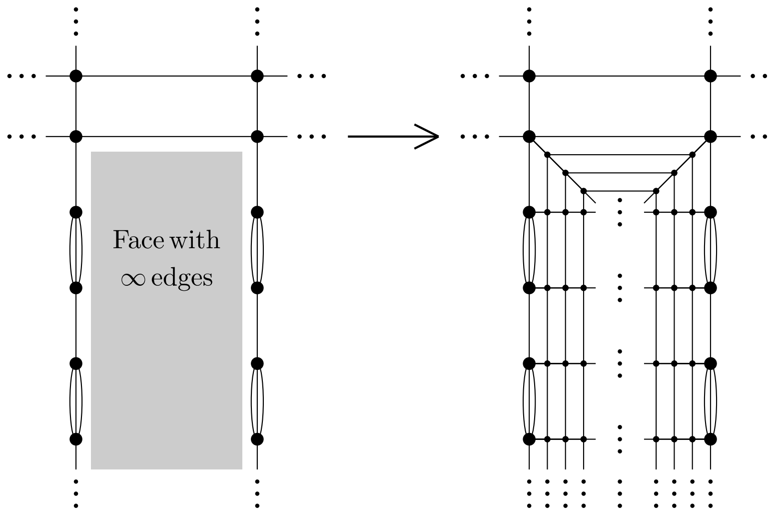

Figure 4: A folded half-plane lattice

Theorem 2.17 ([4, C.1]). *Let T and T_i for $i \in I$ be families of paths in graph G, where I is an indexing set that is at most countable. We assume that $E_{T_i} \cap E_{T_j} = \emptyset$ for $i \neq j$ and that for every $t \in T$ and $i \in I$ there exists a path $t_i \in T_i$ that is a subpath of t. Then,*

$$\lambda(T) \geq \sum_{i \in I} \lambda(T_i).$$

Given a graph G, we will construct families of paths T and T_i by using annuli, as defined below.

Definition 2.18. Given a graph G, suppose we divide G into a countable number of roughly ring-like concentric sections. We call each such section an *annulus*. The inner boundary of an annulus is called an *inner ring* and the outer boundary is called an *outer ring*. We designate one of the vertices of the graph inside all of the annuli as the *origin*.

An example of annuli is depicted in Figure 5 by the regions between an inner blue ring and an outer red ring. Note that the annuli need not cover the entire graph. For example, the outer red ring of the second annulus (the second red ring from the center) and the inner blue ring of the third annulus (the third blue ring from the center) are not the same.

Definition 2.19. Given a graph G sectioned into annuli, let T be the family of paths from the origin to infinity. We define the *extremal length of G* as the extremal length of T. Let T_i be the family of all paths starting at the inner ring of the ith annulus from the origin and ending at the outer ring of the ith annulus from the origin. We define the *extremal length of the ith annulus* as the extremal length of T_i.

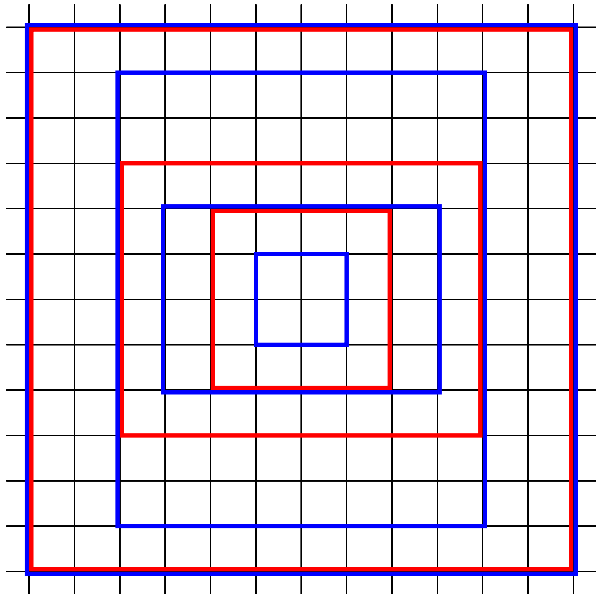

Figure 5: A graph separated into annuli

The families of paths formed by the annuli satisfy the conditions for Theorem 2.17, so the extremal length of G is greater than or equal to the sum of the extremal lengths of the annuli. Applying Theorem 2.16 leads to the following theorem.

Theorem 2.20 (Method of Extremal Length)**.** *To prove a graph G is recurrent, it suffices to prove that the sum of the extremal lengths of its annuli (or a lower bound of this sum) diverges.*

2.4.2 The Method of Flows

The other method we discuss in this section is the method of flows, which is a method is used to prove transience.

Definition 2.21. Let G be a locally-finite connected graph. A *flow* j on G from a vertex v out to infinity assigns a value j_{xy} to each directed edge xy such that

1. $j_{xy} = j_{yx}$
2. $\sum_y j_{xy} = 0$ if $x \neq v$.

Definition 2.22. A flow j from a vertex v out to infinity is called a *unit flow* if $\sum_y j_{vy} = 1$.

Definition 2.23. The *dissipation rate* of a flow is defined as

$$\frac{1}{2}\sum_{x,y} j_{xy}^2 R_{xy},$$

where R_{xy} is the resistance of the edge between x and y.

Theorem 2.24 (Thomson's Principle [2, pg.111])**.** *The dissipation rate of a unit flow gives an upper bound on a quantity called the effective resistance of the graph.*

Theorem 2.25 ([2, pg.111])**.** *The effective resistance of a graph is infinite if and only if the graph is recurrent.*

By Theorem 2.24 and Theorem 2.25, in order to prove a graph is transient, it suffices to find a unit flow with finite dissipation rate.

3 Proving Directly that the Extended Speiser Graph of a Parabolic Surface is Recurrent

In this section, we examine the Speiser graph in Figure 6 of a surface proven to be parabolic by [8] using cutting and pasting methods. Our main result is proving that the extended Speiser graph, shown in Figure 7 is recurrent.

Theorem 3.1. *The extended Speiser graph depicted in Figure 7 is proven to be recurrent using only discrete methods on the graph, rather than employing methods on the covering surface. In particular, by Theorem 2.13, the covering surface corresponding to the Speiser graph in Figure 6 is parabolic.*

We provide an alternate proof of the parabolicity of the covering surface corresponding through the recurrence of its extended Speiser graph. The extended Speiser graph is depicted in Figure 7, with all multi-edges being replaced by single edges, as according to Definition 2.4 and Theorem 2.8, such a moderation does not affect the type of the graph.

The extended Speiser graph is made up of a "top half," which is a half-plane lattice, and a "bottom half," which consists of infinitely many "folded half-plane lattices," each of which are attached to the top half-plane lattice. We will assign a vertex on the boundary between the top half and bottom half of the graph to be the origin. To prove that the extended Speiser graph is recurrent, we construct concentric annuli around the origin and use the method of extremal length.

3.1 Constructing the Annuli

We describe how to construct the annuli in three different regions: the top half of the graph, in the bottom half of the graph where the annulus connects to the top half, and in the rest of the bottom half of the graph.

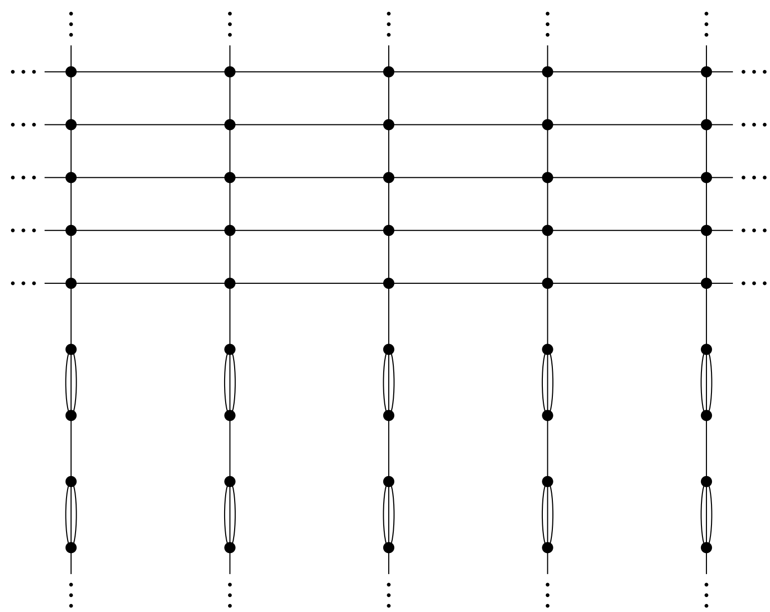

Figure 6: The Speiser graph of a parabolic surface

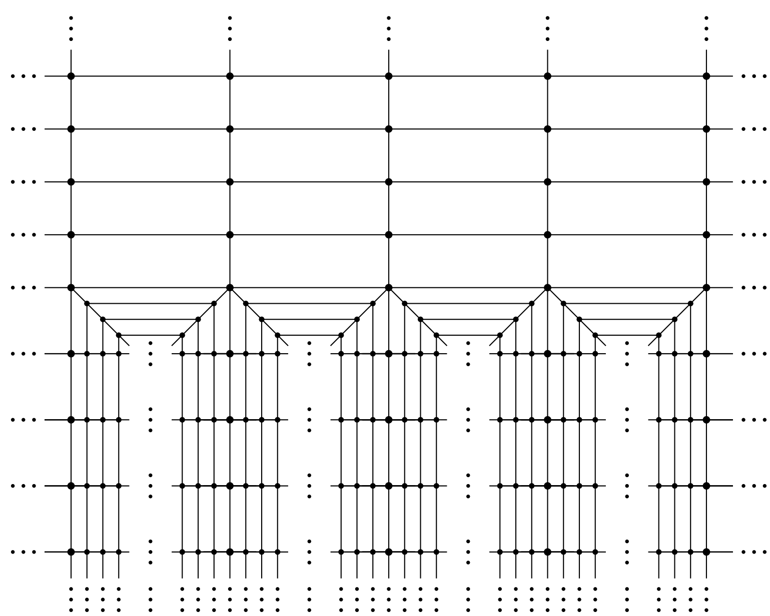

Figure 7: The extended Speiser graph of a parabolic surface

3.1.1 The Annuli in the Top Half of the Graph

We first describe how to construct the top half of the ith annulus from the origin. We let the concentric annuli have width 1 in the top half of the graph. The inner ring of ith annulus consists of $4i$ edges and meets the boundary of the top half and bottom half of the graph at a distance of i from the origin. The second annulus is depicted by the region between the thick blue lines in Figure 8.

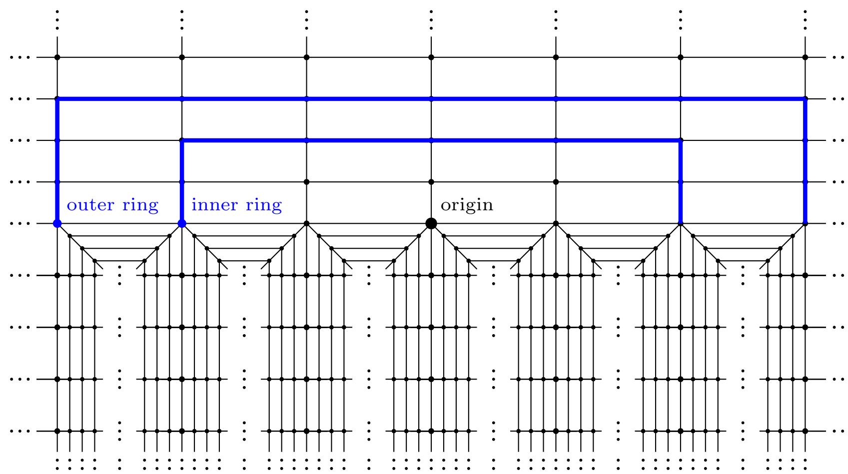

Figure 8: The top half of the annulus

3.1.2 The Annuli Entering the Bottom Half of the Graph

The diagram in Figure 9 shows the top half of the ith annulus entering a folded half-plane lattice in the bottom half of the graph. We call this region the "entry bottom half-plane lattice." We let the inner ring of the annulus consist of i edges going "down" in the lattice, 2^{i-1} edges going "right" in the lattice, and then i edges going "up" in the lattice into the next folded half-plane. Meanwhile, let the outer ring consist of 2^i edges going "down" in the lattice, $2^i + 1$ edges going "right" in the lattice, and then 2^i edges going "up" in the lattice into the next folded half-plane.

The other end of the top half of the annulus also enters a folded half-plane lattice. The part of the annulus in this entry bottom half plane is constructed similarly, but mirrors the other entry bottom half plane, with left and right directions reversed.

3.1.3 The Annuli in the Rest of the Bottom Half of the Graph

The ith annulus in all other folded half-planes lattices is constructed as in Figure 10. The inner ring enters the half-plane lattice a distance of 2^{i-1} away from

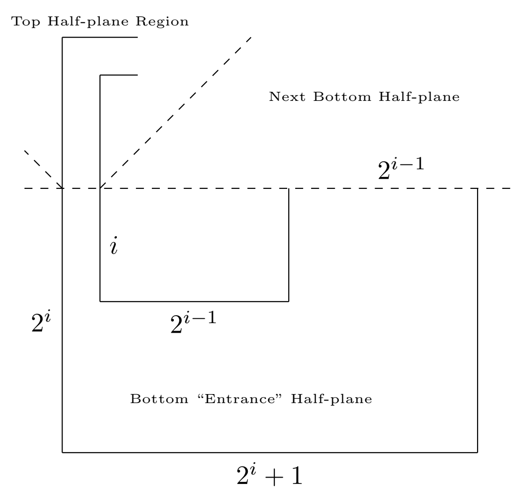

Figure 9: The ith annulus entering a folded half-plane lattice in the bottom half of the graph

where the bottom half-plane lattice is attached to the top half-plane lattice. The inner ring then travels 2^{i-1} edges “down” in the lattice, $2^i + 1$ “right” in the lattice, and 2^{i-1} “up” in the lattice into the next half-plane lattice. Meanwhile the outer ring enters a distance of 2^i away and travels 2^i edges “down” in the lattice, then $2^{i+1} + 1$ “right” in the lattice, then 2^i “up” in the lattice.

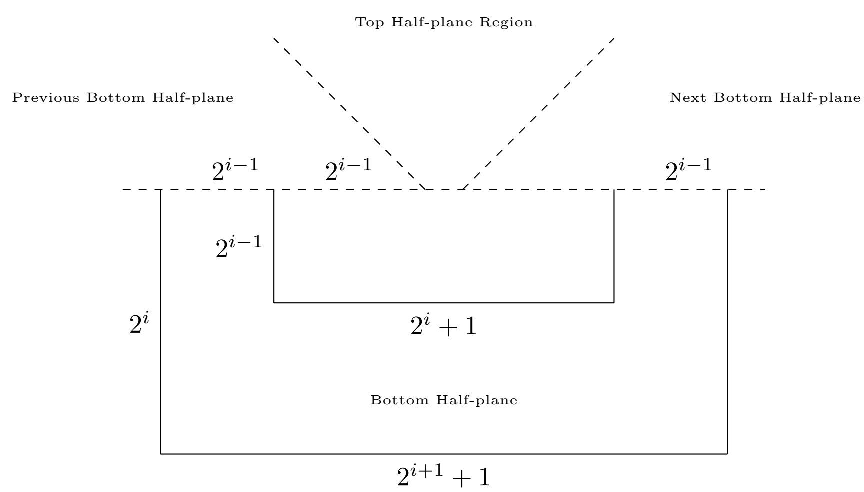

Figure 10: Middle half-planes of the bottom half of the annulus

Note that the outer ring of the ith annulus and inner ring of the $(i+1)$th annulus do not coincide in some places. However, since they are concentric and never intersect, we can still use Theorem 2.17 and the method of extremal length.

3.2 Bounding the Mass of the Annuli

Now that we have constructed our annuli, we will introduce a density function $\mu(e)$ on the edges to compute an upper bound the mass of each annulus.

For the ith annulus from the origin, we let each edge in the top half-plane have weight 1. For the entry bottom half-plane, we let the edges in the strip of width 1 have weight 1. Then, we let the weights halve in the regions depicted in Figure 11 until they reach weight $\frac{1}{2^{i-1}}$. All other edges in the bottom half-planes have weight $\frac{1}{2^{i-1}}$. With such a density function, we can check that the condition

$$\sum_{e \in E_t} \mu(e) \geq 1$$

is satisfied for all paths t contained in the annulus that start from the inner ring and end at the outer ring. Thus, we can use this density function to compute an upper bound on the mass of the annulus.

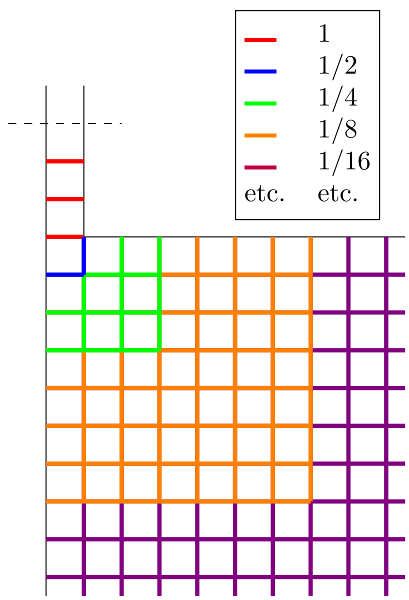

Figure 11: The density function in the first half-plane of the bottom half of the annulus

Note that since

$$\lambda(T_i)^{-1} = \inf \left\{ \sum_{e \in E_{T_i}} \mu(e)^2 \right\} \leq \sum_{e \in E_{T_i}} \mu(e)^2,$$

we can use an upper bound on $\sum_{e\in ET_i} \mu(e)^2$ as an upper bound for the mass of the ith annulus. We will bound the mass of the part of the annulus in each of the following regions separately: the top half-plane lattice, the two entry bottom half-plane lattices, and rest of the half-plane lattices.

Lemma 3.2. *The mass of the top half of the ith annulus is bounded above by $4i + 3$.*

Proof. As the density function $\mu(e)$ equal to 1 in for all edges in the top, the mass is bounded above by

$$\lambda^{-1}(T)_{\text{top}} \leq \sum_{e\in E_{T_{\text{top}}}} \mu(e)^2 = \left|E_{T_{\text{top}}}\right| = 4i + 3,$$

where the edge set $E_{T_{\text{top}}}$ includes the edges on the boundary between the top half and bottom half of the graph. □

Lemma 3.3. *The mass of the section of the ith annulus in the two entry bottom half plane lattices can be bounded above by $6i + 50$.*

Proof. The two entry bottom half-planes are mirror images of each other, so we look at one of them. For a single entry bottom half-plane, we first find the number of edges that have each weight. The number of edges with weight 1 is $i + 1$. Meanwhile, for $1 \leq k \leq i - 2$, the number of edges with weight $\frac{1}{2^k}$ is

$$n_k = 2 \cdot \left(\left(2^k - 1\right)^2 - \left(2^{k-1} - 1\right)^2\right).$$

We bound n_k above by $n_k < 2 \cdot (2^k)^2$. All other edges in the entry bottom half-plane lattice have weight $\frac{1}{2^{i-1}}$. For simplicity, we bound the number of edges with weight $\frac{1}{2^{i-1}}$ by the total number of edges in the entry bottom half-plane lattice part of the annulus. We break up the count for total number edges into horizontal and vertical edges.

$$\begin{aligned}\text{Horizontal: } h &= (2^i - i - 1)(2^i + 1) + (i + 1) \cdot (2^{i-1}) + (i + 1)\\ \text{Vertical: } v &= (2^i - i)(2^{i-1} + 1) + 2^i \cdot (2^{i-1} - 1)\end{aligned}$$

Then, we take an upper bound on h and v again, yielding

$$\begin{aligned}h &< (2 \cdot 2^i)(2 \cdot 2^i) + (2^i)(2^i) = 5 \cdot 2^{2i}\\ v &< (2^i)(2^i) + (2^i)(2^i) = 2 \cdot 2^{2i}.\end{aligned}$$

Overall, the mass from the entry bottom half-plane lattice can be bounded above by

$$\begin{aligned}\sum_{e\in ET_i} \mu(e)^2 &< (i + 1) \cdot (1)^2 + \sum_{k=1}^{i-2} \left(n_k \cdot \left(\frac{1}{2^k}\right)^2\right) + (h + v) \cdot \left(\frac{1}{2^{i-1}}\right)^2\\ &< (i + 1) + \sum_{k=1}^{i-2} \left(2 \cdot (2^k)^2 \cdot \left(\frac{1}{2^k}\right)^2\right) + 7 \cdot 2^{2i} \cdot \left(\frac{1}{2^{i-1}}\right)^2.\\ &= (i + 1) + 2(i - 2) + 28 = 3i + 25.\end{aligned}$$

Since there are two entry bottom half-plane lattices, their combined mass can be bounded above by $6i + 50$, as desired.

$\square$

Lemma 3.4. *The mass of the ith annulus in each folded half-plane lattice other than the entry half-plane lattices can be bounded above by* 48.

Proof. We first find an upper bound on the number of edges in the bottom half-plane lattice part of the ith annulus. Consider all edges in the bottom half-plane lattice within the outer ring of the ith annulus. There are at most $(2^i+1)(2^{i+1}+1)$ horizontal edges at at most $(2^i)(2^{i+1}+2)$ vertical edges within the outer ring of the ith annulus. Thus, we will take $2 \cdot (2^i + 1)(2^{i+1} + 2)$ as an upper bound on the number of edges.

Then, we can compute an upper bound for the mass of the ith annulus in a folded half-plane lattice as

$$\sum_{e \in ET_i} \mu(e)^2 < \left(\frac{1}{2^{i-1}}\right)^2 \cdot 2 \cdot (2^i + 1)(2^{i+1} + 2) < \left(\frac{1}{2^{i-1}}\right)^2 \cdot 2 \cdot (2 \cdot 2^i)(3 \cdot 2^i) = 48.$$

$\square$

With our upper bounds on the mass of the different parts of the annulus, we can complete our proof that the graph is recurrent.

Proof of Theorem 3.1. We compute a lower bound for the extremal length of each annulus using the fact that

$$\lambda(T_i)^{-1} = \inf \left\{ \sum_{e \in E_{T_i}} \mu(e)^2 \right\} \leq \sum_{e \in E_{T_i}} \mu(e)^2,$$

where T_i is the family of paths contained in the annulus that start from the inner ring and end at the outer ring, and E_{T_i} is the edge set of those paths.

From Lemma 3.2, Lemma 3.3, and Lemma 3.4, the mass of the ith annulus is bounded by

$$\sum_{e \in ET_i} \mu(e)^2 < (4i + 3) + 6i + 50 + 2i \cdot (48) = 106i + 53.$$

Thus, the extremal length of the ith annulus is bounded below by

$$\lambda(T_i) > \frac{1}{106i + 53}.$$

By Theorem 2.17, the extremal length of the whole graph is bounded below by the sum of the extremal lengths of all the annuli, which is a harmonic series. Since the harmonic series is well known to be divergent, the extremal length is infinite, proving that the graph is recurrent. $\square$

4 Approach For Proving Directly that the Extended Speiser Graph of a Hyperbolic Surface is Transient

In this section, we examine a surface proven to be hyperbolic in [3]. The Speiser graph and extended Speiser graph (with multi-edges replaced with single edges) are shown in Figure 12 and Figure 13, respectively.

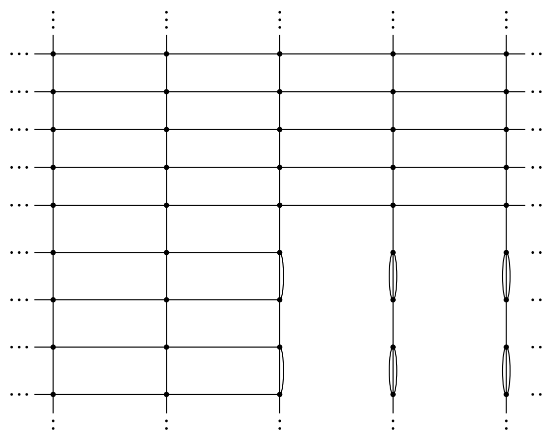

Figure 12: The Speiser graph of a hyperbolic surface

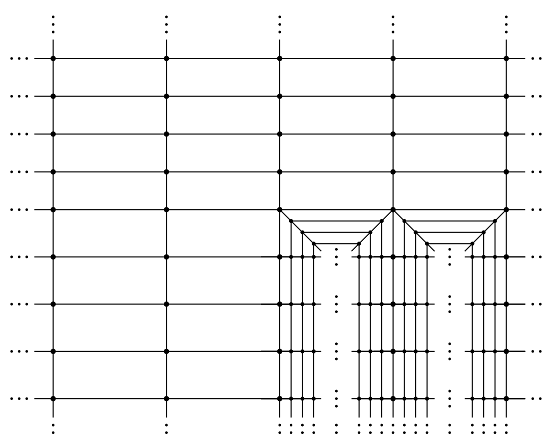

Figure 13: The extended Speiser graph of a hyperbolic surface

Conjecture 4.1. *The extended Speiser graph depicted in Figure 13 can be proven to be transient using only discrete methods on the graph, rather than employing methods on the covering surface. In particular, by Theorem 2.13, the* *covering surface corresponding to the Speiser graph in Figure 12 can be proven to be hyperbolic by determining the type of its extended Speiser graph.*

We were not able to prove the extended Speiser graph in Figure 13 is transient. However, we explain a possible approach that may potentially prove transience. Our suggested approach is to use the method of flows described in Section 2.4.2, that is, finding a unit flow on the graph with finite dissipation rate.

4.1 Shape of the Flow

Taking inspiration from the surface methods used in [3] to prove the surface is hyperbolic, we want to construct a discrete analog for the graph in the form of a flow that "spirals" clockwise outwards from the origin (the corner of the boundary between the 3/4 plane lattice and the folded half-plane lattices). We want the width of the flow to be a_n on the nth time spiraling around, where a_0 is some arbitrary positive real number and $a_{n+1} = 2^{2^{a_n}}$. The width of the flow should grow in the region of folded half-plane lattices and stay constant in the 3/4 plane lattice.

4.2 Distribution of Flow

4.2.1 Uniform Flow

Suppose the spiral has width n. We look at a "corner," as depicted in Figure 14, which is a pattern that will likely appear often as the flow turns in the spiral. If the flow in every edge is $\frac{1}{n}$, then the energy dissipation of a corner is

$$\frac{1}{2}\sum_{x,y} j_{xy}^2 R_{xy} = \frac{1}{2}\sum_{x,y}\left(\frac{1}{n}\right)^2 \cdot (1) = \frac{n(n+1)}{2n^2} > \frac{1}{2},$$

where we assume $R = 1$ for all edges. The number of edges in a corner is $n(n+1)$, which yields the calculation above.

The energy dissipation is bounded below, and because there will be an infinite number of corners, the total energy dissipation will be infinite. However, we want to find a flow for which the total energy dissipation is finite, and thus we cannot use a uniform flow to turn corners.

4.2.2 An Example of Nonuniform Flow

Consider a corner of width $2^{2^{a_n}}$. Notice the number of edges traversed by a flow in the same strip when rounding the corner is smaller for flow that is closer to the origin. In fact the number of edges traversed are $0, 2, 4, \ldots, 2 \cdot 2^{2^{a_n}}$. Suppose for these edges, we assign flow of $0, \frac{1}{S_n}, \frac{\frac{1}{2}}{S_n}, \ldots, \frac{\frac{1}{2^{2^{a_n}}}}{S_n}$, respectively as shown in Figure 15, where $S_n = \sum_{k=1}^{2^{2^{a_n}}} \frac{1}{k}$. Computing the energy dissipation

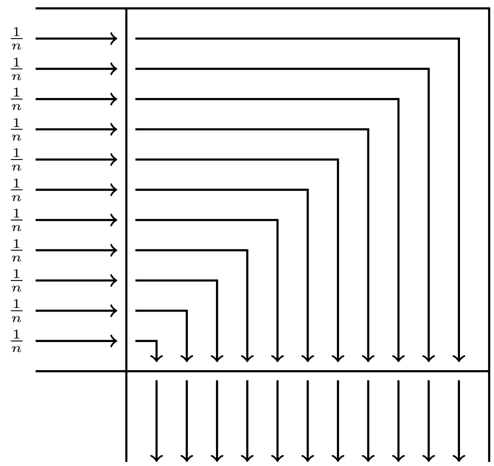

Figure 14: A uniform flow around a corner

gives

$$\frac{1}{2}\sum_{x,y} j_{xy}^2 R_{xy} = \frac{1}{2}\sum_{k=1}^{2^{2^{a_n}}} 2k \cdot \left(\frac{\frac{1}{k}}{S_n}\right)^2 \cdot (1) = \sum_{k=1}^{2^{2^{a_n}}} \frac{1}{kS_n^2} = \frac{1}{S_n}.$$

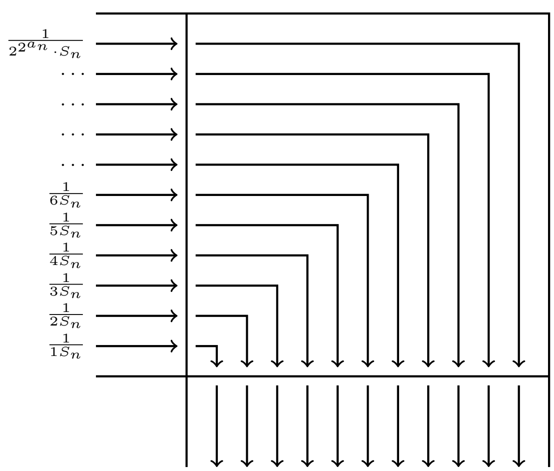

Figure 15: A nonuniform flow around a corner

Notice that

$$S_n = \sum_{k=1}^{2^{2^{a_n}}} \frac{1}{k} \geq \sum_{j=1}^{2^{a_n}} \left(2^{j-1} \cdot \frac{1}{2^j}\right) \geq 2^{a_n - 1},$$

so $\frac{1}{S_n} \leq \frac{1}{2^{a_n-1}}$. With this nonuniform distribution of flow, we could potentially get a finite total energy dissipation bounded above by the sum of a geometric series. However, the geometric series bound requires the width of each consecutive corner to increase more than exponentially, a condition which we have not been able to satisfy given the frequency with which the spiral turns.

4.2.3 Finishing the Flow: Next Steps

Wwe believe that the most important issues to resolve for using the flow method on this problem are the following:

- Growth: How should the the width of the flow grow in the region of folded half-planes as to yield a finite energy dissipation.
- Corner Flow: Section 4.2.2 gives an example of flow around a corner that could potentially give a bounded energy dissipation, but requires the size of the corners to increase faster than exponentially. Are there other corner flows that could work better? Are there other ways for the flow to turn besides using a corner?
- Transitions: The optimal distribution of flow in "straightaways," where the number of edges traversed by each line of the flow are the same is a uniform distribution. This differs from the distribution of flow in corners. How should the flow transition from one distribution to another?

4.3 An Alternative Approach Using Trees

Another potential approach for those tackling this problem is to find a transient treelike subgraph.

Trees are a common type of graph which are often transient. Random walks on a tree often have a nonzero probability of escape due to the way that trees branch outwards. However, not all trees are transient.

Consider a tree that branches n ways each time it splits and the length of whose branches scale by k after each split. Figure 16 shows such a tree with $n = 2$ and $k = 3$. By looking at a quantity called effective resistance, one can find that trees with $n > k$ are transient.

We did not extensively try embedding a tree in our graph, so we cannot speak to its feasibility. However, we will provide a few heuristic observations that may give insight into some potential challenges.

- Given the graph in Section 3 is recurrent, by Theorem 2.10, we know the half-plane lattices by themselves must be recurrent. Thus, the embedding must somehow take advantage of the 3/4 plane, likely in a similar spiral pattern as for the flow.
- The maximal degree of any vertex in our extended Speiser graph is 6, on the boundary between the 3/4 plane and the half-plane lattices. Thus, the number of ways our tree splits is at most 5. More realistically, the

maximum number of ways our tree splits is 3, as 2 of the directions are along the boundary. Thus, for our tree to work, we need to ensure the length of the branches at most doubles between splittings of the tree.

- Unlike in our example, the number of ways the tree splits and the lengths of its branches are not necessarily constant throughout the entire tree. This allows for a more diverse variety of trees, but also makes computations of effective resistance to determine the type more difficult.

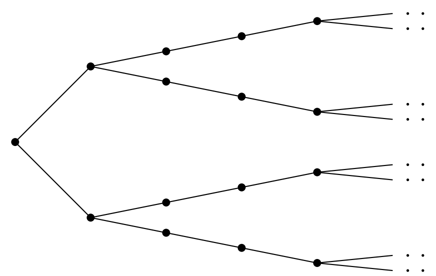

Figure 16: A Tree That Splits 2 Ways and Scales by 3 at Each Node

4.4 Project Extensions

We believe that the approach outlined in this paper will lead to a proof of the transience of the graph in Section 4. If successful, one could use similar methods to investigate the type of the extended Speiser graphs where the half-plane lattices are no longer only folded into a strip that is one edge wide, but also into strips of longer widths as well. For example, one could consider extended Speiser graphs were the widths of strips containing the folded half-plane lattices grow in an arithmetic or a geometric sequence. This extension can also be studied for the recurrent graph.

A possible computational aid that could help this project would be computing discrete harmonic functions for finite versions of these extended Speiser graphs (including the graph for the parabolic surface). Such computations could potentially be carried out through algorithms using energy minimizing properties. The behavior of the discrete harmonic functions as the size of the graph approaches infinity could give insight into how construct annuli to prove recurrence or construct flows to prove transience. Hopefully, finding these functions will help shed more light on the relation between surfaces and Speiser graphs.

Acknowledgments

First, I would like to thank my mentor Dr. Sergiy Merenkov for his support throughout the PRIMES program. It is because of his guidance that I have learned so much over the past year.

Thank you to Thomas Rud and Tanya Khovanova for reviewing my paper. I would also like to thank the entire PRIMES program for this amazing research opportunity, with special thanks to Pavel Etingof and Slava Gerovitch.

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